Iranian Journal of Mathematical Sciences and Informatics
Vol. 17, No. 1 (2022), pp 111-123
DOI: 10.52547/ijmsi.17.1.111

On Bernstein Type Inequalities for Complex Polynomial

Elahe Khojastehnezhad, Mahmood Bidkham*

Department of Mathematics, University of Semnan, Semnan, Iran

E-mail: ekhojastehnejadelahe@semnan.ac.ir
E-mail: mbidkham@semnan.ac.ir; mdbidkham@gmail.com

Abstract. In this paper, we establish some Bernstein type inequalities for the complex polynomial. Our results constitute generalizations and refinements of some well-known polynomial inequalities.

Keywords: Inequality, Polynomial, Derivative, Maximum modulus, Restricted zeros.

2000 Mathematics subject classification: 30A10, 30C10, 30D15.

[DOI: 10.52547/ijmsi.17.1.111]

1. Introduction and statement of results

Let p be a polynomial of degree atmost n. Then, according to a famous result known as Bernsteins inequality [8]

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|, \tag{1.1}$$

whereas concerning the maximum modulus of p on a large circle $|z| = R > 1$, we have [20]

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)|. \tag{1.2}$$

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then inequalities (1.1) and (1.2) can be sharpened. In fact, if $p(z) \neq 0$ in $|z| < 1$, then (1.1) and (1.2) can respectively be replaced by

*Corresponding Author

Received 11 August 2018; Accepted 16 August 2020

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{1.3}$$

and

$$\max_{|z|=R}|p(z)| \leq \frac{R^n+1}{2}\max_{|z|=1}|p(z)|,\ R>1. \tag{1.4}$$

Inequality (1.3) was conjectured by Erdös and later verified by Lax [19], whereas Ankeny and Rivlin [5] used (1.3) to prove (1.4).

In the literature, there are already various generalizations and refinements of (1.3) and (1.4), for example (see Aziz [6], Bidkham et al. [9, 10, 11], Khojastehnezhad and Bidkham [17], Zireh [21], etc).

Inequalities (1.3) and (1.4) were sharpened by Dewan et.al [12, 13] proving that under the same hypothesis, for every real or complex number β with $|\beta| \leq 1,\ R>1$ and $|z|=1$, we have

$$|zp'(z)+\frac{n\beta}{2}p(z)| \leq \frac{n}{2}\{(|1+\frac{\beta}{2}|+|\frac{\beta}{2}|)\max_{|z|=1}|p(z)|-(|1+\frac{\beta}{2}|-|\frac{\beta}{2}|)\min_{|z|=1}|p(z)|\}, \tag{1.5}$$

and

$$|p(Rz)+\beta(\frac{R+1}{2})^n p(z)| \leq \frac{1}{2}\{(|R^n+\beta(\frac{R+1}{2})^n|+|1+\beta(\frac{R+1}{2})^n|)\max_{|z|=1}|p(z)|- (|R^n+\beta(\frac{R+1}{2})^n|-|1+\beta(\frac{R+1}{2})^n|)\min_{|z|=1}|p(z)|\}. \tag{1.6}$$

Also they [12] proved if p has all its zeros in $|z| \leq 1$, then for every real or complex number β with $|\beta| \leq 1$, we have

$$\min_{|z|=1}|zp'(z)+\frac{n\beta}{2}p(z)| \geq n|1+\frac{\beta}{2}|\min_{|z|=1}|p(z)|. \tag{1.7}$$

In this paper, we first prove an interesting result which is a compact generalization of inequality (1.7).

Theorem 1.1. *If p is a polynomial of degree n having all its zeros in $|z| \leq k,\ k>0$, then for all $\alpha,\ \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R \geq r,\ rR \geq k^2$ and $|z|=1$, we have*

$$|p(Rz)-\alpha p(rz)+\beta\{(\frac{R+k}{r+k})^n-|\alpha|\}p(rz)| \geq \frac{1}{k^n}|R^n-\alpha r^n+\beta\{(\frac{R+k}{r+k})^n-|\alpha|\}r^n|\min_{|z|=k}|p(z)|. \tag{1.8}$$

Assuming $\alpha=1$ in Theorem 1.1, we have the following result.

[DOI: 10.52547/ijmsi.17.1.111]

Corollary 1.2. *Let p be a polynomial of degree n such that does not vanish in $|z| > k,\ k > 0$, then for all $\beta \in \mathbb{C}$ with $|\beta| \leq 1,\ R > r,\ rR \geq k^2$ and $|z| = 1$, we get*

$$|p(Rz) - p(rz) + \beta\{(\frac{R+k}{r+k})^n - 1\}p(rz)| \geq \frac{1}{k^n}|R^n - r^n + \beta\{(\frac{R+k}{r+k})^n - 1\}r^n| \min_{|z|=k} |p(z)|. \tag{1.9}$$

By dividing the two sides of the inequality (1.9) by $(R-r)$ and letting $R \to r$, we get the following interesting result.

Corollary 1.3. *Let p be a polynomial of degree n such that does not vanish in $|z| > k,\ k > 0$. Then for all $\beta \in \mathbb{C}$ with $|\beta| \leq 1,\ r \geq k$ and $|z| = 1$, we get*

$$|zp'(rz) + \frac{n\beta}{r+k}p(rz)| \geq \frac{n}{k^n}|r^{n-1} + \frac{\beta}{r+k}r^n| \min_{|z|=k} |p(z)|. \tag{1.10}$$

Assuming $k = 1,\ r = 1$ in Corollary (1.3), we have the inequality (1.7). Using Theorem 1.1, we prove the following theorem, which provides a compact generalization of inequalities (1.5), (1.6).

Theorem 1.4. *Let p be a polynomial of degree n such that it does not vanish in $|z| < k,\ k > 0$. Then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R \geq r,\ rR \geq \frac{1}{k^2}$ and $|z| = 1$,*

$$\begin{aligned}&|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)| \leq \\ &\frac{1}{2}\{[k^n|R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n| + |1 - \alpha + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}|] \max_{|z|=k} |p(z)| - \\ &[k^n|R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n| - |1 - \alpha + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}|] \min_{|z|=k} |p(z)|\}.\end{aligned} \tag{1.11}$$

Equality holds for the polynomials $az^n + bk^n$, $|a| = |b|$.

Assuming $\alpha = 1$ in Theorem 1.4, we have the following result.

Corollary 1.5. *Let p be a polynomial of degree n such that does not vanish in $|z| < k,\ k > 0$. Then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1,\ R \geq r,\ rR \geq \frac{1}{k^2}$ and $|z| = 1$, we get*

$$\begin{aligned}&|p(Rk^2z) - p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - 1\}p(rk^2z)| \leq \frac{1}{2}\{ \\ &[k^n|R^n - r^n + \beta\{(\frac{Rk+1}{rk+1})^n - 1\}r^n| + |\beta\{(\frac{Rk+1}{rk+1})^n - 1\}|] \max_{|z|=k} |p(z)| - \\ &[k^n|R^n - r^n + \beta\{(\frac{Rk+1}{rk+1})^n - 1\}r^n| - |\beta\{(\frac{Rk+1}{rk+1})^n - 1\}|] \min_{|z|=k} |p(z)|\}.\end{aligned} \tag{1.12}$$

By dividing the two sides of the inequality (1.12) by $(R-r)$ and letting $R \to r$, we get the following interesting result.

Corollary 1.6. *Let p be a polynomial of degree n such that does not vanish in $|z|<k,\ k>0$. Then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1,\quad r \geq \frac{1}{k}$ and $|z|=1$, we have*

$$|k^2zp'(rk^2z) + \frac{n\beta k}{rk+1}p(rk^2z)| \leq \frac{n}{2}\{[k^n|r^{n-1} + \frac{\beta k}{rk+1}r^n| + |\frac{\beta k}{rk+1}|]\max_{|z|=k}|p(z)| - [k^n|r^{n-1} + \frac{\beta k}{rk+1}r^n| - |\frac{\beta k}{rk+1}|]\min_{|z|=k}|p(z)|\}. \tag{1.13}$$

Remark 1.7. Assuming $k=1$ and $r=1$ in Corollary 1.6 we have the inequality (1.5).

2. Lemmas

To prove of these theorems, we need the following lemmas. The first lemma is due to Aziz and Zargar [7].

Lemma 2.1. *Let p be a polynomial of degree n having all its zeros in $|z| \leq k$, $k>0$. Then for every $R \geq r$ and $rR \geq k^2$, we have*

$$|p(Rz)| \geq (\frac{R+k}{r+k})^n|p(rz)|, \quad |z|=1. \tag{2.1}$$

Lemma 2.2. *Let p be a polynomial of degree n such that does not vanish in $|z|<k,\ k>0$, and $q(z)=z^n\overline{p(\frac{1}{\bar{z}})}$. Then for all $\alpha,\ \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R \geq r,\ rR \geq \frac{1}{k^2}$ and $|z|=1$, we have*

$$\begin{aligned}|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)| \leq \\ k^n|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)|.\end{aligned} \tag{2.2}$$

Proof. Based on the hypotheses that the polynomial p has no zeros in $|z|<k$, therefore the polynomial $q(z)=z^n\overline{p(\frac{1}{\bar{z}})}$ has all its zeros in $|z| \leq \frac{1}{k}$. Since $\frac{1}{k^n}|p(k^2z)| = |q(z)|$ for $|z|=\frac{1}{k}$, therefore the function $\phi(z)=\frac{p(k^2z)}{k^nq(z)}$ is analytic in the disc $|z| \geq \frac{1}{k}$ and $|\phi(z)|=1$ on $|z|=\frac{1}{k}$. Hence based on the the maximum modulus principle $|\phi(z)|<1$ for $|z|>\frac{1}{k}$, or equivalently

$$|p(k^2z)| \leq k^n|q(z)|,\ |z| \geq \frac{1}{k}. \tag{2.3}$$

Since $\frac{1}{k^n}|p(k^2z)| = |q(z)|$ for $|z|=\frac{1}{k}$, therefore for every real or complex number δ with $|\delta|<1$ and $|z|=\frac{1}{k}$, $|\delta p(k^2z)| < |k^nq(z)|$. Now using Rouche's theorem it follows that all the zeros of $H(z) := k^nq(z)+\delta p(k^2z)$ lie in $|z| \leq \frac{1}{k}$. While applying Lemma 2.1, we have

$$|H(Rz)| \geq (\frac{Rk+1}{rk+1})^n|H(rz)| > |H(rz)|, \quad |z|=1, \tag{2.4}$$

[DOI: 10.52547/ijmsi.17.1.111]

where $R > r,\ rR \geq \frac{1}{k^2}$.
It follows that for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, we get

$$|H(Rz) - \alpha H(rz)| \geq |H(Rz)| - |\alpha||H(rz)| \geq \{(\frac{Rk+1}{rk+1})^n - |\alpha|\}|H(rz)|,\ |z| = 1$$

i.e.

$$|H(Rz) - \alpha H(rz)| \geq \{(\frac{Rk+1}{rk+1})^n - |\alpha|\}|H(rz)| \text{ for } |z| = 1. \tag{2.5}$$

Since $H(Rz)$ has all its zeros in $|z| \leq \frac{1}{Rk} < 1$, and $|H(rz)| < |H(Rz)|$ for $|z| = 1$, a direct application of Rouche's theorem shows that the polynomial $H(Rz) - \alpha H(rz)$ has all its zeros in $|z| < 1$. Using Rouche's theorem again, it follows that for every $\beta \in \mathbb{C}$ with $|\beta| < 1$ and $R > r,\ rR \geq \frac{1}{k^2}$, all the zeros of the polynomial

$$T(z) = H(Rz) - \alpha H(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}H(rz)$$

lie in $|z| < 1$.
Replacing $H(z)$ by $k^n q(z) + \delta p(k^2 z)$, we conclude that all the zeros of

$$\begin{aligned} T(z) = k^n[q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)]+ \\ \delta\{p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)\} \end{aligned} \tag{2.6}$$

lie in $|z| < 1$, for every $R > r,\ rR \geq \frac{1}{k^2},\ |\alpha| \leq 1,\ |\beta| < 1$ and $|\delta| < 1$. We now show that (2.6) implies (2.2). Indeed, suppose otherwise. Then, there is a point $z = z_0$ with $|z_0| = 1$ such that

$$|p(Rk^2z_0) - \alpha p(rk^2z_0) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z_0)| >$$
$$k^n|q(Rz_0) - \alpha q(rz_0) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz_0)|.$$

We take

$$\delta = -\frac{k^n[q(Rz_0) - \alpha q(rz_0) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz_0)]}{p(Rk^2z_0) - \alpha p(rk^2z_0) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z_0)},$$

then $|\delta| < 1$ and with this choice of δ, we have, $T(z_0) = 0$ for $|z_0| = 1$. But this contradicts that T has all its zeros in $|z| < 1$. For the case β, with $|\beta| = 1$, (2.2) follows by continuity. For $R = r$ inequality (2.2) follows by inequality (2.3). This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let p be a polynomial of degree n. Then for all $\alpha,\ \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R \geq r,\ rR \geq k^2,\ k > 0$ and $|z| = 1$, we have*

$$\begin{aligned} |p(Rz) - \alpha p(rz) + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}p(rz)| \leq \\ \frac{1}{k^n}|R^n - \alpha r^n + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}r^n|\max_{|z|=k}|p(z)|. \end{aligned} \tag{2.7}$$

[DOI: 10.52547/ijmsi.17.1.111]

Proof. Let $M = \max_{|z|=k} |p(z)|$, then for δ with $|\delta| > 1$, we can conclude from Rouche's theorem that all zeros of polynomial $H(z) = p(z) - \delta M(\frac{z}{k})^n$ lie in the closed disk $|z| \le k,\ k > 0$. Using Lemma 2.1, we have

$$|H(Rz)| \ge (\frac{R+k}{r+k})^n |H(rz)| > |H(rz)|, \quad |z| = 1, \tag{2.8}$$

where $R > r,\ rR \ge k^2$.
It follows that for every $\alpha \in \mathbb{C}$ with $|\alpha| \le 1$, we get

$$|H(Rz) - \alpha H(rz)| \ge |H(Rz)| - |\alpha||H(rz)| \ge \{(\frac{R+k}{r+k})^n - |\alpha|\}|H(rz)|,\ |z| = 1,$$

i.e.

$$|H(Rz) - \alpha H(rz)| \ge \{(\frac{R+k}{r+k})^n - |\alpha|\}|H(rz)|,\ |z| = 1. \tag{2.9}$$

Since $H(Rz)$ has all its zeros in $|z| \le \frac{k}{R} < 1$, and $|H(rz)| < |H(Rz)|$, a direct application of Rouche's theorem shows that the polynomial $H(Rz) - \alpha H(rz)$ has all its zeros in $|z| < 1$. Using Rouche's theorem again ,implies that for every $\beta \in \mathbb{C}$ with $|\beta| < 1$ and $R > r,\ rR \ge k^2$, all the zeros of the polynomial

$$T(z) = H(Rz) - \alpha H(rz) + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}H(rz)$$

lie in $|z| < 1$.
Replacing $H(z)$ by $p(z) - \delta M(\frac{z}{k})^n$, we conclude that all the zeros of

$$\begin{aligned} T(z) = [p(Rz) - \alpha p(rz) + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}p(rz)] + \\ \delta \frac{Mz^n}{k^n}\{R^n - \alpha r^n + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}r^n\} \end{aligned} \tag{2.10}$$

lie in $|z| < 1$, for every $R > r,\ rR \ge k^2,\ |\alpha| \le 1,\ |\beta| < 1$ and $|\delta| > 1$. This implies

$$\begin{aligned} |p(Rz) - \alpha p(rz) + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}p(rz)| \le \\ |R^n - \alpha r^n + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}r^n| \frac{M}{k^n}, \end{aligned} \tag{2.11}$$

where $|z| = 1$.
For β, with $|\beta| = 1$, (2.11) follows by continuity. For $R = r$ inequality (2.11) reduces to $|p(rz)| \le \frac{r^n}{k^n} \max_{|z|=k} |p(z)|$ which it follows by taking $p(kz)$ and $|z| = \frac{r}{k}$ where $\frac{r}{k} \ge 1$ in inequality (1.2). This completes the proof of Lemma 2.3. □

[DOI: 10.52547/ijmsi.17.1.111]

Lemma 2.4. *If p is a polynomial of degree n, then for all $\alpha,\ \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R \geq r,\ \ rR \geq \frac{1}{k^2}$ and $|z| = 1$,*

$$\begin{aligned}&|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)| + \\ &k^n|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)| \leq \\ &\{k^n|R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n| + |1 - \alpha + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}|\} \max_{|z|=k} |p(z)|,\end{aligned} \tag{2.12}$$

where $q(z) = z^n\overline{p(1/\overline{z})}$.

Proof. Assume that $M = \max_{|z|=k} |p(z)|$. Then, for δ with $|\delta| > 1$, we can conclude from Rouche's theorem that the polynomial $G(z) = p(z) - \delta M$ does not vanish in $|z| < k$. If we take $H(z) = z^n\overline{G(1/\overline{z})}$, then $|G(k^2z)| = k^n|H(z)|$ for $|z| = \frac{1}{k}$. Using Lemma 2.2, for all $\alpha,\ \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R \geq r\ \ rR \geq \frac{1}{k^2}$ and $|z| = 1$, we have

$$\begin{aligned}&|G(Rk^2z) - \alpha G(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}G(rk^2z)| \leq \\ &k^n|H(Rz) - \alpha H(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}H(rz)|.\end{aligned} \tag{2.13}$$

Therefore, by using the equality

$$\begin{aligned}H(z) &= z^n\overline{G(\frac{1}{\overline{z}})} = z^n\overline{p(\frac{1}{\overline{z}})} - \overline{\delta}Mz^n \\ &= q(z) - \overline{\delta}Mz^n,\end{aligned}$$

we get

$$\begin{aligned}&|\{p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)\} - \\ &\delta\{1 - \alpha + \beta\{(\frac{R+1}{r+1})^n - |\alpha|\}\}M| \leq \\ &k^n|\{q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)\} - \\ &\overline{\delta}\{R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n\}M|.\end{aligned} \tag{2.14}$$

Since $\frac{1}{k^n}|p(k^2z)| = |q(z)|$ for $|z| = \frac{1}{k}$, therefore

$$\begin{aligned}\max_{|z|=\frac{1}{k}} |q(z)| &= \frac{1}{k^n} \max_{|z|=k} |p(z)|, \\ \max_{|z|=\frac{1}{k}} |q(z)| &= \frac{M}{k^n}.\end{aligned}$$

[DOI: 10.52547/ijmsi.17.1.111]

Now by applying Lemma 2.3 to $q(z)$ for $\frac{1}{k} > 0$, we have

$$|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)| \leq$$
$$\{R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n\}k^n \max_{|z|=\frac{1}{k}} |q(z)|.$$

i.e.

$$|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)| \leq$$
$$\{R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n\}M.$$

Now by suitable choice of argument of δ, we get

$$\begin{aligned}|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)\}- \\ \overline{\delta}\{R^n - \alpha r^n + \beta\{(\frac{R+1}{r+1})^n - |\alpha|\}r^n\}M| = \\ |\delta||R^n - \alpha r^n + \beta\{(\frac{R+1}{r+1})^n - |\alpha|\}r^n|M- \\ |q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)|.\end{aligned} \tag{2.15}$$

Combining right hand sides of (2.14) and (2.15) we can obtain

$$\begin{aligned}|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)|- \\ |\delta||1 - \alpha + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}|M \leq \\ |\delta|k^n|R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n|M||- \\ k^n|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)|,\end{aligned}$$

which implies

$$\begin{aligned}&|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)|+ \\ &k^n|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)| \leq \\ &|\delta|\{k^n|R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n| + |1 - \alpha + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}|\}M.\end{aligned}$$

Making $|\delta| \to 1$, we have the result. □

Lemma 2.5. *Let p be a polynomial of degree n having no zeros in $|z| < k,\ k > 0$. Then for all $\alpha,\ \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R \geq r,\ \ Rr \geq \frac{1}{k^2}$ and $|z| = 1$,* *we have*

$$|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)| \leq \frac{1}{2}$$
$$\{k^n|R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n| + |1 - \alpha + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}|\} \max_{|z|=k}|p(z)| \tag{2.16}$$

Proof. Since p does not vanish in $|z| < k,\ k > 0$, Lemma 2.2, yields

$$\begin{aligned}|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)| \leq \\ k^n|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)|,\end{aligned} \tag{2.17}$$

Now by combining the inequalities (2.12) and (2.17), we have

$$\begin{aligned}&2|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)| \leq \\ &|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)| + \\ &k^n|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)| \leq \\ &\{k^n|R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n| + |1 - \alpha + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}|\} \max_{|z|=k}|p(z)|.\end{aligned} \tag{2.18}$$

This gives the result. □

3. Proofs of the theorems

Proof of Theorem 1.1. If p has a zero on $|z| = k$, then inequality is trivial. Therefore, we assume that $p(z)$ has all its zeros in $|z| < k$. If $m = \min_{|z|=k}|p(z)|$, then $m > 0$ and $|p(z)| \geq m$ for $|z| = k$. If $|\lambda| < 1$, then it follows by Rouche's theorem that the polynomial $p(z) - \lambda m(\frac{z}{k})^n$, has all its zeros in $|z| < k,\ k > 0$. Proceeding similarly as in the proof of Lemma 2.3, it follows that all the zeros of

$$\begin{aligned}p(Rz) - \alpha p(rz) + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}p(rz) + \\ \lambda m(\frac{z}{k})^n\{R^n - \alpha r^n + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}r^n\}\end{aligned} \tag{3.1}$$

lie in $|z| < 1$, for every $R \geq r,\ Rr \geq k^2,\ |\alpha| \leq 1,\ |\beta| < 1$ and $|\lambda| < 1$. This implies

$$\begin{aligned}\frac{m}{k^n}|R^n - \alpha r^n + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}r^n| \leq \\ |p(Rz) - \alpha p(rz) + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}p(rz)|,\end{aligned} \tag{3.2}$$

[DOI: 10.52547/ijmsi.17.1.111]

where $|z|=1$. This completes the proof. $\square$

Proof of Theorem 1.4. If $p(z)$ has a zero on $|z|=k$, then $\min_{|z|=k}|p(z)|=0$ and in this case the result follows from Lemma 2.5. Hence we assume that $p(z)\neq 0$ in $|z|\leq k$. In this case we have $m=\min_{|z|=k}|p(z)|>0$ and for γ with $|\gamma|<1$, we get $|\gamma m|<m\leq|p(z)|$, where $|z|=k$. Now we conclude from Rouche's theorem that the polynomial $G(z)=p(z)-\gamma m$ does not vanish in $|z|<k$. If we take $H(z)=z^n\overline{G(1/\overline{z})}$, then by using the polynomials $G(z)$ and $H(z)$ in Lemma 2.2, we have

$$\begin{aligned}|G(Rk^2z)-\alpha G(rk^2z)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}G(rk^2z)|\leq\\ k^n|H(Rz)-\alpha H(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}H(rz)|.\end{aligned}\tag{3.3}$$

Using the fact that

$$H(z)=z^n\overline{G(\frac{1}{\overline{z}})}=z^n\overline{p(\frac{1}{\overline{z}})}-\overline{\gamma}mz^n=q(z)-\overline{\gamma}mz^n,$$

or

$$H(z)=q(z)-\overline{\gamma}mz^n,$$

and substituting $G(z)$ and $H(z)$ in (3.3), we get

$$\begin{aligned}&|\{p(Rk^2z)-\alpha p(rk^2z)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}p(rk^2z)\}-\\ &\gamma\{1-\alpha+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}\}m|\leq\\ &k^n|\{q(Rz)-\alpha q(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}q(rz)\}-\\ &\overline{\gamma}\{R^n-\alpha r^n+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}r^n\}mz^n|.\end{aligned}\tag{3.4}$$

Since the polynomial $q(z)=z^n\overline{p(\frac{1}{\overline{z}})}$ has all zeros in $|z|\leq\frac{1}{k}$ and $m=\min_{|z|=k}|p(z)|=k^n\min_{|z|=\frac{1}{k}}|q(z)|$, hence by applying Theorem 1.1 for the polynomial $q(z)$ with $\frac{1}{k}$, we obtain

$$\begin{aligned}&|R^n-\alpha r^n+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}r^n|m\leq\\ &|q(Rz)-\alpha q(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}q(rz)|.\end{aligned}$$

[DOI: 10.52547/ijmsi.17.1.111]

Therefore, by suitable choice of argument of γ, we get

$$\begin{aligned}|\{q(Rz)-\alpha q(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}q(rz)\}-\\ \overline{\gamma}\{R^n-\alpha r^n+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}r^n m|=\\ |q(Rz)-\alpha q(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}q(rz)|-\\ |\gamma||R^n-\alpha r^n+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}r^n|m.\end{aligned} \tag{3.5}$$

Now combining (3.4) and (3.5), we get

$$\begin{aligned}|p(Rk^2z)-\alpha p(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}p(rk^2z)|-\\ |\gamma||1-\alpha+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}|m\leq\\ k^n|q(Rz)-\alpha q(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}q(rz)|-\\ |\gamma|k^n|R^n-\alpha r^n+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}r^n|m.\end{aligned}$$

This implies

$$\begin{aligned}&|p(Rk^2z)-\alpha p(rk^2z)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}p(rk^2z)|\leq\\ &k^n|q(Rz)-\alpha q(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}q(rz)|-\\ &|\gamma|\{k^n|R^n-\alpha r^n+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}r^n|z^n|-|1-\alpha+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}|\}m.\end{aligned}$$

Letting $|\gamma|\to 1$, we have

$$\begin{aligned}&|p(Rk^2z)-\alpha p(rk^2z)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}p(rk^2z)|\leq\\ &k^n|q(Rz)-\alpha q(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}q(rz)|-\\ &\{k^n|R^n-\alpha r^n+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}r^n|-|1-\alpha+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}|\}m.\end{aligned} \tag{3.6}$$

On the other hand, based on Lemma 2.4, we have

$$\begin{aligned}&|p(Rk^2z)-\alpha p(rk^2z)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}p(rk^2z)|+\\ &k^n|q(Rz)-\alpha q(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}q(rz)|\leq\\ &\{k^n|R^n-\alpha r^n+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}r^n|+|1-\alpha+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}|\}\max_{|z|=k}|p(z)|.\end{aligned} \tag{3.7}$$

[DOI: 10.52547/ijmsi.17.1.111]

Combining (3.6) and (3.7), we get (1.11) and this completes the proof of Theorem 1.4. □

Acknowledgments

The authors wish to thank the referees for their comments and sugestions.

References

1. F. G. Abdullayev, On Some Properties of Orthogonal Polynomials Over an Area in Domains of the Complex Plane III, *Ukrainian Math. J.*, **53** (12), (2001), 1588-1599.
2. F. G. Abdullayev, The Properties of the Orthogonal Polynomials with Weight Having Singulerity on the Boundary Contour, *J. Comput. Anal. Appl.*,**6** (1), (2004), 43-59.
3. F. G. Abdullayev, D. Aral, The Relation Between Different Norms of Algebraic Polynomials in the Regions of Complex Plane, *Azerb. J. Math.*, **1** (2), (2011), 1-16.
4. F. G. Abdullayev, N. P. Ozkartepe, An Analogue of the Bernstein-Walsh Lemma in Jordan Regions, *J. Inequal. Appl.*, (2013), 2013:570, 7 pp
5. N. C. Ankeny, T. J. Rivlin, On a Theorem of S. Bernstein, *Pacific J. Math.*, **5**, (1955), 849-852.
6. A. Aziz, Inequalities for the Polar Derivative of a Polynomial, *J. Approx. Theory.*, **55**, (1998), 183-193.
7. A. Aziz, B. A. Zargar, Inequalities for a Polynomial and its Derivative, *Math. Inequal. Appl.*, **1** (4), (1998), 543-550.
8. S. Bernstein, Sur La Limitation Des Derivees Des Polnomes, *C. R. Acad. Sci. Paris.*, **190**, (1930), 338-341.
9. M. Bidkham, K. K. Dewan, Inequalities for a Polynomial and its Derivative, *J. Math. Anal. Appl.*, **166** (2), (1992), 319-324.
10. M. Bidkham, H. A. Soleiman Mezerji, Some Inequalities for the Polar Derivative of Polynomials in Complex Domain, *Complex Anal. Oper. Theory*, **7** (4), (2013), 1257-1266.
11. K. K. Dewan, M. Bidkham, On the Enestrom-Kakeya Theorem, *J. Math. Anal. Appl.*, **180** (1), (1993), 29-36.
12. K. K. Dewan and S. Hans, Generalization of Certain Well-known Polynomial Inequalities, *J. Math. Anal. Appl.*, **363** (1), (2010), 38-41.
13. K. K. Dewan, S. Hans, Some Polynomial Inequalities in the Complex Domain, *Anal. Theory Appl.*, **26** (1), (2010), 1-6.
14. K. K. Dewan, N. Singh, A. Mir, Extension of Some Polynomial Inequalities to the Polar Derivative, *J. Math. Anal. Appl.*, **352** (2), (2009), 807-815.
15. N. K. Govil, Some Inequalities for Derivative of Polynomials, *J. Approx. Theory*, **66** (1), (1991), 29-35.
16. N. K. Govil, G. N. McTume, Some Generalization Involving the Polar Derivative for an Inequality of Paul Turan, *Acta. Math. Hungar.*, **104** (2), (2004), 115-126.
17. E. Khojastehnezhad, M. Bidkham, Inequalities for the Polar Derivative of a Polynomial with S-fold Zeros at the Origin, *Bull. Iran. Math. Soc.*, **43** (7), (2017), 2153-2167.
18. M. A. Malik, On the Derivative of a Polynomial, *J. London. Math. Soc.*, **1** (2), (1969), 57-60.
19. P. D. Lax, Proof of a Conjecture of P. Erdös on the Derivative of a Polynomial, *Bull. Amer. Math. Soc.*, **50**, (1944), 509-513.
20. M. Marden, *Geometry of Polynomials*, Math. Surveys, No. 3, Amer. Math. Soc. Providence, RI, 1966.

[DOI: 10.52547/ijmsi.17.1.111]

21. A. Zireh, Generalization of Certain Well-known Inequalities for the Derivative of Polynomials, *Anal. Math.*, **41** (2), (2015), 117-132.
22. J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, AMS, 1960.

[DOI: 10.52547/ijmsi.17.1.111]