Iranian Journal of Mathematical Sciences and Informatics

Vol. 17, No. 1 (2022), pp 111-123

DOI: 10.52547/ijmsi.17.1.111

On Bernstein Type Inequalities for Complex Polynomial

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ABSTRACT. In this paper, we establish some Bernstein type inequalities for the complex polynomial. Our results constitute generalizations and refinements of some well-known polynomial inequalities.

Keywords: Inequality, Polynomial, Derivative, Maximum modulus, Restricted zeros.

2000 Mathematics subject classification: 30A10, 30C10, 30D15.

1. Introduction and statement of results

Let p be a polynomial of degree at most n. Then, according to a famous result known as Bernsteins inequality [8]

$$\max_{|z|=1} |p'(z)| \le n \max_{|z|=1} |p(z)|, \tag{1.1}$$

whereas concerning the maximum modulus of p on a large circle |z| = R > 1, we have [20]

$$\max_{|z|=R} |p(z)| \le R^n \max_{|z|=1} |p(z)|. \tag{1.2}$$

If we restrict ourselves to the class of polynomials having no zeros in |z| < 1, then inequalities (1.1) and (1.2) can be sharpened. In fact, if $p(z) \neq 0$ in |z| < 1, then (1.1) and (1.2) can respectively be replaced by

Received 11 August 2018; Accepted 16 August 2020 ©2022 Academic Center for Education, Culture and Research TMU

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$$\max_{|z|=1} |p'(z)| \le \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.3}$$

and

$$\max_{|z|=R} |p(z)| \le \frac{R^n + 1}{2} \max_{|z|=1} |p(z)|, \ R > 1.$$
 (1.4)

Inequality (1.3) was conjectured by Erdös and later verified by Lax [19], whereas Ankeny and Rivlin [5] used (1.3) to prove (1.4).

In the literature, there are already various generalizations and refinements of (1.3) and (1.4), for example (see Aziz [6], Bidkham et al. [9, 10, 11], Khojastehnezhad and Bidkham [17], Zireh [21], etc).

Inequalities (1.3) and (1.4) were sharpened by Dewan et.al [12, 13] proving that under the same hypothesis, for every real or complex number β with $|\beta| \le 1$, R > 1 and |z| = 1, we have

$$|zp'(z) + \frac{n\beta}{2}p(z)| \le \frac{n}{2}\{(|1 + \frac{\beta}{2}| + |\frac{\beta}{2}|) \max_{|z|=1}|p(z)| - (|1 + \frac{\beta}{2}| - |\frac{\beta}{2}|) \min_{|z|=1}|p(z)|\},\tag{1.5}$$

and

$$|p(Rz) + \beta(\frac{R+1}{2})^{n}p(z)| \le \frac{1}{2} \{ (|R^{n} + \beta(\frac{R+1}{2})^{n}| + |1 + \beta(\frac{R+1}{2})^{n}|) \max_{|z|=1} |p(z)| - (|R^{n} + \beta(\frac{R+1}{2})^{n}| - |1 + \beta(\frac{R+1}{2})^{n}|) \min_{|z|=1} |p(z)| \}.$$
(1.6)

Also they [12] proved if p has all its zeros in $|z| \leq 1$, then for every real or complex number β with $|\beta| \leq 1$, we have

$$\min_{|z|=1} |zp'(z) + \frac{n\beta}{2}p(z)| \ge n|1 + \frac{\beta}{2}|\min_{|z|=1} |p(z)|. \tag{1.7}$$

In this paper, we first prove an interesting result which is a compact generalization of inequality (1.7).

Theorem 1.1. If p is a polynomial of degree n having all its zeros in $|z| \le k$, k > 0, then for all α , $\beta \in \mathbb{C}$ with $|\alpha| \le 1$, $|\beta| \le 1$, $R \ge r$, $rR \ge k^2$ and |z| = 1, we have

$$|p(Rz) - \alpha p(rz) + \beta \{ (\frac{R+k}{r+k})^n - |\alpha| \} p(rz) | \ge \frac{1}{k^n} |R^n - \alpha r^n + \beta \{ (\frac{R+k}{r+k})^n - |\alpha| \} r^n | \min_{|z|=k} |p(z)|.$$
(1.8)

Assuming $\alpha = 1$ in Theorem 1.1, we have the following result.

Corollary 1.2. Let p be a polynomial of degree n such that does not vanish in |z| > k, k > 0, then for all $\beta \in \mathbb{C}$ with $|\beta| \le 1$, R > r, $rR \ge k^2$ and |z| = 1, we get

$$|p(Rz) - p(rz) + \beta \{ (\frac{R+k}{r+k})^n - 1 \} p(rz) | \ge \frac{1}{k^n} |R^n - r^n + \beta \{ (\frac{R+k}{r+k})^n - 1 \} r^n | \min_{|z|=k} |p(z)|.$$
(1.9)

By dividing the two sides of the inequality (1.9) by (R-r) and letting $R \to r$, we get the following interesting result.

Corollary 1.3. Let p be a polynomial of degree n such that does not vanish in |z| > k, k > 0. Then for all $\beta \in \mathbb{C}$ with $|\beta| \le 1$, $r \ge k$ and |z| = 1, we get

$$|zp'(rz) + \frac{n\beta}{r+k}p(rz)| \ge \frac{n}{k^n}|r^{n-1} + \frac{\beta}{r+k}r^n|\min_{|z|=k}|p(z)|.$$
 (1.10)

Assuming k = 1, r = 1 in Corollary (1.3), we have the inequality (1.7). Using Theorem 1.1, we prove the following theorem, which provides a compact generalization of inequalities (1.5), (1.6).

Theorem 1.4. Let p be a polynomial of degree n such that it does not vanish in |z| < k, k > 0. Then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \le 1$, $|\beta| \le 1$, $R \ge r$, $rR \ge \frac{1}{k^2}$ and |z| = 1,

$$|p(Rk^{2}z) - \alpha p(rk^{2}z) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} p(rk^{2}z) | \leq \frac{1}{2} \{ [k^{n}|R^{n} - \alpha r^{n} + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} r^{n} | + |1 - \alpha + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} |] \max_{|z|=k} |p(z)| - [k^{n}|R^{n} - \alpha r^{n} + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} r^{n} | - |1 - \alpha + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} |] \min_{|z|=k} |p(z)| \}.$$

$$(1.11)$$

Equality holds for the polynomials $az^n + bk^n$, |a| = |b|.

Assuming $\alpha = 1$ in Theorem 1.4, we have the following result.

Corollary 1.5. Let p be a polynomial of degree n such that does not vanish in |z| < k, k > 0. Then for every $\beta \in \mathbb{C}$ with $|\beta| \le 1$, $R \ge r$, $rR \ge \frac{1}{k^2}$ and |z| = 1, we get

$$|p(Rk^{2}z) - p(rk^{2}z) + \beta\{\left(\frac{Rk+1}{rk+1}\right)^{n} - 1\}p(rk^{2}z)| \leq \frac{1}{2}\{$$

$$[k^{n}|R^{n} - r^{n} + \beta\{\left(\frac{Rk+1}{rk+1}\right)^{n} - 1\}r^{n}| + |\beta\{\left(\frac{Rk+1}{rk+1}\right)^{n} - 1\}|] \max_{|z|=k} |p(z)| - [k^{n}|R^{n} - r^{n} + \beta\{\left(\frac{Rk+1}{rk+1}\right)^{n} - 1\}r^{n}| - |\beta\{\left(\frac{Rk+1}{rk+1}\right)^{n} - 1\}|] \min_{|z|=k} |p(z)|\}.$$

$$(1.12)$$

By dividing the two sides of the inequality (1.12) by (R-r) and letting $R \to r$, we get the following interesting result.

Corollary 1.6. Let p be a polynomial of degree n such that does not vanish in $|z| < k, \ k > 0$. Then for every $\beta \in \mathbb{C}$ with $|\beta| \le 1, \ r \ge \frac{1}{k}$ and |z| = 1, we have

$$|k^2zp'(rk^2z) + \frac{n\beta k}{rk+1}p(rk^2z)| \leq \frac{n}{2}\{[k^n|r^{n-1} + \frac{\beta k}{rk+1}r^n| + |\frac{\beta k}{rk+1}|]\max_{|z|=k}|p(z)| - [k^n|r^{n-1} + \frac{\beta k}{rk+1}r^n| - |\frac{\beta k}{rk+1}|]\min_{|z|=k}|p(z)|\}. \tag{1.13}$$

Remark 1.7. Assuming k = 1 and r = 1 in Corollary 1.6 we have the inequality (1.5).

2. Lemmas

To prove of these theorems, we need the following lemmas. The first lemma is due to Aziz and Zargar [7].

Lemma 2.1. Let p be a polynomial of degree n having all its zeros in $|z| \le k$, k > 0. Then for every $R \ge r$ and $rR \ge k^2$, we have

$$|p(Rz)| \ge \left(\frac{R+k}{r+k}\right)^n |p(rz)|, \quad |z| = 1.$$
 (2.1)

Lemma 2.2. Let p be a polynomial of degree n such that does not vanish in $|z| < k, \ k > 0$, and $q(z) = z^n \overline{p(\frac{1}{z})}$. Then for all $\alpha, \ \beta \in \mathbb{C}$ with $|\alpha| \le 1, \ |\beta| \le 1, \ R \ge r, \ rR \ge \frac{1}{k^2}$ and |z| = 1, we have

$$|p(Rk^{2}z) - \alpha p(rk^{2}z) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} p(rk^{2}z) | \leq k^{n} |q(Rz) - \alpha q(rz) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} q(rz) |.$$
(2.2)

Proof. Based on the hypotheses that the polynomial p has no zeros in |z| < k, therefore the polynomial $q(z) = z^n \overline{p(\frac{1}{z})}$ has all its zeros in $|z| \le \frac{1}{k}$. Since $\frac{1}{k^n} |p(k^2 z)| = |q(z)|$ for $|z| = \frac{1}{k}$, therefore the function $\phi(z) = \frac{p(k^2 z)}{k^n q(z)}$ is analytic in the disc $|z| \ge \frac{1}{k}$ and $|\phi(z)| = 1$ on $|z| = \frac{1}{k}$. Hence based on the maximum modulus principle $|\phi(z)| < 1$ for $|z| > \frac{1}{k}$, or equivalently

$$|p(k^2z)| \le k^n |q(z)|, \ |z| \ge \frac{1}{k}.$$
 (2.3)

Since $\frac{1}{k^n}|p(k^2z)|=|q(z)|$ for $|z|=\frac{1}{k}$, therefore for every real or complex number δ with $|\delta|<1$ and $|z|=\frac{1}{k},$ $|\delta p(k^2z)|<|k^nq(z)|$. Now using Rouche's theorem it follows that all the zeros of $H(z):=k^nq(z)+\delta p(k^2z)$ lie in $|z|\leq \frac{1}{k}$. While applying Lemma 2.1, we have

$$|H(Rz)| \ge \left(\frac{Rk+1}{rk+1}\right)^n |H(rz)| > |H(rz)|, \quad |z| = 1,$$
 (2.4)

where R > r, $rR \ge \frac{1}{k^2}$.

It follows that for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, we get

$$|H(Rz) - \alpha H(rz)| \ge |H(Rz)| - |\alpha||H(rz)| \ge \{(\frac{Rk+1}{rk+1})^n - |\alpha|\}|H(rz)|, \ |z| = 1$$
 i.e.

$$|H(Rz) - \alpha H(rz)| \ge \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} |H(rz)| \text{ for } |z| = 1.$$
 (2.5)

Since H(Rz) has all its zeros in $|z| \leq \frac{1}{Rk} < 1$, and |H(rz)| < |H(Rz)| for |z| = 1, a direct application of Rouche's theorem shows that the polynomial $H(Rz) - \alpha H(rz)$ has all its zeros in |z| < 1. Using Rouche's theorem again, it follows that for every $\beta \in \mathbb{C}$ with $|\beta| < 1$ and R > r, $rR \geq \frac{1}{k^2}$, all the zeros of the polynomial

$$T(z) = H(Rz) - \alpha H(rz) + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} H(rz)$$

lie in |z| < 1.

Replacing H(z) by $k^n q(z) + \delta p(k^2 z)$, we conclude that all the zeros of

$$T(z) = k^{n} [q(Rz) - \alpha q(rz) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} q(rz)] +$$

$$\delta \{ p(Rk^{2}z) - \alpha p(rk^{2}z) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} p(rk^{2}z) \}$$
(2.6)

lie in |z| < 1, for every R > r, $rR \ge \frac{1}{k^2}$, $|\alpha| \le 1$, $|\beta| < 1$ and $|\delta| < 1$. We now show that (2.6) implies (2.2). Indeed, suppose otherwise. Then, there is a point $z = z_0$ with $|z_0| = 1$ such that

$$|p(Rk^{2}z_{0}) - \alpha p(rk^{2}z_{0}) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} p(rk^{2}z_{0})| > k^{n}|q(Rz_{0}) - \alpha q(rz_{0}) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} q(rz_{0})|.$$

We take

$$\delta = -\frac{k^n [q(Rz_0) - \alpha q(rz_0) + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} q(rz_0)]}{p(Rk^2 z_0) - \alpha p(rk^2 z_0) + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} p(rk^2 z_0)},$$

then $|\delta| < 1$ and with this choice of δ , we have, $T(z_0) = 0$ for $|z_0| = 1$. But this contradicts that T has all its zeros in |z| < 1. For the case β , with $|\beta| = 1$, (2.2) follows by continuity. For R = r inequality (2.2) follows by inequality (2.3). This completes the proof of Lemma 2.2.

Lemma 2.3. Let p be a polynomial of degree n. Then for all α , $\beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1, R \geq r, rR \geq k^2, k > 0$ and |z| = 1, we have

$$|p(Rz) - \alpha p(rz) + \beta \{ (\frac{R+k}{r+k})^n - |\alpha| \} p(rz) | \le \frac{1}{k^n} |R^n - \alpha r^n + \beta \{ (\frac{R+k}{r+k})^n - |\alpha| \} r^n |\max_{|z|=k} |p(z)|.$$
(2.7)

Proof. Let $M = \max_{|z|=k} |p(z)|$, then for δ with $|\delta| > 1$, we can conclude from Rouche's theorem that all zeros of polynomial $H(z) = p(z) - \delta M(\frac{z}{k})^n$ lie in the closed disk $|z| \le k$, k > 0. Using Lemma 2.1, we have

$$|H(Rz)| \ge \left(\frac{R+k}{r+k}\right)^n |H(rz)| > |H(rz)|, \quad |z| = 1,$$
 (2.8)

where R > r, $rR \ge k^2$.

It follows that for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, we get

$$|H(Rz) - \alpha H(rz)| \ge |H(Rz)| - |\alpha||H(rz)| \ge \{(\frac{R+k}{r+k})^n - |\alpha|\}|H(rz)|, \ |z| = 1,$$

i.e.

$$|H(Rz) - \alpha H(rz)| \ge \{ (\frac{R+k}{r+k})^n - |\alpha| \} |H(rz)|, \ |z| = 1.$$
 (2.9)

Since H(Rz) has all its zeros in $|z| \leq \frac{k}{R} < 1$, and |H(rz)| < |H(Rz)|, a direct application of Rouche's theorem shows that the polynomial $H(Rz) - \alpha H(rz)$ has all its zeros in |z| < 1. Using Rouche's theorem again ,implies that for every $\beta \in \mathbb{C}$ with $|\beta| < 1$ and R > r, $rR \geq k^2$, all the zeros of the polynomial

$$T(z) = H(Rz) - \alpha H(rz) + \beta \left\{ \left(\frac{R+k}{r+k} \right)^n - |\alpha| \right\} H(rz)$$

lie in |z| < 1.

Replacing H(z) by $p(z) - \delta M(\frac{z}{k})^n$, we conclude that all the zeros of

$$T(z) = [p(Rz) - \alpha p(rz) + \beta \{ (\frac{R+k}{r+k})^n - |\alpha| \} p(rz)] + \delta \frac{Mz^n}{k^n} \{ R^n - \alpha r^n + \beta \{ (\frac{R+k}{r+k})^n - |\alpha| \} r^n \}$$
(2.10)

lie in |z|<1, for every $R>r,\ rR\geq k^2,\ |\alpha|\leq 1,\ |\beta|<1$ and $|\delta|>1.$ This implies

$$|p(Rz) - \alpha p(rz) + \beta \{ (\frac{R+k}{r+k})^n - |\alpha| \} p(rz) | \le$$

$$|R^n - \alpha r^n + \beta \{ (\frac{R+k}{r+k})^n - |\alpha| \} r^n | \frac{M}{k^n},$$
(2.11)

where |z|=1.

For β , with $|\beta| = 1$, (2.11) follows by continuity. For R = r inequality (2.11) reduces to $|p(rz)| \leq \frac{r^n}{k^n} \max_{|z|=k} |p(z)|$ which it follows by taking p(kz) and $|z| = \frac{r}{k}$ where $\frac{r}{k} \geq 1$ in inequality (1.2). This completes the proof of Lemma 2.3.

Lemma 2.4. If p is a polynomial of degree n, then for all α , $\beta \in \mathbb{C}$ with $|\alpha| \leq 1, \ |\beta| \leq 1, \ R \geq r, \ rR \geq \frac{1}{k^2}$ and |z| = 1,

$$\begin{split} |p(Rk^2z) - \alpha p(rk^2z) + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} p(rk^2z) | + \\ k^n |q(Rz) - \alpha q(rz) + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} q(rz) | \leq \\ \{ k^n |R^n - \alpha r^n + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} r^n | + |1 - \alpha + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} | \} \max_{|z| = k} |p(z)|, \\ (2.12) \end{split}$$

where $q(z) = z^n \overline{p(1/\overline{z})}$.

Proof. Assume that $M=\max_{|z|=k}|p(z)|$. Then, for δ with $|\delta|>1$, we can conclude from Rouche's theorem that the polynomial $G(z)=p(z)-\delta M$ does not vanish in |z|< k. If we take $H(z)=z^n\overline{G(1/\overline{z})}$, then $|G(k^2z)|=k^n|H(z)|$ for $|z|=\frac{1}{k}$. Using Lemma 2.2, for all $\alpha,\ \beta\in\mathbb{C}$ with $|\alpha|\leq 1,\ |\beta|\leq 1,\ R\geq r$ $rR\geq\frac{1}{k^2}$ and |z|=1, we have

$$|G(Rk^{2}z) - \alpha G(rk^{2}z) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} G(rk^{2}z) | \le k^{n} |H(Rz) - \alpha H(rz) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} H(rz) |.$$
(2.13)

Therefore, by using the equality

$$\begin{split} H(z) &= z^n \overline{G(\frac{1}{\overline{z}})} = z^n \overline{p(\frac{1}{\overline{z}})} - \overline{\delta} M z^n \\ &= q(z) - \overline{\delta} M z^n, \end{split}$$

we get

$$|\{p(Rk^{2}z) - \alpha p(rk^{2}z) + \beta\{(\frac{Rk+1}{rk+1})^{n} - |\alpha|\}p(rk^{2}z)\} - \delta\{1 - \alpha + \beta\{(\frac{R+1}{r+1})^{n} - |\alpha|\}\}M| \le k^{n}|\{q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^{n} - |\alpha|\}q(rz)\} - \overline{\delta}\{R^{n} - \alpha r^{n} + \beta\{(\frac{Rk+1}{rk+1})^{n} - |\alpha|\}r^{n}\}M|.$$
(2.14)

Since $\frac{1}{k^n}|p(k^2z)| = |q(z)|$ for $|z| = \frac{1}{k}$, therefore

$$\max_{|z|=\frac{1}{k}} |q(z)| = \frac{1}{k^n} \max_{|z|=k} |p(z)|,$$

$$\max_{|z|=\frac{1}{k}} |q(z)| = \frac{M}{k^n}.$$

Now by applying Lemma 2.3 to q(z) for $\frac{1}{k} > 0$, we have

$$|q(Rz) - \alpha q(rz) + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} q(rz) | \le$$

$$\{ R^n - \alpha r^n + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} r^n \} k^n \max_{|z| = \frac{1}{k}} |q(z)|.$$

i.e.

$$|q(Rz) - \alpha q(rz) + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} q(rz) | \le$$

$$\{ R^n - \alpha r^n + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} r^n \} M.$$

Now by suitable choice of argument of δ , we get

$$|q(Rz) - \alpha q(rz) + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} q(rz) \} - \overline{\delta} \{ R^n - \alpha r^n + \beta \{ (\frac{R+1}{r+1})^n - |\alpha| \} r^n \} M | = |\delta| |R^n - \alpha r^n + \beta \{ (\frac{R+1}{r+1})^n - |\alpha| \} r^n | M - |q(Rz) - \alpha q(rz) + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} q(rz) |.$$
(2.15)

Combining right hand sides of (2.14) and (2.15) we can obtain

$$|p(Rk^{2}z) - \alpha p(rk^{2}z) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} p(rk^{2}z) | - |\beta| |1 - \alpha + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} |M| \le |\delta| k^{n} |R^{n} - \alpha r^{n} + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} r^{n} |M| | - k^{n} |q(Rz) - \alpha q(rz) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} q(rz) |,$$

which implies

$$\begin{split} &|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)| + \\ &k^n|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)| \leq \\ &|\delta|\{k^n|R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n| + |1 - \alpha + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}|\}M. \end{split}$$

Making $|\delta| \to 1$, we have the result.

Lemma 2.5. Let p be a polynomial of degree n having no zeros in |z| < k, k > 0. Then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \le 1, |\beta| \le 1, R \ge r, Rr \ge \frac{1}{k^2}$ and |z| = 1,

we have

$$|p(Rk^{2}z) - \alpha p(rk^{2}z) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} p(rk^{2}z) | \leq \frac{1}{2}$$

$$\{ k^{n} | R^{n} - \alpha r^{n} + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} r^{n} | + |1 - \alpha + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} | \} \max_{|z|=k} |p(z)|$$

$$(2.16)$$

Proof. Since p does not vanish in |z| < k, k > 0, Lemma 2.2, yields

$$|p(Rk^{2}z) - \alpha p(rk^{2}z) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} p(rk^{2}z) | \le k^{n} |q(Rz) - \alpha q(rz) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} q(rz) |,$$
(2.17)

Now by combining the inequalities (2.12) and (2.17), we have

$$\begin{split} &2|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)| \leq \\ &|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)| + \\ &k^n|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)| \leq \\ &\{k^n|R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n| + |1 - \alpha + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}|\} \max_{|z|=k}|p(z)|. \end{split}$$

This gives the result.

3. Proofs of the theorems

Proof of Theorem 1.1. If p has a zero on |z| = k, then inequality is trivial. Therefore, we assume that p(z) has all its zeros in |z| < k. If $m = \min_{|z| = k} |p(z)|$, then m > 0 and $|p(z)| \ge m$ for |z| = k. If $|\lambda| < 1$, then it follows by Rouche's theorem that the polynomial $p(z) - \lambda m(\frac{z}{k})^n$, has all its zeros in |z| < k, k > 0. Proceeding similarly as in the proof of Lemma 2.3, it follows that all the zeros of

$$p(Rz) - \alpha p(rz) + \beta \left\{ \left(\frac{R+k}{r+k} \right)^n - |\alpha| \right\} p(rz) +$$

$$\lambda m \left(\frac{z}{k} \right)^n \left\{ R^n - \alpha r^n + \beta \left\{ \left(\frac{R+k}{r+k} \right)^n - |\alpha| \right\} r^n \right\}$$
(3.1)

lie in |z|<1, for every $R\geq r,$ $Rr\geq k^2,$ $|\alpha|\leq 1,$ $|\beta|<1$ and $|\lambda|<1.$ This implies

$$\frac{m}{k^{n}}|R^{n} - \alpha r^{n} + \beta\{(\frac{R+k}{r+k})^{n} - |\alpha|\}r^{n}| \leq |p(Rz) - \alpha p(rz) + \beta\{(\frac{R+k}{r+k})^{n} - |\alpha|\}p(rz)|,$$
(3.2)

DOI: 10.52547/ijmsi.17.1.111]

where |z| = 1. This completes the proof.

Proof of Theorem 1.4. If p(z) has a zero on |z| = k, then $\min_{|z|=k} |p(z)| = 0$ and in this case the result follows from Lemma 2.5. Hence we assume that $p(z) \neq 0$ in $|z| \leq k$. In this case we have $m = \min_{|z|=k} |p(z)| > 0$ and for γ with $|\gamma| < 1$, we get $|\gamma m| < m \le |p(z)|$, where |z| = k. Now we conclude from Rouche's theorem that the polynomial $G(z) = p(z) - \gamma m$ does not vanish in |z| < k. If we take $H(z) = z^n \overline{G(1/\overline{z})}$, then by using the polynomials G(z) and H(z) in Lemma 2.2, we have

$$|G(Rk^{2}z) - \alpha G(rk^{2}z) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} G(rk^{2}z) | \leq k^{n} |H(Rz) - \alpha H(rz) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} H(rz) |.$$
(3.3)

Using the fact that

$$H(z) = z^n \overline{G(\frac{1}{\overline{z}})} = z^n \overline{p(\frac{1}{\overline{z}})} - \overline{\gamma} m z^n = q(z) - \overline{\gamma} m z^n,$$

or

$$H(z) = q(z) - \overline{\gamma} m z^n,$$

and substituting G(z) and H(z) in (3.3), we get

$$|\{p(Rk^{2}z) - \alpha p(rk^{2}z) + \beta\{(\frac{Rk+1}{rk+1})^{n} - |\alpha|\}p(rk^{2}z)\} - \gamma\{1 - \alpha + \beta\{(\frac{Rk+1}{rk+1})^{n} - |\alpha|\}\}m| \le k^{n}|\{q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^{n} - |\alpha|\}q(rz)\} - \overline{\gamma}\{R^{n} - \alpha r^{n} + \beta\{(\frac{Rk+1}{rk+1})^{n} - |\alpha|\}r^{n}\}mz^{n}|.$$
(3.4)

Since the polynomial $q(z)=z^n\overline{p(\frac{1}{\overline{z}})}$ has all zeros in $|z|\leq \frac{1}{k}$ and $m=\min_{|z|=k}|p(z)|=1$ $k^n \min_{|z|=\frac{1}{k}} |q(z)|,$ hence by applying Theorem 1.1 for the polynomial q(z) with $\frac{1}{k},$ we obtain

$$|R^{n} - \alpha r^{n} + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} r^{n} | m \le |q(Rz) - \alpha q(rz) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} q(rz) |.$$

Therefore, by suitable choice of argument of γ , we get

$$|\{q(Rz) - \alpha q(rz) + \beta \{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)\} - \overline{\gamma} \{R^n - \alpha r^n + \beta \{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n m| = |q(Rz) - \alpha q(rz) + \beta \{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)| - |\gamma||R^n - \alpha r^n + \beta \{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n|m.$$
(3.5)

Now combining (3.4) and (3.5), we get

$$\begin{split} |p(Rk^2z) - \alpha p(rz) + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} p(rk^2z) | - \\ |\gamma| |1 - \alpha + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} |m \leq \\ k^n |q(Rz) - \alpha q(rz) + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} q(rz) | - \\ |\gamma| k^n |R^n - \alpha r^n + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} r^n |m. \rangle \end{split}$$

This implies

$$\begin{split} |p(Rk^2z) - \alpha p(rk^2z) + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} p(rk^2z) | &\leq \\ k^n |q(Rz) - \alpha q(rz) + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} q(rz) | - \\ |\gamma| \{ k^n |R^n - \alpha r^n + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} r^n |z^n| - |1 - \alpha + \beta \{ (\frac{Rk+1}{rk+1})^n - |\alpha| \} | \} m. \end{split}$$

Letting $|\gamma| \to 1$, we have

$$|p(Rk^{2}z) - \alpha p(rk^{2}z) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} p(rk^{2}z) | \leq k^{n} |q(Rz) - \alpha q(rz) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} q(rz) | - \{ k^{n} |R^{n} - \alpha r^{n} + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} r^{n} | - |1 - \alpha + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} | \} m.$$

$$(3.6)$$

On the other hand, based on Lemma 2.4, we have

$$|p(Rk^{2}z) - \alpha p(rk^{2}z) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} p(rk^{2}z) | + k^{n}|q(Rz) - \alpha q(rz) + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} q(rz) | \le \{ k^{n}|R^{n} - \alpha r^{n} + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} r^{n}| + |1 - \alpha + \beta \{ (\frac{Rk+1}{rk+1})^{n} - |\alpha| \} | \} \max_{|z|=k} |p(z)|.$$

$$(3.7)$$

[DOI: 10.52547/ijmsi.17.1.111]

Combining (3.6) and (3.7), we get (1.11) and this completes the proof of Theorem 1.4.

Acknowledgments

The authors wish to thank the referees for their comments and sugestions.

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