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Coincidence Quasi-Best Proximity Points for Quasi-Cyclic-Noncyclic Mappings in Convex Metric Spaces

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Abstract. We introduce the notion of quasi-cyclic-noncyclic pair and its relevant new notion of coincidence quasi-best proximity points in a convex metric space. In this way we generalize the notion of coincidence-best proximity point already introduced by M. Gabeleh et al [14]. It turns out that under some circumstances this new class of mappings contains the class of cyclic-noncyclic mappings as a subclass. The existence and convergence of coincidence-best and coincidence quasi-best proximity points in the setting of convex metric spaces are investigated.

Keywords: Coincidence-best proximity point, Cyclic-noncyclic contraction, Quasi-cyclic-noncyclic contraction, Uniformly convex metric space.

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1. Introduction

Let (X, d) be a metric space, and let A, B be subsets of X. A mapping $T : A \cup B \to A \cup B$ is said to be *cyclic* provided that $T(A) \subseteq B$ and $T(B) \subseteq A$; similarly, a mapping $S : A \cup B \to A \cup B$ is said to be *noncyclic* if $S(A) \subseteq A$ and $S(B) \subseteq B$. The following theorem is an extension of Banach contraction principle.

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Theorem 1.1. *([18]) Let A and B be nonempty closed subsets of a complete metric space (X, d). Suppose that T is a cyclic mapping such that*

$$d(Tx, Ty) \le \alpha d(x, y),$$

for some $\alpha \in (0, 1)$ and for all $x \in A,\ y \in B$. Then T has a unique fixed point in $A \cap B$.

Let A and B be nonempty subsets of a metric space X. A mapping $T : A \cup B \to A \cup B$ is said to be a *cyclic contraction* if T is cyclic and

$$d(Tx, Ty) \le \alpha d(x, y) + (1 - \alpha)\text{dist}(A, B)$$

for some $\alpha \in (0, 1)$ and for all $x \in A,\ y \in B$, where

$$\text{dist}(A, B) := \inf\{d(x, y) : (x, y) \in A \times B\}.$$

For a cyclic mapping $T : A \cup B \to A \cup B$, a point $x \in A \cup B$ is said to be a best proximity point provided that

$$d(x, Tx) = \text{dist}(A, B).$$

The following existence, uniqueness and convergence result of a best proximity point for cyclic contractions is the main result of [8].

Theorem 1.2. *([8]) Let A and B be nonempty closed convex subsets of a uniformly convex Banach space X and let $T : A \cup B \to A \cup B$ be a cyclic contraction map. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \ge 0$. Then there exists a unique $x \in A$ such that $x_{2n} \to x$ and*

$$\|x - Tx\| = \text{dist}(A, B).$$

In the theory of best proximity points, one usually considers a cyclic mapping T defined on the union of two (closed) subsets of a given metric space. Here the objective is to minimize the expression $d(x, Tx)$ where x runs through the domain of T; that is $A \cup B$. In other words, we want to find

$$\min\{d(x, Tx) : x \in A \cup B\}.$$

If A and B intersect, the solution is clearly a fixed point of T; otherwise we have

$$d(x, Tx) \ge \text{dist}(A, B), \quad \forall x \in A \cup B,$$

so that the point at which the equality occurs is called a best proximity point of T. This point of view dominates the literature.

Very recently, M. Gabeleh, O. Olela Otafudu, and N. Shahzad [14] considered two mappings T and S simultaneously and established interesting results. For technical reasons, the first map should be cyclic and the second one should be noncyclic. According to [14], for a nonempty pair of subsets (A, B), and a cyclic-noncyclic pair $(T; S)$ on $A \cup B$ (that is, $T : A \cup B \to A \cup B$ is cyclic and

$S : A \cup B \to A \cup B$ is noncyclic); they called a point $p \in A \cup B$ a *coincidence best proximity point* for $(T; S)$ provided that

$$d(Sp, Tp) = \text{dist}(A, B).$$

Note that if $S = I$, the identity map on $A \cup B$, then $p \in A \cup B$ is a best proximity point for T. Also, if $\text{dist}(A, B) = 0$, then p is called a *coincidence point* for $(T; S)$ (see [12] and [15] for more information). With the definition just given, and depending on the situation as to whether S equals the identity map, or if the distance between the underlying sets is zero, one obtains a best proximity point for T, or a coincidence point for the pair $(T; S)$. This was in fact the philosophy behind the phrase *coincidence-best proximity point* coined by Gabeleh et al. They then defined the notion of a cyclic-noncyclic contraction.

Definition 1.3. ([14]) Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and $T, S : A \cup B \to A \cup B$ be two mappings. The pair $(T; S)$ is called a cyclic-noncyclic contraction pair if it satisfies the following conditions:
(1) $(T; S)$ is a cyclic-noncyclic pair on $A \cup B$.
(2) For some $r \in (0, 1)$ we have

$$d(Tx, Ty) \leq rd(Sx, Sy) + (1 - r)\text{dist}(A, B), \ \forall (x, y) \in A \times B.$$

To state the main result of [14], we need to recall the notion of convexity in the framework of metric spaces. In [26], Takahashi introduced the notion of convexity in metric spaces as follows (see also [24]).

Definition 1.4. Let (X, d) be a metric space and $I := [0, 1]$. A mapping $\mathcal{W} : X \times X \times I \to X$ is said to be a convex structure on X provided that for each $(x, y; \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, \mathcal{W}(x, y; \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

A metric space (X, d) together with a convex structure $\mathcal{W}$ is called a *convex metric space*, and is denoted by $(X, d, \mathcal{W})$. A Banach space and each of its convex subsets are convex metric spaces.

A subset K of a convex metric space $(X, d, \mathcal{W})$ is said to be a convex set provided that $\mathcal{W}(x, y; \lambda) \in K$ for all $x, y \in K$ and $\lambda \in I$.

Similarly, a convex metric space $(X, d, \mathcal{W})$ is said to be uniformly convex if for any $\varepsilon > 0$, there exists $\alpha = \alpha(\varepsilon)$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \leq r, \ d(z, y) \leq r$ and $d(x, y) \geq r\varepsilon$,

$$d(z, \mathcal{W}(x, y; \frac{1}{2})) \leq r(1 - \alpha) < r.$$

For example every uniformly convex Banach space is a uniformly convex metric space.

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Definition 1.5. ([14]) Let (A, B) be a nonempty pair of subsets of a metric space (X, d). A mapping $S : A \cup B \to A \cup B$ is said to be a relatively anti-Lipschitzian mapping if there exists $c > 0$ such that

$$d(x, y) \leq cd(Sx, Sy), \ \forall (x, y) \in A \times B.$$

The main result of M. Gabeleh et al reads as follows:

Theorem 1.6. *([14]) Let (A, B) be a nonempty, closed pair of subsets of a complete uniformly convex metric space $(X, d, \mathcal{W})$ such that A is convex. Let $(T; S)$ be a cyclic-noncyclic contraction pair defined on $A \cup B$ such that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$ and that S is continuous on A and relatively anti-Lipschitzian on $A \cup B$. Then $(T; S)$ has a coincidence best proximity point in A. Further, if $x_0 \in A$ and $Sx_{n+1} := Tx_n$, then (x_{2n}) converges to the coincidence-best proximity point of $(T; S)$.*

Existence of best proximity pairs was first studied in [9] by using a geometric property on a nonempty pair of subsets of a Banach space, called *proximal normal structure*, for noncyclic relatively nonexpansive mappings (Theorem 2.2 of [9]). Some existence results of best proximity pairs can be found in [1, 2, 5, 6, 7, 10, 11, 13, 17, 23, 25].

In the current paper, we study sufficient conditions which ensure the existence and convergence of *coincidence-best and quasi-best proximity point* for a pair of quasi-cyclic-noncyclic contraction mappings in the setting of convex metric spaces.

2. Coincidence quasi-best proximity point

In this section, we introduce the class of quasi-cyclic-noncyclic mappings that contains the class of cyclic-noncyclic mappings as a subclass. Next, we introduce the new notion of quasi-best proximity points for this mappings. Finally, we study the existence and convergence of coincidence quasi-best proximity points for quasi-cyclic-noncyclic contraction mappings in the setting of convex metric spaces.

Definition 2.1. Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and $T, S : X \to X$ be two mappings. The pair $(T; S)$ is called a quasi-cyclic-noncyclic (**QCN**) contraction pair if it satisfies the following conditions:

(1) $(T; S)$ is a quasi-cyclic-noncyclic pair on X; that is,

$$T(A) \subseteq S(B), \ T(B) \subseteq S(A).$$

(2) For some $\alpha \in (0, 1)$ and for each $(x, y) \in A \times B$ we have

$$d(Tx, Ty) \leq \alpha d(Sx, Sy) + (1 - \alpha)\text{dist}(S(A), S(B)).$$

Note that if $S(A) = A$ and $S(B) = B$, then the above definition reduces to Definition 1.3; that is, the pair $(T; S)$ is a cyclic-noncyclic pair.

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EXAMPLE 2.2. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, -1]$ and $B = [1, +\infty)$ define $T, S : X \to X$ by

$$Tx := \begin{cases} -x, \ if \ x \in A \cup B \\ 0, \ ow. \end{cases} \quad \text{and} \quad Sx := \begin{cases} 2x+1, \ if \ x \in A \\ 2x-1, \ if \ x \in B \\ 0, \ ow. \end{cases}$$

Then $(T; S)$ is a QCN contraction pair with $\alpha = \frac{1}{2}$. Indeed, for all $(x, y) \in A \times B$ we have

$$\begin{aligned} |Tx - Ty| = (y - x) &\leq \frac{1}{2}(2y - 2x - 2) + \frac{1}{2}(2) \\ &= \alpha|Sx - Sy| + (1 - \alpha)\text{dist}(S(A), S(B)). \end{aligned}$$

Also, $T(A) = B \subseteq S(B)$ and $T(B) = A \subseteq S(A)$.

The next example shows that there is a QCN mapping that is not a cyclic-noncyclic mapping.

EXAMPLE 2.3. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, -1]$ and $B = [1, +\infty)$ define $T, S : X \to X$ by

$$Tx := \begin{cases} -x, \ if \ x \in A \cup B \\ 0, \ ow. \end{cases} \quad \text{and} \quad Sx := \begin{cases} x+1, \ if \ x \in A \\ x-1, \ if \ x \in B \\ 0, \ ow. \end{cases}$$

Then $(T; S)$ is a quasi-cyclic-noncyclic pair that is not a cyclic-noncyclic pair.

Remark 2.4. Notice that (2) implies that

$$d(Tx, Ty) \leq d(Sx, Sy), \ \forall (x, y) \in A \times B.$$

Moreover, if S is a noncyclic relatively nonexpansive mapping; meaning that

$$d(Sx, Sy) \leq d(x, y), \ \forall (x, y) \in A \times B,$$

then T is a cyclic contraction. In addition, if in the above definition S is assumed to be continuous, then T would be continuous too.

Definition 2.5. Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and $T, S : X \to X$ be a quasi-cyclic-noncyclic pair on X. A point $p \in A \cup B$ is said to be a coincidence quasi-best proximity point for $(T; S)$ provided that

$$d(Sp, Tp) = \text{dist}(S(A), S(B)).$$

Note that if $S = I$, then p reduces to a coincidence-best proximity point for $(T; S)$.

To prove the main result of this section, we need some preparations.

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Lemma 2.6. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a quasi-cyclic-noncyclic pair defined on X. Then there exists a sequence $\{x_n\}$ in X such that for all $n \geq 0$ we have $Tx_n = Sx_{n+1}$ where $\{x_{2n}\}$, $\{x_{2n+1}\}$ are subsequences in A and B respectively.*

Proof. Let $x_0 \in A$. Since $Tx_0 \in S(B)$, there exists $x_1 \in B$ such that $Tx_0 = Sx_1$. Again, since $Tx_1 \in S(A)$, there exists $x_2 \in A$ such that $Tx_1 = Sx_2$.

Continuing this process, we obtain a sequence $\{x_n\}$, such that $\{x_{2n}\}$, $\{x_{2n+1}\}$ are in A and B respectively and $Tx_n = Sx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. □

Lemma 2.7. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a QCN contraction pair defined on X. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. Then we have*

$$d(Sx_{2n}, Sx_{2n+1}) \to \operatorname{dist}(S(A), S(B)).$$

Proof.

$$\begin{aligned}
d(Sx_{2n+1}, Sx_{2n+2}) &= d(Tx_{2n}, Tx_{2n+1}) \\
&\leq \alpha d(Sx_{2n}, Sx_{2n+1}) + (1-\alpha)\operatorname{dist}(S(A), S(B)) \\
&= \alpha d(Tx_{2n-1}, Tx_{2n}) + (1-\alpha)\operatorname{dist}(S(A), S(B)) \\
&\leq \alpha[\alpha d(Sx_{2n-1}, Sx_{2n}) + (1-\alpha)\operatorname{dist}(S(A), S(B))] \\
&+ (1-\alpha)\operatorname{dist}(S(A), S(B)) \\
&= \alpha^2 d(Sx_{2n-1}, Sx_{2n}) + (1-\alpha^2)\operatorname{dist}(S(A), S(B)) \\
&= \alpha^2 d(Tx_{2n-2}, Tx_{2n-1}) + (1-\alpha^2)\operatorname{dist}(S(A), S(B)) \\
&\leq \cdots \\
&\leq \alpha^{2n} d(Tx_0, Tx_1) + (1-\alpha^2)\operatorname{dist}(S(A), S(B)).
\end{aligned}$$

Now, if $n \to \infty$ in above relation, we conclude that

$$d(Sx_{2n}, Sx_{2n+1}) \to \operatorname{dist}(S(A), S(B)).$$

□

Theorem 2.8. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a QCN contraction pair defined on X. Assume that S is continuous on A. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. If $\{x_{2n}\}$ has a convergent subsequence in A, then the pair $(T; S)$ has a coincidence quasi-best proximity point in A.*

Proof. Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ such that $x_{2n_k} \to p \in A$. We have

$$\begin{aligned}
\operatorname{dist}(S(A), S(B)) &\leq d(Tx_{2n_k-1}, Tp) \leq d(Sx_{2n_k-1}, Sp) \\
&\leq d(Sp, Sx_{2n_k}) + d(Sx_{2n_k}, Sx_{2n_k-1}).
\end{aligned}$$

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By Lemma 2.7, if $k \to \infty$, we obtain that

$$d(Tx_{2n_k-1}, Tp) \to \text{dist}(S(A), S(B)).$$

Moreover, we have

$$\begin{aligned}\text{dist}(S(A), S(B)) &\leq d(Sp, Tp)\\ &\leq d(Sp, Tx_{2n_k-1}) + d(Tx_{2n_k-1}, Tp)\\ &= d(Sp, Sx_{2n_k}) + d(Tx_{2n_k-1}, Tp)\\ &\to \text{dist}(S(A), S(B)),\end{aligned}$$

that is,

$$d(Sp, Tp) = \text{dist}(S(A), S(B)).$$

□

Lemma 2.9. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a QCN contraction pair defined on X. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. Then $\{Sx_{2n}\}$, and $\{Sx_{2n+1}\}$ are bounded sequences in $S(A)$ and $S(B)$ respectively.*

Proof. Since

$$d(Sx_{2n}, Sx_{2n+1}) \to \text{dist}(S(A), S(B)),$$

it suffices to show that $\{Sx_{2n}\}$ is bounded in $S(A)$. Assume to the contrary that there exists $N_0 \in \mathbb{N}$ such that

$$d(Sx_2, Sx_{2N_0+1}) > M,\ d(Sx_2, Sx_{2N_0-1}) \leq M,$$

where,

$$M > \max\left\{\frac{\alpha^2}{1-\alpha^2} d(Sx_0, Sx_2) + \text{dist}(S(A), S(B)),\ d(Sx_1, Sx_0)\right\}.$$

By the above assumption, we have

$$\begin{aligned}\frac{M - \text{dist}(S(A), S(B))}{\alpha^2} &+ \text{dist}(S(A), S(B))\\ &< \frac{d(Sx_2, Sx_{2N_0+1}) - \text{dist}(S(A), S(B))}{\alpha^2}\\ &+ \text{dist}(S(A), S(B))\\ &\leq \frac{d(Sx_2, Sx_{2N_0+1}) + (\alpha^2 - 1)d(Sx_2, Sx_{2N_0+1})}{\alpha^2}\\ &= d(Sx_2, Sx_{2N_0+1}) = d(Tx_1, Tx_{2N_0})\\ &\leq d(Sx_1, Sx_{2N_0}) = d(Tx_0, Tx_{2N_0-1})\\ &= d(Sx_0, Sx_{2N_0-1})\\ &\leq d(Sx_0, Sx_2) + d(Sx_2, Sx_{2N_0-1})\\ &\leq d(Sx_0, Sx_2) + M.\end{aligned}$$

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This implies that

$$\frac{M - \text{dist}(S(A), S(B))}{\alpha^2} + \text{dist}(S(A), S(B)) < d(Sx_0, Sx_2) + M,$$

hence,

$$M - (1 - \alpha^2)\text{dist}(S(A), S(B)) < \alpha^2[d(Sx_0, Sx_2) + M],$$

and,

$$(1 - \alpha^2)M < \alpha^2 d(Sx_0, Sx_2) + (1 - \alpha^2)\text{dist}(S(A), S(B)).$$

Now, it follows that

$$M < \frac{\alpha^2}{1 - \alpha^2} d(Sx_0, Sx_2) + \text{dist}(S(A), S(B)),$$

which contradicts the choice of M. □

Before we state the following theorem, we recall that a subset $A \subseteq X$ is said to be boundedly compact if the closure of every bounded subset of A is compact and is contained in A.

Theorem 2.10. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) such that $S(A)$ is boundedly compact and let $(T; S)$ be a QCN contraction pair defined on X. If S is relatively anti-Lipschitzian and continuous on A, then there exists $p \in A$ such that*

$$d(Sp, Tp) = \text{dist}(S(A), S(B)).$$

Proof. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. By Lemma 2.9, $\{Sx_{2n}\}$ is bounded in $S(A)$. On the other hand, $S(A)$ is boundedly compact, so that there exists a subsequence $\{Sx_{2n_k}\}$ of $\{Sx_{2n}\}$ such that

$$Sx_{2n_k} \to Sp,$$

for some $p \in A$. We know that S is relatively anti-Lipschitzian, therefore

$$d(x_{2n_k}, p) \leq c\, d(Sx_{2n_k}, Sp) \to 0, \ k \to \infty.$$

This implies that $\{x_{2n_k}\}$ is a convergent subsequence of $\{x_{2n}\}$. Now, the result follows from Theorem 2.8. □

Example 2.11. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, 0]$ and $B = [0, +\infty)$ define $T, S : X \to X$ by

$$Tx := \begin{cases} -x, \ if \ x \in A \cup B \\ 0, \ ow. \end{cases} \quad \text{and} \quad Sx := \begin{cases} 2x, \ if \ x \in A \cup B \\ 0, \ ow. \end{cases}$$

Then $(T; S)$ is a QCN contraction pair with $\alpha = \frac{1}{2}$. Indeed, for all $(x, y) \in A \times B$ we have

$$\begin{aligned} |Tx - Ty| &= (y - x) \leq \frac{1}{2}(2y - 2x) + \frac{1}{2}(0) \\ &= \alpha|Sx - Sy| + (1 - \alpha)\text{dist}(S(A), S(B)). \end{aligned}$$

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Also, $T(A) = B \subseteq S(B)$ and $T(B) = A \subseteq S(A)$. Moreover, S is continuous on A and $S(A)$ is boundedly compact in X. Besides, S is relatively anti-Lipschitzian on $A \cup B$ with $c = 1$. In fact, for all $(x, y) \in A \times B$ we have

$$|Sx - Sy| = 2y - 2x \geq |x - y|.$$

Finally, the existence of coincidence quasi-best proximity point of the pair $(T; S)$ follows from Theorem 2.10; that is, there exists $p \in A$ such that

$$|Tp - Sp| = \mathrm{dist}(S(A), S(B)) = 0 \ or \ -p - 2p = 0,$$

which implies that $p = 0$. In this case, p is a fixed point of S.

In the following we supply an example which shows that there exists a coincidence quasi-best proximity point that is not a fixed point of S.

EXAMPLE 2.12. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, 0]$ and $B = [0, +\infty)$ define $T, S : X \to X$ by

$$Tx := \begin{cases} -(x+1), \ if \ x \in A \cup B \\ 0, \ ow. \end{cases} \quad \text{and} \quad Sx := \begin{cases} 2x, \ if \ x \in A \cup B \\ 0, \ ow. \end{cases}$$

Then $(T; S)$ is a QCN contraction pair with $\alpha = \frac{1}{2}$. Indeed, for all $(x, y) \in A \times B$ we have

$$\begin{aligned} |Tx - Ty| = (y - x) &\leq \frac{1}{2}(2y - 2x) + \frac{1}{2}(0) \\ &= \alpha|Sx - Sy| + (1 - \alpha)\mathrm{dist}(S(A), S(B)). \end{aligned}$$

Also, $T(A) = [1, +\infty) \subseteq S(B)$ and $T(B) = (-\infty, -1] \subseteq S(A)$. Moreover, S is continuous on A and $S(A)$ is boundedly compact in X. Besides, S is relatively anti-Lipschitzian on $A \cup B$ with $c = 1$. In fact, for all $(x, y) \in A \times B$ we have

$$|Sx - Sy| = 2y - 2x \geq |x - y|.$$

Finally, the existence of coincidence quasi-best proximity point of the pair $(T; S)$ follows from Theorem 2.10; that is, there exists $p \in A$ such that

$$|Tp - Sp| = \mathrm{dist}(S(A), S(B)) = 0 \ or \ -(p+1) - 2p = 0,$$

which implies that $p = -\frac{1}{3}$.

Lemma 2.13. *Let (A, B) be a nonempty pair of subsets of a uniformly convex metric space $(X, d, \mathcal{W})$ such that $S(A)$ is convex. Let $(T; S)$ be a QCN contraction pair defined on X. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. Then*

$$d(Sx_{2n+2}, Sx_{2n}) \to 0, \ d(Sx_{2n+3}, Sx_{2n+1}) \to 0.$$

Proof. We prove that $d(Sx_{2n+2}, Sx_{2n}) \to 0$. To the contrary, assume that there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$, there exists $n_k \geq k$ such that

$$d(Sx_{2n_k+2}, Sx_{2n_k}) \geq \varepsilon_0.$$

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Choose $0 < \gamma < 1$ such that $\frac{\varepsilon_0}{\gamma} > \operatorname{dist}(S(A), S(B))$ and choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \min\left\{\frac{\varepsilon_0}{\gamma} - \operatorname{dist}(S(A), S(B)), \frac{\operatorname{dist}(S(A), S(B))\alpha(\gamma)}{1-\alpha(\gamma)}\right\}.$$

By Lemma 2.7, since $d(Sx_{2n_k}, Sx_{2n_k+1}) \to \operatorname{dist}(S(A), S(B))$, there exists $N \in \mathbb{N}$ such that

$$d(Sx_{2n_k}, Sx_{2n_k+1}) \leq \operatorname{dist}(S(A), S(B)) + \varepsilon,$$
$$d(Sx_{2n_k+2}, Sx_{2n_k+1}) \leq \operatorname{dist}(S(A), S(B)) + \varepsilon$$

and

$$d(Sx_{2n_k}, Sx_{2n_k+2}) \geq \varepsilon_0 > \gamma(\operatorname{dist}(S(A), S(B)) + \varepsilon).$$

It now follows from the uniform convexity of X and the convexity of $S(A)$ that

$$\begin{aligned}\operatorname{dist}(S(A), S(B)) &\leq d(Sx_{2n_k+1}, \mathcal{W}(Sx_{2n_k}, Sx_{2n_k+2}, \frac{1}{2}))\\ &\leq (\operatorname{dist}(S(A), S(B)) + \varepsilon)(1-\alpha(\gamma))\\ &< \operatorname{dist}(S(A), S(B)) + \frac{\operatorname{dist}(S(A), S(B))\alpha(\gamma)}{1-\alpha(\gamma)}(1-\alpha(\gamma))\\ &= \operatorname{dist}(S(A), S(B)),\end{aligned}$$

which is a contradiction. Similarly, we see that $d(Sx_{2n+3}, Sx_{2n+1}) \to 0$. □

The following Theorem guarantees the existence and convergence of coincidence quasi-best proximity points for QCN contraction mappings in the setting of uniformly convex metric spaces.

Theorem 2.14. *Let (A, B) be a nonempty, closed pair of subsets of a complete uniformly convex metric space $(X, d; \mathcal{W})$ such that $S(A)$ is convex. Let $(T; S)$ be a QCN contraction pair defined on X such that S is continuous on A and relatively anti-Lipschitzian on $A \cup B$. Then there exists $p \in A$ such that*

$$d(Sp, Tp) = \operatorname{dist}(S(A), S(B)).$$

Further, if $x_0 \in A$ and $Tx_n = Sx_{n+1}$, then $\{x_{2n}\}$ converges to the coincidence quasi-best proximity point of $(T; S)$.

Proof. For $x_0 \in A$ define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. We prove that $\{Sx_{2n}\}$ and $\{Sx_{2n+1}\}$ are Cauchy sequences. First, we verify that for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$d(Sx_{2l}, Sx_{2n+1}) < \operatorname{dist}(S(A), S(B)) + \varepsilon,\ \forall l > n \geq N_0. \qquad (*)$$

Assume to the contrary that there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$ there exists $l_k > n_k \geq k$ satisfying

$$d(Sx_{2l_k}, Sx_{2n_k+1}) \geq \operatorname{dist}(S(A), S(B)) + \varepsilon_0$$

and

$$d(Sx_{2l_k-2}, Sx_{2n_k+1}) < \operatorname{dist}(S(A), S(B)) + \varepsilon_0.$$

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We have

$$\begin{aligned}\operatorname{dist}(S(A),S(B))+\varepsilon_0 &\leq d(Sx_{2l_k},Sx_{2n_k+1})\\ &\leq d(Sx_{2l_k},Sx_{2l_k-2})+d(Sx_{2l_k-2},Sx_{2n_k+1})\\ &\leq d(Sx_{2l_k},Sx_{2l_k-2})+\operatorname{dist}(S(A),S(B))+\varepsilon_0.\end{aligned}$$

Letting $k\to\infty$, we obtain

$$d(Sx_{2l_k},Sx_{2n_k+1})\to \operatorname{dist}(S(A),S(B))+\varepsilon_0.$$

Moreover, we have

$$\begin{aligned}\operatorname{dist}(S(A),S(B))+\varepsilon_0 &\leq d(Sx_{2l_k},Sx_{2n_k+1})=d(Tx_{2l_k-1},Tx_{2n_k})\\ &\leq \alpha d(Sx_{2l_k-1},Sx_{2n_k})+(1-\alpha)\operatorname{dist}(S(A),S(B))\\ &=\alpha d(Tx_{2l_k-2},Tx_{2n_k-1})+(1-\alpha)\operatorname{dist}(S(A),S(B))\\ &\leq \alpha d(Sx_{2l_k-2},Sx_{2n_k-1})+(1-\alpha)\operatorname{dist}(S(A),S(B)).\end{aligned}$$

Therefore, by letting $k\to\infty$ we obtain

$$\begin{aligned}\operatorname{dist}(S(A),S(B))+\varepsilon_0 &\leq \alpha(\operatorname{dist}(S(A),S(B))+\varepsilon_0)+(1-\alpha)\operatorname{dist}(S(A),S(B))\\ &\leq \operatorname{dist}(S(A),S(B))+\varepsilon_0.\end{aligned}$$

This implies that $\alpha=1$, which is a contradiction. That is, $(*)$ holds. Now, assume $\{Sx_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon_0>0$ such that for each $k\geq 1$ there exists $l_k>n_k\geq k$ such that

$$d(Sx_{2l_k},Sx_{n_k})\geq\varepsilon_0.$$

Choose $0<\gamma<1$ such that $\frac{\varepsilon_0}{\gamma}>\operatorname{dist}(S(A),S(B))$ and choose $\varepsilon>0$ such that

$$0<\varepsilon<\min\left\{\frac{\varepsilon_0}{\gamma}-\operatorname{dist}(S(A),S(B)),\frac{\operatorname{dist}(S(A),S(B))\alpha(\gamma)}{1-\alpha(\gamma)}\right\}.$$

Let $N\in\mathbb{N}$ be such that

$$d(Sx_{2n_k},Sx_{2n_k+1})\leq\operatorname{dist}(S(A),S(B))+\varepsilon,\ \forall n_k\geq N$$

and

$$d(Sx_{2l_k},Sx_{2n_k+1})\leq\operatorname{dist}(S(A),S(B))+\varepsilon,\ \forall l_k>n_k\geq N.$$

Uniform convexity of X implies that

$$\begin{aligned}\operatorname{dist}(S(A),S(B)) &\leq d(Sx_{2n_k+1},\mathcal{W}(Sx_{2n_k},Sx_{2l_k},\frac{1}{2}))\\ &\leq(\operatorname{dist}(S(A),S(B))+\varepsilon)(1-\alpha(\gamma))<\operatorname{dist}(S(A),S(B)),\end{aligned}$$

which is a contradiction. Therefore, $\{Sx_{2n}\}$ is a Cauchy sequence in $S(A)$. By the fact that S is relatively anti-Lipschitzian on $A\cup B$, we have

$$d(x_{2l},x_{2n})\leq cd(Sx_{2l},Sx_{2n})\to 0,\ l,n\to\infty,$$

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that is, $\{x_{2n}\}$ is a Cauchy sequence. Since A is complete, there exists $p \in A$ such that $x_{2n} \to p$. Now, the result follows from a similar argument as in Theorem 2.8. □

3. QUASI-CYCLIC-NONCYCLIC RELATIVELY CONTRACTION MAPPINGS

In this section, we introduce the class of quasi-cyclic-noncyclic relatively contraction mappings that contains the class of cyclic-noncyclic contraction mappings as a subclass. Next, we study the existence and convergence of coincidence best proximity points in the setting of convex metric spaces for quasi-cyclic-noncyclic relatively contraction mappings.

Definition 3.1. Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and $T, S : X \to X$ be two mappings. The pair $(T; S)$ is called a quasi-cyclic-noncyclic relatively contraction pair if it satisfies the following conditions:
(1) $(T; S)$ is a quasi-cyclic-noncyclic pair on X; that is,

$$T(A) \subseteq S(B),\ T(B) \subseteq S(A).$$

(2) For some $\alpha \in (0, 1)$ and for each $(x, y) \in A \times B$ we have

$$d(Tx, Ty) \le \alpha d(Sx, Sy) + (1 - \alpha)\mathrm{dist}(A, B).$$

Note that in the above definition we do not have the inequality

$$\mathrm{dist}(A, B) \le d(Sx, Sy),$$

that is,

$$d(Tx, Ty) \le d(Sx, Sy)$$

is not always true.
We emphasize that if $S = I$ or if $S(A) = A$ and $S(B) = B$, then the above definition reduces to Definition 1.3.

EXAMPLE 3.2. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, -3]$ and $B = [3, +\infty)$ define $T, S : X \to X$ by

$$Tx := \begin{cases} -(x+1), & if\ x \in A \cup B \\ 0, & ow. \end{cases} \qquad \text{and} \qquad Sx := \begin{cases} 3x+5, & if\ x \in A \\ 3x-7, & if\ x \in B \\ 0, & ow. \end{cases}$$

Then $(T; S)$ is a QCN relatively contraction pair with $\alpha = \frac{1}{3}$. Indeed, for all $(x, y) \in A \times B$ we have

$$\begin{aligned} |Tx - Ty| = (y - x) &\le \frac{1}{3}(3y - 3x - 12) + \frac{2}{3}(6) \\ &= \alpha|Sx - Sy| + (1 - \alpha)\mathrm{dist}(A, B). \end{aligned}$$

Also, $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$.

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Lemma 3.3. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a QCN relatively contraction pair defined on X and $dist(A, B) \leq dist(S(A), S(B))$. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. Then we have*

$$d(Sx_{2n}, Sx_{2n+1}) \to \text{dist}(A, B).$$

Proof. We note that

$$\begin{aligned}
\text{dist}(A, B) \leq \text{dist}(S(A), S(B)) &\leq d(Sx_{2n+1}, Sx_{2n+2}) = d(Tx_{2n}, Tx_{2n+1}) \\
&\leq \alpha d(Sx_{2n}, Sx_{2n+1}) + (1-\alpha)\text{dist}(A, B) \\
&= \alpha d(Tx_{2n-1}, Tx_{2n}) + (1-\alpha)\text{dist}(A, B) \\
&\leq \alpha[\alpha d(Sx_{2n-1}, Sx_{2n}) + (1-\alpha)\text{dist}(A, B)] \\
&+ (1-\alpha)\text{dist}(A, B) \\
&= \alpha^2 d(Sx_{2n-1}, Sx_{2n}) + (1-\alpha^2)\text{dist}(A, B) \\
&= \alpha^2 d(Tx_{2n-2}, Tx_{2n-1}) + (1-\alpha^2)\text{dist}(A, B) \\
&\leq \cdots \\
&\leq \alpha^{2n} d(Tx_0, Tx_1) + (1-\alpha^2)\text{dist}(A, B).
\end{aligned}$$

Now, if $n \to \infty$, we conclude that

$$d(Sx_{2n}, Sx_{2n+1}) \to \text{dist}(A, B).$$

□

Remark 3.4. If the pair $(T; S)$ is a QCN relatively contraction pair such that

$$S(A) \subseteq A \text{ and } S(B) \subseteq B,$$

then we have

$$dist(A, B) \leq \text{dist}(S(A), S(B)).$$

Thus, by this assumption, the Lemma holds true.

Theorem 3.5. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a QCN relatively contraction pair defined on X and* $\text{dist}(A, B) \leq \text{dist}(S(A), S(B))$. *Assume S is continuous on A. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. If $\{x_{2n}\}$ has a convergent subsequence in A, then the pair $(T; S)$ has a coincidence best proximity point in A.*

Proof. Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ such that $x_{2n_k} \to p \in A$. we have

$$\begin{aligned}
\text{dist}(A, B) \leq \text{dist}(S(A), S(B)) \leq d(Tx_{2n_k-1}, Tp) &\leq d(Sx_{2n_k-1}, Sp) \\
&\leq d(Sp, Sx_{2n_k}) + d(Sx_{2n_k}, Sx_{2n_k-1}).
\end{aligned}$$

By Lemma 3.3, if $k \to \infty$, we obtain that

$$d(Tx_{2n_k-1}, Tp) \to \text{dist}(A, B).$$

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Moreover,

$$\begin{aligned}\operatorname{dist}(A,B) \le \operatorname{dist}(S(A),S(B)) &\le d(Sp,Tp)\\ &\le d(Sp,Tx_{2n_k-1}) + d(Tx_{2n_k-1},Tp)\\ &= d(Sp,Sx_{2n_k}) + d(Tx_{2n_k-1},Tp)\\ &\to \operatorname{dist}(A,B),\end{aligned}$$

that is,

$$d(Sp,Tp) = \operatorname{dist}(A,B).$$

□

Lemma 3.6. *Let (A,B) be a nonempty pair of subsets of a metric space (X,d). Suppose $(T;S)$ is a QCN relatively contraction pair defined on X and $\operatorname{dist}(A,B) \le \operatorname{dist}(S(A),S(B))$. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \ge 0$. Then $\{Sx_{2n}\}$, and $\{Sx_{2n+1}\}$ are bounded sequences in $S(A)$ and $S(B)$ respectively.*

Proof. Since

$$d(Sx_{2n},Sx_{2n+1}) \to \operatorname{dist}(A,B),$$

it suffices to verify that $\{Sx_{2n}\}$ is bounded in $S(A)$. Assume to the contrary that there exists $N_0 \in \mathbb{N}$ such that

$$d(Sx_2,Sx_{2N_0+1}) > M, \;\; d(Sx_2,Sx_{2N_0-1}) \le M,$$

where,

$$M > \max\left\{\frac{\alpha^2}{1-\alpha^2} d(Sx_0,Sx_2) + \operatorname{dist}(A,B), \;\; d(Sx_1,Sx_0)\right\}.$$

By the above assumption, we have

$$\begin{aligned}\frac{M - \operatorname{dist}(A,B)}{\alpha^2} + \operatorname{dist}(A,B) &< \frac{d(Sx_2,Sx_{2N_0+1}) - \operatorname{dist}(A,B)}{\alpha^2} + \operatorname{dist}(A,B)\\ &\le \frac{d(Sx_2,Sx_{2N_0+1}) + (\alpha^2-1)d(Sx_2,Sx_{2N_0+1})}{\alpha^2}\\ &= d(Sx_2,Sx_{2N_0+1}) = d(Tx_1,Tx_{2N_0})\\ &\le d(Sx_1,Sx_{2N_0}) = d(Tx_0,Tx_{2N_0-1})\\ &= d(Sx_0,Sx_{2N_0-1})\\ &\le d(Sx_0,Sx_2) + d(Sx_2,Sx_{2N_0-1})\\ &\le d(Sx_0,Sx_2) + M.\end{aligned}$$

This implies that

$$\frac{M - \operatorname{dist}(A,B)}{\alpha^2} + \operatorname{dist}(A,B) < d(Sx_0,Sx_2) + M,$$

or,

$$M - (1-\alpha^2)\operatorname{dist}(A,B) < \alpha^2[d(Sx_0,Sx_2) + M].$$

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and finally,

$$(1-\alpha^2)M < \alpha^2 d(Sx_0, Sx_2) + (1-\alpha^2)\mathrm{dist}(A,B).$$

Now, we conclude that

$$M < \frac{\alpha^2}{1-\alpha^2} d(Sx_0, Sx_2) + \mathrm{dist}(A,B),$$

which is a contradiction by the choice of M. □

Theorem 3.7. *Let (A,B) be a nonempty pair of subsets of a metric space (X,d) such that $S(A)$ is boundedly compact. Suppose $(T;S)$ is a QCN relatively contraction pair defined on X and $\mathrm{dist}(A,B) \leq \mathrm{dist}(S(A),S(B))$. If S is relatively anti-Lipschitzian and continuous on A, then there exists $p \in A$ such that*

$$d(Sp, Tp) = \mathrm{dist}(A,B).$$

Proof. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. According to Lemma 3.6, $\{Sx_{2n}\}$ is bounded in $S(A)$, on the other hand $S(A)$ is boundedly compact, so that there exists a subsequence $\{Sx_{2n_k}\}$ of $\{Sx_{2n}\}$ such that

$$Sx_{2n_k} \to Sp,$$

for some $p \in A$. We know that S is relatively anti-Lipschitzian, therefore

$$d(x_{2n_k}, p) \leq c d(Sx_{2n_k}, Sp) \to 0, \ k \to \infty.$$

This implies that $\{x_{2n_k}\}$ is a convergent subsequence of $\{x_{2n}\}$, hence the result follows from Theorem 3.5. □

In the following we give examples to show that there exists a coincidence best proximity point that is not a fixed point for S.

Example 3.8. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, -3]$ and $B = [3, +\infty)$ define $T, S : X \to X$ by

$$Tx := \begin{cases} 3-x, & if\ x \in A \cup B \\ 0, & ow. \end{cases} \quad \text{and} \quad Sx := \begin{cases} 2x+6, & if\ x \in A \\ 2x, & if\ x \in B \\ 0, & ow. \end{cases}$$

Then $(T;S)$ is a QCN relatively contraction pair with $\alpha = \frac{1}{2}$. Indeed, for all $(x,y) \in A \times B$ we have

$$\begin{aligned} |Tx - Ty| = (y-x) &\leq \frac{1}{2}(2y - 2x - 6) + \frac{1}{2}(6) \\ &= \alpha|Sx - Sy| + (1-\alpha)\mathrm{dist}(A,B). \end{aligned}$$

Also, $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. Finally, the existence of coincidence best proximity point of the pair $(T;S)$ follows from Theorem 3.7; that is, there exists $p \in A$ such that

$$|Tp - Sp| = \mathrm{dist}(A,B) = 0 \ or \ 3 - p - 2p - 6 = 6,$$

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which implies that $p = -3$.

EXAMPLE 3.9. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, -4]$ and $B = [4, +\infty)$ define $T, S : X \to X$ by

$$Tx := \begin{cases} 4 - x, \ if \ x \in A \cup B \\ 0, \ ow. \end{cases} \quad \text{and} \quad Sx := \begin{cases} 4x + 16, \ if \ x \in A \\ 4x - 8, \ if \ x \in B \\ 0, \ ow. \end{cases}$$

Then $(T; S)$ is a QCN relatively contraction pair with $\alpha = \frac{1}{4}$. Indeed, for all $(x, y) \in A \times B$ we have

$$\begin{aligned} |Tx - Ty| = (y - x) &\leq \frac{1}{4}(4y - 4x - 24) + \frac{3}{4}(8) \\ &= \alpha|Sx - Sy| + (1 - \alpha)\text{dist}(A, B). \end{aligned}$$

Also, $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. Finally, the existence of coincidence best proximity point of the pair $(T; S)$ follows from Theorem 3.7; that is, there exists $p \in A$ such that

$$|Tp - Sp| = \text{dist}(A, B) = 8 \ or \ 4 - p - 4p - 16 = 8,$$

which implies that $p = -4$.

Lemma 3.10. *Let (A, B) be a nonempty pair of subsets of a uniformly convex metric space $(X, d, \mathcal{W})$ such that $S(A)$ is convex. Suppose $(T; S)$ is a QCN relatively contraction pair defined on X and* $\text{dist}(A, B) \leq \text{dist}(S(A), S(B))$. *For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. Then*

$$d(Sx_{2n+2}, Sx_{2n}) \to 0, \ d(Sx_{2n+3}, Sx_{2n+1}) \to 0.$$

Proof. We prove that $d(Sx_{2n+2}, Sx_{2n}) \to 0$. Assume to the contrary that there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$, there exists $n_k \geq k$ such that

$$d(Sx_{2n_k+2}, Sx_{2n_k}) \geq \varepsilon_0.$$

Choose $0 < \gamma < 1$ such that $\frac{\varepsilon_0}{\gamma} > \text{dist}(A, B)$ and choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \min\left\{\frac{\varepsilon_0}{\gamma} - \text{dist}(A, B), \frac{\text{dist}(A, B)\alpha(\gamma)}{1 - \alpha(\gamma)}\right\}.$$

By Lemma 3.3, we know that $d(Sx_{2n_k}, Sx_{2n_k+1}) \to \text{dist}(A, B)$, so there exists $N \in \mathbb{N}$ such that

$$d(Sx_{2n_k}, Sx_{2n_k+1}) \leq \text{dist}(A, B) + \varepsilon,$$

$$d(Sx_{2n_k+2}, Sx_{2n_k+1}) \leq \text{dist}(A, B) + \varepsilon$$

and

$$d(Sx_{2n_k}, Sx_{2n_k+2}) \geq \varepsilon_0 > \gamma(\text{dist}(A, B) + \varepsilon).$$

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It now follows from the uniformly convexity of X and the convexity of $S(A)$ that

$$\begin{aligned}\operatorname{dist}(A,B) \leq \operatorname{dist}(S(A),S(B)) &\leq d(Sx_{2n_k+1}, \mathcal{W}(Sx_{2n_k}, Sx_{2n_k+2}, \frac{1}{2}))\\ &\leq (\operatorname{dist}(A,B)+\varepsilon)(1-\alpha(\gamma))\\ &< \operatorname{dist}(A,B) + \frac{\operatorname{dist}(A,B)\alpha(\gamma)}{1-\alpha(\gamma)}(1-\alpha(\gamma))\\ &= \operatorname{dist}(A,B),\end{aligned}$$

which is a contradiction. Similarly, we see that $d(Sx_{2n+3}, Sx_{2n+1}) \to 0$. □

The following Theorem guarantees the existence and convergence of coincidence best proximity points for QCN relatively contraction mappings in the setting of uniformly convex metric spaces.

Theorem 3.11. *Let (A,B) be a nonempty, closed pair of subsets of a complete uniformly convex metric space $(X,d;\mathcal{W})$ such that $S(A)$ is convex. Suppose $(T;S)$ is a QCN relatively contraction pair defined on X such that S is continuous on A and relatively anti-Lipschitzian on $A \cup B$. Assume that* $\operatorname{dist}(A,B) \leq \operatorname{dist}(S(A),S(B))$. *Then there exists $p \in A$ such that*

$$d(Sp,Tp) = \operatorname{dist}(A,B).$$

Further, if $x_0 \in A$ and $Tx_n = Sx_{n+1}$, then $\{x_{2n}\}$ converges to the coincidence best proximity point of $(T;S)$.

Proof. For $x_0 \in A$ define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. We prove that $\{Sx_{2n}\}$ and $\{Sx_{2n+1}\}$ are Cauchy sequences. First, we verify that for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$d(Sx_{2l}, Sx_{2n+1}) < \operatorname{dist}(A,B) + \varepsilon,\ \forall l > n \geq N_0. \qquad (*)$$

Assume the contrary. Then there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$ there exists $l_k > n_k \geq k$ satisfying

$$d(Sx_{2l_k}, Sx_{2n_k+1}) \geq \operatorname{dist}(A,B)+\varepsilon_0,\ d(Sx_{2l_k-2}, Sx_{2n_k+1}) < \operatorname{dist}(A,B)+\varepsilon_0.$$

Note that

$$\begin{aligned}\operatorname{dist}(A,B)+\varepsilon_0 &\leq d(Sx_{2l_k}, Sx_{2n_k+1})\\ &\leq d(Sx_{2l_k}, Sx_{2l_k-2}) + d(Sx_{2l_k-2}, Sx_{2n_k+1})\\ &\leq d(Sx_{2l_k}, Sx_{2l_k-2}) + \operatorname{dist}(A,B)+\varepsilon_0.\end{aligned}$$

Letting $k \to \infty$, we obtain

$$d(Sx_{2l_k}, Sx_{2n_k+1}) \to \operatorname{dist}(A,B)+\varepsilon_0.$$

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Moreover, we have

$$\begin{aligned}\operatorname{dist}(A,B)+\varepsilon_0 &\le d(Sx_{2l_k},Sx_{2n_k+1}) = d(Tx_{2l_k-1},Tx_{2n_k})\\ &\le \alpha d(Sx_{2l_k-1},Sx_{2n_k})+(1-\alpha)\operatorname{dist}(A,B)\\ &= \alpha d(Tx_{2l_k-2},Tx_{2n_k-1})+(1-\alpha)\operatorname{dist}(A,B)\\ &\le \alpha d(Sx_{2l_k-2},Sx_{2n_k-1})+(1-\alpha)\operatorname{dist}(A,B).\end{aligned}$$

Therefore, by letting $k\to\infty$ we obtain

$$\operatorname{dist}(A,B)+\varepsilon_0 \le \alpha(\operatorname{dist}(A,B)+\varepsilon_0)+(1-\alpha)\operatorname{dist}(A,B) \le \operatorname{dist}(A,B)+\varepsilon_0.$$

This implies that $\alpha=1$, which is a contradiction. That is, $(*)$ holds. Now, assume that $\{Sx_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon_0>0$ such that for each $k\ge 1$ there exists $l_k>n_k\ge k$ such that

$$d(Sx_{2l_k},Sx_{n_k})\ge \varepsilon_0.$$

Choose $0<\gamma<1$ such that $\frac{\varepsilon_0}{\gamma}>\operatorname{dist}(A,B)$ and choose $\varepsilon>0$ such that

$$0<\varepsilon<\min\left\{\frac{\varepsilon_0}{\gamma}-\operatorname{dist}(A,B),\frac{\operatorname{dist}(A,B)\alpha(\gamma)}{1-\alpha(\gamma)}\right\}.$$

Let $N\in\mathbb{N}$ be such that

$$d(Sx_{2n_k},Sx_{2n_k+1})\le \operatorname{dist}(A,B)+\varepsilon,\ \forall n_k\ge N$$

and

$$d(Sx_{2l_k},Sx_{2n_k+1})\le \operatorname{dist}(A,B)+\varepsilon,\ \forall l_k>n_k\ge N.$$

Uniformly convexity of X implies that

$$\begin{aligned}\operatorname{dist}(A,B)\le \operatorname{dist}(S(A),S(B)) &\le d(Sx_{2n_k+1},\mathcal{W}(Sx_{2n_k},Sx_{2l_k},\frac{1}{2}))\\ &\le (\operatorname{dist}(A,B)+\varepsilon)(1-\alpha(\gamma))<\operatorname{dist}(A,B),\end{aligned}$$

which is a contradiction. Therefore, $\{Sx_{2n}\}$ is a Cauchy sequence in $S(A)$. By the fact that S is relatively anti-Lipschitzian on $A\cup B$, we have

$$d(x_{2l},x_{2n})\le c\,d(Sx_{2l},Sx_{2n})\to 0,\ l,n\to\infty,$$

that is, $\{x_{2n}\}$ is Cauchy. Since A is complete, there exists $p\in A$ such that $x_{2n}\to p$. Now, the result follows from a similar argument as in the proof of Theorem 3.5. □

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References

1. A. Abkar, M. Gabeleh, Best Proximity Points for Cyclic Mappings in Ordered Metric Spaces, *J. Optim. Theory. Appl.*, **150**, (2011), 188–193.
2. M. A. Al-Thagafi, N. Shahzad, Convergence and Existence Results for Best Proximity Points, *Nonlinear Anal.*, **70**, (2009), 3665–3671.
3. M. Borcut, V. Berinde, Tripled Fixed Point Theorems for Contractive Type Mappings in Partially Ordered Metric Spaces, *Nonlinear Anal.*, **74**, (2011), 4889–4897.
4. Y. J. Cho, A. Gupta, E. Karapinar, P. Kumam, W. Sintunawarat, Tripled Best Proximity Point Theorem in Metric Spaces, *Math. Ineq. Appl.*, **16**, (2013), 1197–1216.
5. M. De la Sen, Some Results on Fixed and Best Proximity Points of Multivalued Cyclic Self Mappings with a Partial Order, *Abst. Appl. Anal.*, **2013**, (2013), Article ID 968492, 11 pages.
6. M. De la Sen, R. P. Agarwal, Some Fixed Point-Type Results for a Class of Extended Cyclic Self Mappings with a More General Contractive Condition, *Fixed Point Theory Appl.*, **59**, (2011), 14 pages.
7. C. Di Bari, T. Suzuki, C. Verto, Best Proximity Points for Cyclic Meir-Keeler Contractions, *Nonlinear Anal.*, **69**, (2008), 3790–3794.
8. A. A. Eldred, P. Veeramani, Existence and Convergence of Best Proximity Points, *J. Math. Anal. Appl.*, **323**, (2006), 1001–1006.
9. A. A. Eldred, W. A. Kirk, P. Veeramani, Proximal Normal Structure and Relatively Nonexpansive Mappings, *Studia Math.*, **171**, (2005), 283–293.
10. R. Espinola, M. Gabeleh, P. Veeramani, On the Structure of Minimal Sets of Relatively Nonexpansive Mappings, *Numer. Funct. Anal. Optim.*, **34**, (2013), 845–860.
11. A. F. Leon, M. Gabeleh, Best Proximity Pair Theorems for Noncyclic Mappings in Banach and Metric Spaces, *Fixed Point Theory*, **17**, (2016), 63–84.
12. H. Fukhar-ud-din, A. R. Khan, Z. Akhtar, Fixed Point Results for a Generalized Nonexpansive Map in Uniformly Convex Metric Spaces, *Nonlinear Anal.*, **75**, (2012), 4747–4760.
13. M. Gabeleh, H. Lakzian, N. Shahzad, Best Proximity Points for Asymptotic Pointwise Contractions, *J. Nonlinear Convex Anal.*, **16**, (2015), 83–93.
14. M. Gabeleh, O. Olela Otafudu, N. Shahzad, Coincidence Best Proximity Points in Convex Metric Spaces, *Filomat*, **32**, (2018), 1–12.
15. J. Garcia Falset, O. Mlesinte, Coincidence Problems for Generalized Contractions, *Applicable Anal. Discrete Math.*, **8**, (2014), 1–15.
16. N. Hussain, A. Latif, P. Salimi, Best Proximity Point Results in G-Metric Spaces, *Abst. Appl. Anal.*, (2014), Article ID 837943.
17. E. Karapinar, Best Proximity Points of Kannan Type Cyclic Weak ϕ-Contractions in Ordered Metric Spaces, *An. St. Univ. Ovidius Constanta.*, **20**, (2012), 51–64.
18. W. A. Kirk, P. S. Srinivasan, P. Veeramani, Fixed Points for Mappings Satisfying Cyclic Contractive Conditions, *Fixed point Theory*, **4**, (2003), 79–86.
19. R. Lashkaripour, J. Hamzehnejadi, Generalization of the Best Proximity Point, *J. Inequalities And Special Functions.*, **4**, (2017), 136–147.
20. Z. Mustafa, *A New Structure for Generalized Metric Spaces with Applications to Fixed Point Theory [Ph.D. Thesis]*, The University of Newcastle, New South Wales, Australia., 2005.
21. Z. Mustafa, H. Obiedat, F. Awawdeh, Some Fixed Point Theorem for Mapping on Complete G-Metric Spaces, *Fixed Point Theory Appl.*, (2008), Article ID 189870.
22. Z. Mustafa, B. Sims, A New Approach to Generalized Metric Spaces, *J. Nonlinear Convex Anal.*, (2006), 289–297.

[DOI: 10.52547/ijmsi.17.1.27]

23. V. Pragadeeswarar, M. Marudai, Best Proximity Points: Approximation and Optimization in Partially Ordered Metric Spaces, *Optim. Lett.*, **7**, (2013), 1883–1892.
24. T. Shimizu, W. Takahashi, Fixed Points of Multivalued Mappings in Certian Convex Metric Spaces, *Topological Methods in Nonlin. Anal.*, **8**, (1996), 197–203.
25. T. Suzuki, M. Kikkawa, C. Vetro, The Existence of Best Proximity Points in Metric Spaces with to Property UC, *Nonlinear Anal.*, **71**, (2009), 2918–2926.
26. W. Takahashi, A Convexity in Metric Space and Nonexpansive Mappings, *Kodai Math. Sem. Rep.*, **22**, (1970), 142–149.
27. T. Van An, N. V. An, V. T. Le Hang, A New Approach to Fixed Point Theorems on G-Metric Spaces, *Topology and its Applications.*, **160**, (2013), 1486–1493.

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