On the 2-Adjointable Operators and Superstability of them between 2-Pre Hilbert $C^*$-Module Spaces

Maryam Ramezani$^{*,a}$, Hamid Baghani$^b$

$^a$Department of Mathematics, University of Bojnord, Bojnord, Iran.
$^b$Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran.

E-mail: m.ramezani@ub.ac.ir
E-mail: h.baghani@gmail.com

Abstract. In this paper, first, we introduce the new concept of 2-inner product on Banach modules over a $C^*$-algebra. Next, we present the concept of 2-linear operators over a $C^*$-algebra. Our result improve the main result of the paper [Z. Lewandowska, On 2-normed sets, Glasnik Mat., 38(58) (2003), 99-110]. In the end of this paper, we define the notions 2-adjointable mappings between 2-pre Hilbert $C^*$-modules and prove superstability of them in the spirit of Hyers-Ulam-Rassias.

Keywords: $C^*$-Algebra, 2-Adjointable mapping, Superstability.

2000 Mathematics subject classification: 46L08, 46L09.

1. Introduction

The concept of 2-inner product has been intensively studied by many authors in the last three decades. The basic definitions and elementary properties of 2-inner product spaces can be found in [1] and [2].

Recently, M.Frank and e.t. defined the notion $\phi$-perturbation of an adjointable mapping and proved the superstability of an adjointable mapping on Hilbert $C^*$-modules(see [3]).

$^*$Corresponding Author

Received 13 August 2016; Accepted 24 September 2017
©2019 Academic Center for Education, Culture and Research TMU
In this paper, first, we introduce the definition 2-pre Hilbert $C^*$-module spaces and give several important properties. Next, we present the concept of 2-linear operators over a $C^*$-algebra which coincides with Lewandowska’s definition (see [4, 5]). Also, we define 2-adjoinable mappings between 2-pre Hilbert $C^*$-modules and prove an analogue of $\phi$-perturbation of adjoinable mappings in paper([3]).

We refer the interested reader to monographs [6, 7, 8, 9] and references therein for more information.

2. 2-Pre Hilbert Modules

Let $X$ be a left module over a $C^*$-algebra $A$. An action of $a \in A$ on $X$ is denoted by $a.x \in X$, $x \in X$.

Definition 2.1. A 2-pre Hilbert $A$-module is a left $A$-module $X$ equipped with $A$-valued function defined on $X \times X \times X$ satisfying the following conditions:

$I_1$ $(x, x|z)$ is a positive element in $A$ for any $x, z \in X$ and $(x, x|z) = 0$ if and only if $x$ and $z$ are linearly dependent;

$I_2$ $(x, x|z) = (z, z|x)$ for any $x, z \in X$;

$I_3$ $(y, x|z) = (x, y|z)^*$ for any $x, y, z \in X$;

$I_4$ $(ax + x', y|z) = \alpha(x, y|z) + (x', y|z)$ for any $\alpha \in C$ and $x, x', y, z \in X$;

$I_5$ $(ax, y|z) = a(x, y|z)$ for any $x, y, z \in X$ and any $a \in A$.

The map $(.,.|.)$ is called $A$-valued 2-inner product and $(X,(.,.|.))$ is called 2-pre Hilbert $C^*$-module space.

Example 2.2. Every 2-inner product space is a 2-pre Hilbert $C$-module.

Example 2.3. Let $A$ be a $C^*$-algebra and $J \subseteq A$ be a left ideal. Then $J$ can be equipped with the structure of 2-pre Hilbert $A$-module with $A$-valued inner product $(x,y|z) := xy^*zz^* - xz^*zy^*$ for any $x, y, z \in A$.

Definition 2.4. Let $X$ be a 2-pre Hilbert $A$-module. we can define a function $\|.|.|_X$ on $X \times X$ by $\|x|z \|_X = \|(x, x|z)\|^\frac{1}{2}$ for all $x, z \in X$.

Lemma 2.5. $\|.|.|_X$ satisfies the following conditions:

N1 $\|ax|z \|_X \leq ||a|| \|x|z \|_X$ for any $x, z \in X$ and $a \in A$;

N2 $(x, y|z) (y, x|z) \leq \|y|z \|^2 (x, x|z)$ for any $x, y, z \in X$;

N3 $\|(x, y|z)\|^2 \leq \|(x, x|z)\| \|(y, y|z)\|$.

Proof. N1 is obvious; N3 follows from N2, so let us prove N2. Let $\phi$ be a positive linear functional on $A$. Then $\phi((.,.|.))$ is usual 2-inner product on $X$. Applying the Schwartz inequality for 2-inner product (see [2],
On the 2-adjointable operators and superstability of · · · 123

page 3) we obtain for all \(x, y, z \in X\),
\[
\phi((x, y|z) (y, x|z)) = \phi((x, y|z)y, x|z))
\]
\[
\leq \phi((x, x|z))^\frac{1}{2} \phi((x, y|z)(y, y|z))^{\frac{1}{2}}
\]
\[
\leq \phi((x, x|z))^\frac{1}{2} \phi((x, y|z)(y, y|z)(x, y|^x))^{\frac{1}{2}}
\]
\[
\leq \phi((x, x|z))^\frac{1}{2} ||(y, y|z)||^{\frac{1}{2}} \phi((x, y|z)(y, x|z))^{\frac{1}{2}}.
\]

Thus, for any positive linear functional \(\phi\), we have
\[
\phi((x, y|z)(y, x|z)) \leq ||y|z||_X^{\frac{1}{2}} \phi((x, x|z))
\]

hence
\[
(x, y|z)(y, x|z) \leq ||y|z||_X^{\frac{1}{2}} (x, x|z).
\]

\(\square\)

**Theorem 2.6.** The function \(||.||_X\) is a 2-norm on \(X\).

**Proof.** Now, we verify that \(||.||_X\) satisfies the following properties of 2-norms:

1) \(I_3\) and \(I_4\) show that \(||\alpha x||_X = ||\alpha x, \alpha y||_X^{\frac{1}{2}} = ||\alpha||_X ||x||_X\) for all \(x, y \in X\) and \(\alpha \in \mathbb{C}\).

2) \(I_1\) follows that \(||x||_X = ||x, x||_X^{\frac{1}{2}} = ||x||_X\) for all \(x, y \in X\).

3) It follows from \(I_2\) that \(||x||_X = ||x, x||_X^{\frac{1}{2}} = ||x||_X\) for all \(x, y \in X\).

4) By proposition 2.5 (\(N3\)), we have
\[
||x + x'||_X^{\frac{1}{2}} = ||(x + x', x + x'|y)|| = ||(x, x|y) + (x', x|y) + (x, x'|y) + (x', x'|y)||
\]
\[
\leq ||(x, x|y)|| + 2||(x, x'|y)|| + ||(x', x'|y)||
\]
\[
\leq (||(x, x|y)||^{\frac{1}{2}} + ||(x', x'|y)||^{\frac{1}{2}})^2 = (||x|y||_X + ||x'|y||_X)^2
\]

for all \(x, x', y \in X\). This show that \((X, ||.||_X)\) is a 2-normed space. \(\square\)

3. 2-ADJOINTABLE MAPPINGS

In continue, we let \(A\) be a \(C^*\)-algebra. Now, we start with following definition.

**Definition 3.1.** Let \(X\) and \(Y\) be two 2-pre Hilbert \(A\)-modules. An operator \(f : X \times X \rightarrow Y\) is said to be \(A\)-2 linear if it satisfies the following conditions:

1) \(f(x + y, z + w) = f(x, z) + f(x, w) + f(y, z) + f(y, w)\) for all \(x, y, z, w \in X\);  
2) \(f(\alpha x, \beta y) = \alpha \beta f(x, y)\) for all \(\alpha, \beta \in \mathbb{C}\) and \(x, y \in X\);  
3) \(f(\alpha x, by) = a \cdot b^* f(x, y)\) for all \(x, y \in X\) and \(a, b \in A\).

**Example 3.2.** Let \(X\) be a 2-pre Hilbert \(A\)-module and \(z \in X\). Define \(f : X \times X \rightarrow A\) by \(f(x, y) = (x, y)|z\). Then \(f\) is a \(A\)-2 linear operator.
**Definition 3.3.** Let \( X \) and \( Y \) be two 2-pre Hilbert \( A \)-modules. A mapping \( f : X \times X \to Y \) is called 2-adjointable if there exists a mapping \( g : Y \times Y \to X \) such that

\[
(f(x, y), s \mid t) = (x, y \mid g(s, t))
\]

(3.1)

for all \( x, y \in X \) and \( s, t \in Y \). The mapping \( g \) is denoted by \( f^* \) and is called the 2-adjointable of \( f \).

**Lemma 3.4.** Let \( X \) be a 2-pre Hilbert \( A \)-module and \( \dim(X) > 1 \). If \( (x, y) = 0 \) for all \( y, z \in X \), then \( x = 0 \).

**Proof.** Suppose \( x \neq 0 \). Let \( x \) and \( y \) be linearly independent. Then by hypothesis \( (x, x, |y) = 0 \) and this is contradiction. \( \square \)

**Lemma 3.5.** Every 2-adjointable mapping is \( A \)-2 linear.

**Proof.** Let \( f : X \times X \to Y \) be a 2-adjointable mapping. Then there exists a mapping \( g : Y \times Y \to X \) such that (3.1) holds. For every \( x, y, z, w \in X \), every \( s, t \in Y \), every \( a, b \in A \), we have

\[
(f(ax + y + b(z + w), s \mid t) = (ax + y + b(z + w) \mid g(s, t))
\]

\[
= a\alpha \beta (x, z \mid g(s, t)) + \alpha a (x, w \mid g(s, t)) + \beta b^* (y, z \mid g(s, t)) + (y, w \mid g(s, t))
\]

\[
= \alpha \beta a (f(x, z), s \mid t) + \alpha a (f(x, w), s \mid t) + \beta b^* (f(y, z), s \mid t) + (f(y, w), s \mid t)
\]

\[
= (\alpha \beta a f(x, z) + \alpha a f(x, w) + \beta b^* f(y, z) + f(y, w), s \mid t).
\]

It follows from lemma 3.4 that \( f \) is \( A \)-2 linear. \( \square \)

### 4. Superstability of 2-Adjointable Mappings

In this section, \( X \) and \( Y \) denote 2-pre Hilbert \( A \)-modules and \( \dim(X) > 1 \), \( \dim(Y) > 1 \) and \( \phi : X^2 \times Y^2 \to [0, \infty) \) is a function. We start our work with following definition.

**Definition 4.1.** \( A \) (not necessarily \( A \)-2 linear) mapping \( f : X \times X \to Y \) is called \( \phi \)-perturbation of an 2-adjointable mapping if there exists a mapping (not necessarily \( A \)-2 linear) \( g : Y \times Y \to X \) such that

\[
\|f(x, y), s \mid t) - (x, y \mid g(s, t))\| \leq \phi(x, y, s, t)
\]

(4.1)

for all \( x, y \in X \) and \( s, t \in Y \).

**Theorem 4.2.** Let \( f : X \times X \to Y \) be a \( \phi \)-perturbation of a 2-adjointable mapping with corresponding mapping \( g : Y \times Y \to X \). Suppose for some sequence \( c_n \) of non-zero complex numbers the following conditions hold:

\[
\lim_{n \to \infty} |c_n|^{-1} \phi(c_n x, y, s, t) = 0 \quad (x, y \in X, s, t \in Y)
\]

(4.2)
\[
\lim_{n \to \infty} |c_n|^{-1} \phi(x, y, c_n s, t) = 0 \quad (x, y \in X, s, t \in Y)
\] (4.3)

Then \( f \) is 2-adjointable and hence \( f \) is \( A \)-2 linear.

\textbf{Proof.} Let \( \lambda \in \mathbb{C} \) be an arbitrary number. Putting \( \lambda x \) instead \( x \) in (4.1), we get
\[
\| (f(\lambda x, y), s | t) - (\lambda f(x, y), s | t) \| \leq |\lambda| \phi(x, y, s, t)
\]
multiplication of (4.1) by \( |\lambda| \), we have
\[
\| (\lambda f(x, y), s | t) - (\lambda f(x, y), s | t) \| \leq |\lambda| \phi(x, y, s, t)
\]

Thus,
\[
\| (f(\lambda x, y), s | t) - (\lambda f(x, y), s | t) \| \leq \phi(\lambda x, y, s, t) + |\lambda| \phi(x, y, s, t)
\] (4.4)

Replacing \( c_n s \) by \( s \) in (4.4), we get
\[
\| (f(\lambda x, y), s | t) - (\lambda f(x, y), s | t) \| \leq |c_n|^{-1} \phi(\lambda x, y, c_n s, t) + |\lambda| |c_n|^{-1} \phi(x, y, c_n s, t)
\]
hence, as \( n \to \infty \), applying (4.3) we obtain
\[
(f(\lambda x, y), s | t) - (\lambda f(x, y), s | t) = 0 \quad (\lambda \in \mathbb{C}, x, y \in X, s, t \in Y).
\]

It follows from proposition 3.4 that
\[
f(\lambda x, y) = \lambda f(x, y) \quad (\lambda \in \mathbb{C}, x, y \in X)
\] (4.5)

Now, we take \( c_n x \) instead \( x \) in (4.1) to get
\[
\| (f(c_n x, y), s | t) - (c_n x, y | g(s, t)) \| \leq \phi(c_n x, y, s, t).
\]

It follows from (4.5) that
\[
\| (f(x, y), s | t) - (x, y | g(s, t)) \| \leq |c_n|^{-1} \phi(c_n x, y, s, t)
\]
hence, as \( n \to \infty \), applying (4.2) we get
\[
(f(x, y), s | t) = (x, y | g(s, t)) \quad (x, y \in X, s, t \in Y).
\]

Therefore \( f \) is 2-adjointable and by Lemma 3.5, \( f \) is \( A \)-2 linear. \( \square \)

In the following, we let \( c_n = a^n \) that \( a > 1 \), we get the following results.

\textbf{Corollary 4.3.} If \( f : X \times X \to Y \) is a \( \phi \)-perturbation of a 2-adjointable mapping, where
\[
\phi(x, y, s, t) = \epsilon \| x \|_X^p \| y \|_Y^q \| s \|_X^r \| t \|_Y^s \quad (\epsilon \geq 0, 0 < p < 1, 0 < q < 1),
\]
then \( f \) is 2-adjointable and hence \( f \) is \( A \)-2 linear.

\textbf{Corollary 4.4.} If \( f : X \times X \to Y \) is a \( \phi \)-perturbation of a 2-adjointable mapping, where
\[
\phi(x, y, s, t) = \epsilon_1 \| x \|_X^p + \epsilon_2 \| s \|_X^r \| t \|_Y^s \quad (\epsilon_1 \geq 0, \epsilon_2 \geq 0, 0 < p < 1, 0 < q < 1).
\]
Then \( f \) is 2-adjointable and hence \( f \) is \( A \)-2 linear.

\textbf{Acknowledgments}

We would like to thank the referee for his/her careful reading of the paper.
References