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On Graded Weakly Classical Prime Submodules

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Abstract. Let R be a G-graded ring and M be a G-gr-R-module. In this article, we introduce the concept of graded weakly classical prime submodules and give some properties of such a submodule.

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1. Introduction

Gr-prime ideals of a commutative graded ring have been introduced and studied by Refai and Al-Zoubi in [14]. Gr-weakly prime ideals of a commutative graded ring have been introduced and studied by Atani in [4]. Gr-prime and gr-weakly prime submodules of graded modules over graded commutative rings have been studied by various authors; (see, for example [5, 6, 7, 12]). Gr-2-absorbing and gr-weakly 2-absorbing submodules have been studied by Al-Zoubi and Abu-Dawwas in [2]. Also, gr-classical prime submodules of graded modules over graded commutative rings have been introduced and studied by various authors; (see [3, 8]). Here we introduce the concept of graded weakly classical prime (gr-weakly classical prime) submodules. A number of results

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concerning of gr-weakly classical prime submodules are given (see sec. 2). First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [9] and [10] for these basic properties and more information on graded rings and modules. Let G be a group with identity e. A ring R is said to be G-graded ring if there exist additive subgroups R_g of R such that $R = \oplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called homogeneous of degree g and R_e (the identity component of R) is a subring of R and $1 \in R_e$. For $x \in R$, x can be written uniquely as $\sum_{g\in G} x_g$ where x_g is the component of x in R_g. Also we write $h(R) = \cup_{g\in G} R_g$ and $supp(R, G) = \{g \in G : R_g \neq 0\}$. Let M be a left R - module. Then M is a G - graded R - module (shortly, M is gr-R- module) if there exist additive subgroups M_g of M indexed by the elements $g \in G$ such that $M = \oplus_{g\in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. The elements of M_g are called homogeneous of degree g. If $x \in M$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in M_g. Clearly, M_g is R_e - submodule of M for all $g \in G$. Also we write $h(M) = \cup_{g\in G} M_g$. and $supp(M, G) = \{g \in G : M_g \neq 0\}$. Let R be a G-graded ring and I be an ideal of R. Then I is called G-graded ideal if $I = \oplus_{g\in G}(I \cap R_g)$, i.e., if $x \in I$ and $x = \sum_{g\in G} x_g$, then $x_g \in I$ for all $g \in G$. An ideal of a G-graded ring need not be G-graded.

Let M be a G-gr-R-module and N be an R-submodule of M. Then N is called G-gr-R-submodule if $N = \oplus_{g\in G}(N \cap M_g)$, i.e., if $x \in N$ and $x = \sum_{g\in G} x_g$, then $x_g \in N$ for all $g \in G$. Also, an R-submodule of a G-graded R-module need not be G-graded.

Let R be a G-graded ring and M a graded R-module. A proper graded ideal P of R is said to be gr-prime (resp. gr-weakly prime) ideal if whenever $r, s \in h(R)$ with $rs \in P$ (resp. $0 \neq rs \in P$), then either $r \in P$ or $s \in P$. A proper graded submodule N of a graded module M is said to be gr-prime (resp. gr-weakly prime) submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$ (resp. $0 \neq rm \in N$), then either $r \in (N :_R M)$ or $m \in N$. A proper graded submodule N of M is called a gr-classical prime submodule if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$, then either $rm \in N$ or $sm \in N$. Of course, every gr-prime submodule is a gr-classical prime submodule, but the converse is not true in general (see [3, Example 2.3]). The annihilator of graded R-module M which is denoted by $Ann_G(M)$ is $(0 : M)$. Furthermore, for every $m \in h(M)$, $(0 : m)$ is denoted by $Ann_G(m)$.

2. Results

Definition 2.1. Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. N is said to be graded weakly classical prime (gr-weakly classical prime) if whenever $a, b \in h(R)$ and $m \in h(M)$ such that $0 \neq abm \in N$, then either $am \in N$ or $bm \in N$.

Proposition 2.2. *Let M be a gr-R-module and N be a gr-R-submodule of M. If $(N:m)$ is a gr-weakly prime ideal of R for every $m \in h(M)-N$, then N is a gr-weakly classical prime R-submodule of M.*

Proof. Let $a, b \in h(R)$ and $m \in h(M)$ such that $0 \neq abm \in N$. If $m \in N$, then we are done. Suppose $m \notin N$. Then $0 \neq ab \in (N:m)$ and since $(N:m)$ is a gr-weakly prime ideal, either $a \in (N:m)$ or $b \in (N:m)$ and then either $am \in N$ or $bm \in N$ and hence N is a gr-weakly classical prime R-submodule of M. $\square$

Proposition 2.3. *Let M be a gr-R-module and N be a gr-R-submodule of M. If N is a gr-weakly classical prime R-submodule of M and $m \in h(M)-N$ such that $Ann_G(m)=0$, then $(N:m)$ is a gr-weakly prime ideal of R.*

Proof. By [5, Lemma 2.1], $(N:m)$ is a graded ideal of R. Let $a, b \in h(R)$ such that $0 \neq ab \in (N:m)$. Then since $Ann_G(m)=0$, $0 \neq abm \in N$ and since N is gr-weakly classical prime, either $am \in N$ or $bm \in N$ and then either $a \in (N:m)$ or $b \in (N:m)$. Hence, $(N:m)$ is a gr-weakly prime ideal of R. $\square$

Let M and L be two gr-R-modules. A homomorphism of gr-R-module $\phi : M \to L$ is a homomorphism of R–modules satisfying $\phi(M_g) \subseteq L_g$ for every $g \in G$(see [10]).

Theorem 2.4. *Let R be a G-graded ring and M, L be two gr-R-modules and $\phi : M \to L$ be an epimorphism of gr-modules. If N is a gr-weakly classical prime R-submodule of M containing $Ker(f)$, then $f(N)$ is a gr-weakly classical prime R-submodule of L.*

Proof. First, we prove that $f(N)$ is a graded R-submodule of L. Clearly, $f(N)$ is an R-submodule of L. Let $y \in f(N)$. Then there exists $x \in N$ such that $f(x)=y$. Let $x=\sum_{i=1}^{n} x_{g_i}$ where $x_{g_i} \in M_{g_i}-0$, $g_i \neq g_j$ for $i \neq j$. Then $y=\sum_{i=1}^{n} f(x_{g_i})$. For each $1 \leq i \leq n$, there exists $h_i \in supp(L,G)$ with $f(x_{g_i}) \in L_{h_i}-0$ and $h_i \neq h_j$ for $i \neq j$. Let $h \in G$. If $h \neq h_i$ for all $1 \leq i \leq n$, then $y_h = 0 = f(0) \in f(N)$. If $h = h_i$ for some $1 \leq i \leq n$, then $y_h = f(x_{g_i})$. Since $x \in N$ and N is graded, $x_{g_i} \in N$ and then $y_h \in f(N)$. Hence, $f(N)$ is a graded R-submodule of L. Secondly, we prove that $f(M_g)=L_g$ for all $g \in G$. Let $g \in G$ and let $r_g \in L_g$. If $r_g=0$, then $r_g=0=f(0) \in f(M_g)$. Suppose $r_g \neq 0$. Since f is onto, there exists $x \in M-0$ such that $f(x)=r_g$. Suppose $x=\sum_{i=1}^{n} x_{g_i}$ where $x_{g_i} \in M_{g_i}-0$, $g_i \neq g_j$ for $i \neq j$. Then $r_g=\sum_{i=1}^{n} f(x_{g_i}) = \sum_{i=1}^{k} f(x_{g_{t_i}})$ where $1 \leq t_i \leq n$ and $f(x_{g_{t_i}}) \neq 0$ for all $1 \leq i \leq k$. Since $f(x_{g_{t_i}}) \in L_{g_{t_i}}$, $r_g \in L_g \bigcap \sum_{i=1}^{k} L_{g_{t_i}}$. Thus, $g = g_{t_1} = ... = g_{t_n}$ and hence $k=1$ and $f(x_{g_{t_i}}) = f(x_g) = r_g$. So, $r_g \in f(M_g)$ and hence $L_g \subseteq f(M_g)$ and as $f(M_g) \subseteq L_g$, $f(M_g)=L_g$. Now, let $a, b \in h(R)$ and $s \in h(L)$ such that $0 \neq abs \in f(N)$. Since $s \in h(L)$, $s \in L_g$ for some $g \in G$ and since $L_g = f(M_g)$, there exists $m \in M_g \subseteq h(M)$ such that $f(m) = s$ and then $0 \neq f(abm) \in f(N)$, it follows that there exists $n \in N \cap h(M)$ such that $f(abm) = f(n)$ and then $f(abm - n) = 0$, so $abm - n \in Ker(f) \subseteq N$ and as $n \in N$, $0 \neq abm \in N$. Since N is gr-weakly classical prime, either $am \in N$ or $bm \in N$ and then either $as \in f(N)$ or $bs \in f(N)$. Hence, $f(N)$ is a gr-weakly classical prime R-submodule of L. $\square$

Let M be a G-graded R-module and K be an R-submodule of M. Then M/K is a graded R-module by putting $(M/K)_g = (M_g + K)/K$.

Proposition 2.5. *Let K and N be two graded proper R-submodules of a gr-R-module M such that $K \subset N$. If K is a gr-weakly classical prime R-submodule of M and N/K is a gr-weakly classical prime R-submodule of M/K, then N is a gr-weakly classical prime R-submodule of M.*

Proof. Let $a, b \in h(R)$ and $m \in h(M)$ such that $0 \neq abm \in N$. If $abm \in K$, then as K is gr-weakly classical prime, either $am \in K \subset N$ or $bm \in K \subset N$ and then we are done. Suppose $abm \notin K$. Since $m \in h(M)$, $m \in M_g$ for some $g \in G$ and then $m + K \in (M_g + K)/K = (M/K)_g \subseteq h(M/K)$. Now, $0 \neq ab(m + K) \in N/K$ and since N/K is gr-weakly classical prime, either $am + K \in N/K$ or $bm + K \in N/K$ and then either $am \in N$ or $bm \in N$. Hence, N is a gr-weakly classical prime R-submodule of M. $\square$

Proposition 2.6. *Let N be a graded R-submodule of a gr-R-module M. If N is a gr-weakly prime R-submodule of M, then N is a gr-weakly classical prime R-submodule of M.*

Proof. Let $a, b \in h(R)$ and $m \in h(M)$ such that $0 \neq abm \in N$. Then since N is gr-weakly prime, either $bm \in N$ or $a \in (N : M)$. If $bm \in N$, then we are done. If $a \in (N : M)$, then $am \in N$. Hence, N is a gr-weakly classical prime R-submodule of M. $\square$

The concept of gr-2-absorbing submodules (respectively, gr-weakly 2-absorbing submodules) of a graded module over a commutative graded ring is studied in [2]. A graded proper R-submodule N of a gr-R-module M is said to be gr-2-absorbing (gr-weakly 2-absorbing) if whenever $a, b \in h(R)$ and $m \in h(M)$ such that $abm \in N$ ($0 \neq abm \in N$), then either $am \in N$, $bm \in N$ or $ab \in (N : M)$.

It is clear that if N is a gr-weakly classical prime R-submodule of M, then N is a gr-weakly 2-absorbing R-submodule of M. We introduce the following:

Proposition 2.7. *If N is a gr-weakly 2-absorbing R-submodule of M and $(N : M)$ is a gr-weakly prime ideal of R, then N is a gr-weakly classical prime R-submodule of M.*

Proof. Let $a, b \in h(R)$ and $m \in h(M)$ such that $0 \neq abm \in N$. Then since N is gr-weakly 2-absorbing, $am \in N$, $bm \in N$ or $ab \in (N : M)$. If $am \in N$

or $bm \in N$, then we are done. Suppose $ab \in (N : M)$. If $ab = 0$, then $abm = 0$ a contradiction. So, $0 \neq ab \in (N : M)$ and since $(N : M)$ is gr-weakly prime, either $a \in (N : M)$ or $b \in (N : M)$ and then either $am \in aM \subseteq N$ or $bm \in bM \subseteq N$. Hence, N is a gr-weakly classical prime R-submodule of M. □

Proposition 2.8. *Let N be a graded R-submodule of a gr-R-module M. If N is a gr-weakly classical prime R-submodule of M, then N_g is a weakly classical prime R_e-submodule of M_g for all $g \in G$.*

Proof. Let $g \in G$. Let $a, b \in R_e$ and $m \in M_g$ such that $0 \neq abm \in N_g$. Since $R_e \subseteq h(R)$ and $M_g \subseteq h(M)$, $a, b \in h(R)$ and $m \in h(M)$. Since $N_g \subseteq N$, $0 \neq abm \in N$ and since N is gr-weakly classical prime, either $am \in N$ or $bm \in N$. If $am \in N$, then $am \in R_eM_g \bigcap N \subseteq M_g \bigcap N = N_g$. Similarly, if $bm \in N$, then $bm \in N_g$. Hence, N_g is a weakly classical prime R_e-submodule of M_g. □

Let M be an R-module and N be an R-submodule of M. Then for every $a \in R$, we define $(N :_M a) = \{m \in M : am \subseteq N\}$. it is easy to prove that $(N :_M a)$ is an R-submodule of M containing N. Moreover, it is easy top prove that if N is a graded R-submodule of a gr-R-module M, then $(N :_M a)$ is a graded R-submodule of M.

The next proposition gives a characterization for gr-weakly classical prime submodules.

Proposition 2.9. *Let M be a gr-R-module and N be a graded R-submodule of M. Then N is a gr-weakly classical prime R-submodule of M if and only if $(N :_{h(M)} ab) = (0 :_{h(M)} ab) \bigcup (N :_{h(M)} a) \bigcup (N :_{h(M)} b)$ for all $a, b \in h(R)$.*

Proof. Suppose N is a gr-weakly classical prime R-submodule of M. Let $a, b \in h(R)$ and let $m \in (N :_{h(M)} ab)$. Then $abm \in N$. If $abm = 0$, then $m \in (0 :_{h(M)} ab)$. Suppose $abm \neq 0$. Since N is gr-weakly classical prime, either $am \in N$ or $bm \in N$ and then either $m \in (N :_{h(M)} a)$ or $(N :_{h(M)} b)$. Conversely, Let $a, b \in h(R)$ and $m \in h(M)$ such that $0 \neq abm \in N$. Then $m \in (N :_{h(M)} ab)$ and then by assumption, either $m \in (N :_{h(M)} a)$ or $m \in (N :_{h(M)} b)$ that is either $am \in N$ or $bm \in N$. Hence, N is a gr-weakly classical prime R-submodule of M. □

Similarly, we introduce the following:

Proposition 2.10. *Let M be a gr-R-module and N be a graded R-submodule of M. If N is a gr-weakly classical prime R-submodule of M, then $(N :_{h(R)} abm) = (0 :_{h(R)} abm) \bigcup (N :_{h(R)} am) \bigcup (N :_{h(R)} bm)$ for all $a, b \in h(R)$ and $m \in h(M)$.*

Proof. Let $a, b \in h(R)$ and $m \in h(M)$. Assume that $r \in (N :_{h(R)} abm)$. Then $rabm \in N$. If $rabm = 0$, then $r \in (0 :_{h(R)} abm)$. Suppose $rabm \neq 0$. Then

$0 \neq ab(rm) \in N$ and since N is gr-weakly classical prime, either $arm \in N$ or $brm \in N$ and then either $r \in (N :_{h(R)} am)$ or $r \in (N :_{h(R)} bm)$. $\square$

Theorem 2.11. *Let M_1, M_2 be two graded R-modules and N_1 be a proper graded R-submodule of M_1. Then the following conditions are equivalent:*

(1) *$N = N_1 \times M_2$ is a gr-weakly classical prime submodule of $M = M_1 \times M_2$.*
(2) *N_1 is a gr-weakly classical prime submodule of M_1 and for each $a, b \in h(R)$ and $m_1 \in h(M_1)$ we have $abm_1 = 0, am_1 \notin N_1, bm_1 \notin N_1 \Rightarrow ab \in Ann_G(M_2)$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $N = N_1 \times M_2$ is a gr-weakly classical prime submodule of $M = M_1 \times M_2$. Let $a, b \in h(R)$ and $m_1 \in h(M_1)$ be such that $0 \neq abm_1 \in N_1$. Then $(0,0) \neq ab(m_1, 0) \in N$. Thus $a(m_1, 0) \in N$ or $b(m_1, 0) \in N$, and so $am_1 \in N_1$ or $bm_1 \in N_1$. Consequently N_1 is a gr-weakly classical prime submodule of M_1. Now, assume that $abm_1 = 0$ for some $a, b \in h(R)$ and $m_1 \in h(M_1)$ such that $am_1 \notin N_1$ and $bm_1 \notin N_1$. Suppose that $ab \notin Ann_G(M_2)$. Therefore there exists $m_2 \in h(M_2)$ such that $abm_2 \neq 0$. Hence $(0,0) \neq ab(m_1, m_2) \in N$, and so $a(m_1, m_2) \in N$ or $b(m_1, m_2) \in N$. Thus $am_1 \in N_1$ or $bm_1 \in N_1$ which is a contradiction. Consequently $ab \in Ann_G(M_2)$.
(2) $\Rightarrow$ (1) Let $a, b \in h(R)$ and $(m_1, m_2) \in h(M) = h(M_1 \times M_2)$ be such that $(0,0) \neq ab(m_1, m_2) \in N = N_1 \times M_2$. First assume that $abm_1 \neq 0$. Then by part (2), $am_1 \in N_1$ or $bm_1 \in N_1$. So $a(m_1, m_2) \in N$ or $b(m_1, m_2) \in N$, and thus we are done. If $abm_1 = 0$, then $abm_2 \neq 0$. Therefore $ab \notin Ann_G(M_2)$, and so part (2) implies that either $am_1 \in N_1$ or $bm_1 \in N_1$. Again we have that $a(m_1, m_2) \in N$ or $b(m_1, m_2) \in N$ which shows N is a gr-weakly classical prime submodule of M. $\square$

The following two propositions have easy verifications.

Proposition 2.12. *Let M_1, M_2 be two graded R-modules and N_1 be a proper graded R-submodule of M_1. Then $N = N_1 \times M_2$ is a gr-classical prime submodule of $M = M_1 \times M_2$ if and only if N_1 is a gr-classical prime submodule of M_1.*

Proposition 2.13. *Let M_1, M_2 be two graded R-modules and N_1, N_2 be two proper graded R-submodules of M_1, M_2, respectively. If $N = N_1 \times N_2$ is a gr-weakly classical prime (resp. gr-classical prime) submodule of $M = M_1 \times M_2$, then N_1 is a gr-weakly classical prime (resp. gr-classical prime) submodule of M_1 and N_2 is a gr-weakly classical prime (resp. gr-classical prime) submodule of M_2.*

Let R_i be a commutative graded ring with unity and M_i be a graded R_i-module, for $i = 1, 2$. Consider the graded ring $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is a graded R-module and each graded submodule of M is in the form of $N = N_1 \times N_2$ for some graded submodules N_1 of M_1 and N_2 of M_2.

Theorem 2.14. *Let $R = R_1 \times R_2$ be a graded ring and $M = M_1 \times M_2$ be a graded R-module where M_1 is a graded R_1-module and M_2 is a graded R_2-module. Suppose that $N = N_1 \times M_2$ is a proper graded submodule of M. Then the following conditions are equivalent:*

(1) *N_1 is a gr-classical prime submodule of M_1;*
(2) *N is a gr-classical prime submodule of M;*
(3) *N is a gr-weakly classical prime submodule of M.*

Proof. (1) $\Rightarrow$ (2) Let $(r_1, r_2)(s_1, s_2)(m_1, m_2) \in N$ for some $(r_1, r_2), (s_1, s_2) \in h(R)$ and $(m_1, m_2) \in h(M)$. Then $r_1 s_1 m_1 \in N_1$ so either $r_1 m_1 \in N_1$ or $s_1 m_1 \in N_1$ which shows that either $(r_1, r_2)(m_1, m_2) \in N$ or $(s_1, s_2)(m_1, m_2) \in N$. Consequently N is a gr-classical prime submodule of M.
(2) $\Rightarrow$ (3) It is clear that every gr-classical prime submodule is a gr-weakly classical prime submodule.
(3) $\Rightarrow$ (1) Let $rsm \in N_1$ for some $r, s \in h(R_1)$ and $m \in h(M_1)$. We may assume that $0 \neq m' \in h(M_2)$. Therefore $0 \neq (r, 1)(s, 1)(m, m') \in N$. So either $(r, 1)(m, m') \in N$ or $(s, 1)(m, m') \in N$. Therefore $rm \in N_1$ or $sm \in N_1$. Hence N_1 is a gr-classical prime submodule of M_1. □

Let R be a G-graded ring, M be a graded R-module and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fraction $S^{-1}R$ is a graded ring which is called graded ring of fractions. Indeed, $S^{-1}R = \underset{g\in G}{\oplus}(S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s : r \in R, s \in S \ \ and \ g = (\deg s)^{-1}(\deg r)\}$. The module of fraction $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module which is called module of fractions, if $S^{-1}M = \underset{g\in G}{\oplus}(S^{-1}M)_g$ where $(S^{-1}M)_g = \{m/s : m \in M, s \in S \ \ and \ g = (\deg s)^{-1}(\deg m)\}$. We write $h(S^{-1}R) = \underset{g\in G}{\cup}(S^{-1}R)_g$ and $h(S^{-1}M) = \underset{g\in G}{\cup}(S^{-1}M)_g$, (see[10]).

A graded zero-divisor on a graded R-module M is an element $r \in h(R)$ for which there exists $m \in h(M)$ such that $m \neq 0$ but $rm = 0$. The set of all graded zero-divisors on M is denoted by $G - Zdv_R(M)$.

The following result studies the behavior of gr-weakly classical prime submodules under localization.

Proposition 2.15. *Let R be a G-graded ring, M a graded R-module and $S \subseteq h(R)$ a multiplication closed subset of R. Then the following hold:*

(1) *If N is a gr-weakly classical prime R-submodule of M and $(N : M) \bigcap S = \phi$, then $S^{-1}N$ is a gr-weakly classical prime R-submodule of $S^{-1}M$.*
(2) *If $S^{-1}N$ is a gr-weakly classical prime R-submodule of $S^{-1}M$ such that $S \bigcap G - Zdv_R(N) = \phi$ and $S \bigcap G - Zdv_R(M/N) = \phi$, then N is a gr-weakly classical prime R-submodule of M.*

Proof. (1) Let N be a gr-weakly classical prime R-submodule of M and $(N : M) \bigcap S = \phi$. Suppose $0 \neq \frac{p}{r}\frac{q}{s}\frac{m}{t} \in S^{-1}N$ for some $\frac{p}{r}, \frac{q}{s} \in h(S^{-1}R)$

and for some $\frac{m}{t} \in h(M)$. Then there exists $u \in S$ such that $upqm \in N$. If $upqm = 0$, then $\frac{p}{r}\frac{q}{s}\frac{m}{t} = \frac{upqm}{urst} = \frac{0}{1}$ a contradiction. Since N is gr-weakly classical prime and $0 \neq upqm \in N$, we conclude that either $pum \in N$ or $qum \in N$. So, $\frac{p}{r}\frac{m}{t} = \frac{upm}{urt} \in S^{-1}N$ or $\frac{q}{s}\frac{m}{t} = \frac{uqm}{ust} \in S^{-1}N$. Thus $S^{-1}N$ is a gr-weakly classical prime R-submodule of $S^{-1}M$.

(2) Suppose $S^{-1}N$ is a gr-weakly classical prime R-submodule of $S^{-1}M$ such that $S \bigcap G - Zdv_R(N) = \phi$ and $S \bigcap G - Zdv_R(M/N) = \phi$. Let $p, q \in h(R)$ and $m \in h(M)$ such that $0 \neq pqm \in N$. Then $\frac{p}{1}\frac{q}{1}\frac{m}{1} \in S^{-1}N$. If $\frac{p}{1}\frac{q}{1}\frac{m}{1} = 0$, then there exists $u \in S$ such that $upqm = 0$ that contradicts $S \bigcap G - Zdv_R(N) = \phi$. Since $S^{-1}N$ is a gr-weakly classical prime R-submodule of $S^{-1}M$ and $0 \neq \frac{p}{1}\frac{q}{1}\frac{m}{1} \in S^{-1}N$, we conclude that either $\frac{p}{1}\frac{m}{1} \in S^{-1}N$ or $\frac{q}{1}\frac{m}{1} \in S^{-1}N$. If $\frac{p}{1}\frac{m}{1} \in S^{-1}N$, then there exists $s \in S$ such that $spm \in N$ and since $S \bigcap G - Zdv_R(M/N) = \phi$, $pm \in N$. Similarly, If $\frac{q}{1}\frac{m}{1} \in S^{-1}N$, then $qm \in N$. Therefore, N is a gr-weakly classical prime R-submodule of M.

□

Theorem 2.16. *Let R be a G-graded ring, M a graded R-module and N a gr-weakly classical prime submodule of M. Then for each $g \in M_g$, either N_g is a classical prime R_e-submodule of M_g or $(N_g :_{R_e} M_g)^2 N_g = 0$.*

Proof. By Proposition 2.8, N_g is a weakly classical prime R_e-submodule of M_g for every $g \in M_g$. It is enough to show that if $(N_g :_{R_e} M_g)^2 N_g \neq 0$ for some $g \in G$, then N_g is a classical prime R_e-submodule of M_g. Let $rsm \in N_g$ where $r, s \in R_e$ and $m \in M_g$. If $rsm \neq 0$, then either $rm \in N_g$ or $sm \in N_g$ since N_g is a weakly classical prime R_e-submodule of M_g. So suppose that $rsm = 0$. If $rsN_g \neq 0$, then there is an element $n \in N_g$ such that $rsn \neq 0$, so $0 \neq rs(m + n) = rsn \in N_g$, so we conclude that $r(m + n) \in N_g$ or $s(m + n) \in N_g$. Thus $rm \in N_g$ or $sm \in N_g$. So we can assume that $rsN_g = 0$. If $r(N_g :_{R_e} M_g)m \neq 0$ then there is an element $w \in (N_g :_{R_e} M_g)$ such that $rwm \neq 0$. Then $r(s + w)m \neq 0$ because $rsm = 0$. Since $wm \in N_g$, $r(s + w)m \in N_g$. Then $rm \in N_g$ or $(s + w)m \in N_g$. Hence $rm \in N_g$ or $sm \in N_g$. So we can assume that $r(N_g :_{R_e} M_g)m = 0$. Similarly, we can assume that $s(N_g :_{R_e} M_g)m = 0$. If $r(N_g :_{R_e} M_g)N_g \neq 0$, then $rka \neq 0$ for some $k \in (N_g :_{R_e} M_g)$ and $a \in N_g$. Since $rsN_g = 0$ and $r(N_g :_{R_e} M_g)m = 0$, we conclude that $0 \neq r(s + k)(m + a) = rka \in N_g$. So $r(m + a) \in N_g$ or $(s + k)(m + a) \in N_g$. Hence $rm \in N_g$ or $sm \in N_g$. So we can assume that $r(N_g :_{R_e} M_g)N_g = 0$. Similarly, we can assume that $s(N_g :_{R_e} M_g)N_g = 0$. Since we assume that $(N_g :_{R_e} M_g)^2 N_g \neq 0$, there are $r_1, r_2 \in (N_g :_{R_e} M_g)$ and $t \in N_g$ such that $r_1 r_2 t \neq 0$. Then $(r + r_1)(s + r_2)(m + t) = r_1 r_2 t \in N_g$. So $(r + r_1)(m + t) \in N_g$ or $(s + r_2)(m + t) \in N_g$. Hence $rm \in N_g$ or $sm \in N_g$. Thus N_g is a classical prime R_e-submodule of M_g □

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