$z_R$-Ideals and $z_R^\circ$-Ideals in Subrings of $\mathbb{R}^X$

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Abstract. Let $X$ be a topological space and $R$ be a subring of $\mathbb{R}^X$. By determining some special topologies on $X$ associated with the subring $R$, characterizations of maximal fixed and maximal $g$-ideals in $R$ of the form $M_x(R)$ are given. Moreover, the classes of $z_R$-ideals and $z_R^\circ$-ideals are introduced in $R$ which are topological generalizations of $z$-ideals and $z^\circ$-ideals of $C(X)$, respectively. Various characterizations of these ideals are established. Also, coincidence of $z_R$-ideals with $z$-ideals and $z_R^\circ$-ideals with $z^\circ$-ideals in $R$ are investigated. It turns out that some fundamental statements in the context of $C(X)$ are extended to the subrings of $\mathbb{R}^X$.

Keywords: $Z(R)$-topology, $Coz(R)$-topology, $g$-ideal, $z_R$-ideal, $z_R^\circ$-ideal, invertible subring.


1. Introduction

For a topological space $X$, $\mathbb{R}^X$ denotes the algebra of all real-valued functions and $C(X)$ (resp., $C^*(X)$) denotes the subalgebra of $\mathbb{R}^X$ consisting of all continuous functions (resp., bounded continuous functions). Moreover, we use $R$ to denote a unital subring of $\mathbb{R}^X$. Note that topological spaces which are considered in this paper are not necessarily Tychonoff. For each $f \in \mathbb{R}^X$,
\[ Z(f) = \{ x \in X : f(x) = 0 \} \] denotes the zero-set of \( f \) and \( \text{Coz}(f) \) denotes the complement of \( Z(f) \) with respect to \( X \). We denote by \( Z(R) \) the collection of all the zero-sets of elements of \( R \), we use \( Z(X) \) instead of \( Z(C(X)) \). We denote by \( M_x(R) \) the set \( \{ f \in R : x \in Z(f) \} \), \( M_x(C(X)) \) is denoted by \( M_x \). The subring \( R \) is called invertible, if \( f \in R \) and \( Z(f) = \emptyset \) implies that \( f \) is invertible in \( R \). Moreover, \( R \) is called a lattice-ordered subring if it is a sublattice of \( \mathbb{R}^X \) (i.e., \( f \land g \) and \( f \lor g \) are in \( R \) for each \( f, g \in R \)). It is clear that \( C(X) \) is an invertible lattice-ordered subring of \( \mathbb{R}^X \). However, the same statement does not hold for \( C^*(X) \). A proper ideal \( I \) of \( R \) is called a growing ideal, briefly, a \( g \)-ideal, if contains no invertible element of \( \mathbb{R}^X \), i.e., \( Z(f) \neq \emptyset \) for each \( f \in I \). It is evident that a subring \( R \) is invertible if and only if every ideal every ideal of \( R \) is a \( g \)-ideal. Clearly, \( M^{*p} \), for each \( p \in \beta X \setminus \nu X \), is not a \( g \)-ideal of \( C^*(X) \). An ideal \( I \) of \( R \) is called fixed if \( \bigcap_{f \in I} Z(f) \neq \emptyset \), otherwise, it is called free. By a maximal fixed ideal of \( R \), we mean a fixed ideal which is maximal in the set of all fixed ideals of \( R \). An ideal \( I \) in a commutative ring \( S \) is called a \( z \)-ideal (resp., \( z^\circ \)-ideal) if \( M_a(S) \subseteq I \) (resp., \( P_a(S) \subseteq I \)), for each \( a \in I \), where \( M_a(S) \) (resp., \( P_a(S) \)) denotes the intersection of all the maximal (resp., minimal prime) ideals of \( S \) containing \( a \). It is well-known that in \( C(X) \) an ideal \( I \) is a \( z \)-ideal (resp., \( z^\circ \)-ideal) if and only if whenever \( Z(f) \subseteq Z(g) \) (resp., \( \text{int}_X Z(f) \subseteq \text{int}_X Z(g) \)), \( f \in I \) and \( g \in C(X) \), then \( g \in I \).

This paper consists of 4 sections. Section 1, as we have already noticed, is the introduction, in which we determine two special topologies on \( X \) which the subring \( R \) generate, namely, \( Z(R) \)-topology and \( \text{Coz}(R) \)-topology. Comparison and coincidence of these topologies are studied. Section 2 deals with maximal ideals in \( R \), specially, maximal fixed and maximal \( g \)-ideals. Using the \( Z(R) \)-topology, characterizations of maximal fixed ideals of \( R \), which are of the form \( M_x(R) \), are given. Moreover, relations between mapping \( "x \rightarrow M_x(R)" \) and the separation properties of the topological space \( (X, \tau_{Z(R)}) \) will be found. In section 3, we introduce the notion of \( z_R \)-ideal in a subring \( R \) as a natural topological generalization of the notion of \( z \)-ideal in \( C(X) \). Various characterizations of these ideals via \( Z(R) \)-topology are given and relations between \( z_R \)-ideals and \( z \)-ideals in \( R \) (by their algebraic descriptions) are discussed. Section 4 deals with \( z^\circ_R \)-ideals of \( R \) which are natural topological generalizations of \( z^\circ \)-ideals of \( C(X) \). Using \( \text{Coz}(R) \)-topology, coincidence of \( z^\circ_R \)-ideals with \( z^\circ \)-ideals of \( R \) (by their algebraic descriptions) are established.

**Definition 1.1.** For each subring \( R \) of \( \mathbb{R}^X \), clearly, \( Z(R) \) and \( \text{Coz}(R) \) constitute bases for some topologies on \( X \). The induced topologies are called \( Z(R) \)-topology and \( \text{Coz}(R) \)-topology, respectively, and are denoted by \( \tau_{Z(R)} \) and \( \tau_{\text{Coz}(R)} \), respectively.

In the next three statements we compare these topologies. Note that two subsets \( S_1, S_2 \) of \( \mathbb{R}^X \) are called zero-set equivalent, if \( Z(S_1) = Z(S_2) \).
Proposition 1.2. Let $R$ be a subring of $\mathbb{R}^X$, if $S$ and $C(\mathbb{R})$ are zero-set equivalent subsets of $\mathbb{R}^R$ and $gof \in R$ for each $f \in R$ and each $g \in S$, then $\tau_{Coz(R)} \subseteq \tau_{Z(R)}$ and the equality does not hold, in general.

Proof. We are to show that $Coz(R) \subseteq \tau_{Z(R)}$. If $x \notin Z(f)$ where $f \in R$, then there is a $g$ in $S$ such that $f(x) \in Z(g)$ and $f^{-1}(Z(g)) \cap Z(f) = \emptyset$. Therefore, $gof \in R$, $x \in Z(gof)$ and $Z(gof) \cap Z(f) = \emptyset$ which proves the inclusion. Now, we show that the inclusion may be proper. Let $(X, \tau_X)$ be a Tychonoff space which has at least one non-open zero-set $Z$. Set $R = C(X)$, then $\tau_{Coz(R)} = \tau_X$, whereas $Z \notin \tau_X$ and hence, $\tau_{Coz(R)} \subset \tau_{Z(R)}$.

Proof of the following proposition is standard.

Proposition 1.3. The following statements are equivalent.

(a) $\tau_{Coz(R)} \subseteq \tau_{Z(R)}$.
(b) Every $Z \in Z(R)$ is clopen under $Z(R)$-topology.

The annihilator of $f \in R$ in $R$ is defined to be the set $\{g \in R : fg = 0\}$ and is denoted by $Ann_R(f)$. A simple reasoning shows that if $X$ is equipped with the $Coz(R)$-topology, then $Ann_R(f) = \{g \in R : Coz(g) \subseteq \text{int}_X(Z(f))\} = \{g \in R : \text{cl}_X(Coz(g)) \subseteq Z(f)\}$.

Proposition 1.4. The following statements are equivalent.

(a) $\tau_{Z(R)} \subseteq \tau_{Coz(R)}$.
(b) $Z(f)$ is clopen in $(X, \tau_{Coz(R)})$ for every $f \in R$.
(c) For each $f \in R$, $Z(f) = \bigcup_{g \in Ann_R(f)} Coz(g)$.
(d) For each $f \in R$, $(Ann_R(f), f)$ is a free ideal.

Proof. The implications (a)⇒(b)⇒(c) are clear.

(c)⇒(d). This clear by the hypothesis and the fact that whenever $f \in R$ and $I$ is an ideal of $R$, then $\bigcap_{h \in \langle f \rangle} Z(h) = \bigcap_{h \notin \langle f \rangle} (Z(f) \cap Z(g))$.

(d)⇒(a). Let $f \in R$ and $x \in Z(f)$. By (d), there exists $g \in Ann_R(f)$ such that $x \notin Z(f) \cap Z(g)$. Hence, $x \notin Z(g)$ and $x \in Coz(g) \subseteq Z(f)$ and so $Z(f) \in \tau_{Coz(R)}$.

An immediate consequence of Propositions 1.3 and 1.4 is that $\tau_{Coz(R)} = \tau_{Z(R)}$ if and only if $Z(f)$ is clopen under both $Z(R)$-topology and $Coz(R)$-topology, for each $f \in R$.

2. Characterization of Maximal Fixed Ideals in Subrings

We remind that maximal fixed ideals of $C(X)$ coincide with its fixed maximal ideals and are of the form $M_x = \{f \in C(X) : f(x) = 0\}$, where $x \in X$. This fact is generalized for some special subalgebras of $C(X)$, such as intermediate subalgebras (subalgebras of $C(X)$ containing $C^*(X)$, see [7]), $C_c(X)$ (the subalgebra of $C(X)$ consisting of all functions with countable image), see [9]) and the subalgebras of the form $R + I$ where $I$ is an ideal of $C(X)$, see [13].
We will show that the same statement does not hold for arbitrary subrings of \( R^X \), in general.

**Remark 2.1.** (a) Every maximal fixed ideal and fixed maximal ideal of \( R \) is of the form \( M_x(R) = \{ f \in R : f(x) = 0 \} \) for some \( x \in X \). However, parts (1) and (2) of Example 2.2 show that the ideals \( M_x(R) \) are not necessarily maximal ideals or even maximal fixed ideals in \( R \).

(b) Every maximal \( g \)-ideal is both a maximal fixed ideal and a maximal \( g \)-ideal. But the converse is not necessarily true, in general, see part (1) of Example 2.2 and Example 2.3.

(c) A maximal fixed ideal need not be a maximal \( g \)-ideal, see Example 2.3.

(d) Every fixed maximal \( g \)-ideal is a maximal fixed ideal.

**Example 2.2.** (1) Let \( X \) be a Tychonoff space, \( x \in X \) and \( R = \mathbb{Z} + M_x \). Then \( M_x(R) = M_x \) is not a maximal ideal in \( R \), since \( 2\mathbb{Z} + M_x \) is a proper ideal of \( R \) and \( M_x \subseteq 2\mathbb{Z} + M_x \). Therefore, \( M_x(R) \) is a maximal fixed ideal and a maximal \( g \)-ideal which is not a maximal ideal.

(2) Let \( X \) be a topological space with more than one point and \( a \in X \). Also, let \( t \in \mathbb{R} \) be a transcendental number and define \( f : X \rightarrow \mathbb{R} \) by \( f(a) = 0 \) and \( f(x) = t \), for every \( x \neq a \). Set \( R = \{ \sum_{i=0}^{n} m_i f^i : n \in \mathbb{N} \cup \{ 0 \}, m_i \in \mathbb{Z} \} \).

Evidently, \( M_a(R) = (f) \) and \( M_x(R) = \{ 0 \} \), for every \( x \neq a \). Therefore, \( M_x(R) \) is not a maximal fixed ideal for any \( x \neq a \).

In the next example we construct a subring \( R \) such that, for some \( x \in X \), \( M_x(R) \) is a maximal fixed ideal which is not a maximal \( g \)-ideal.

**Example 2.3.** Let \( X = \mathbb{R} \), \( a \in \mathbb{R} \setminus \mathbb{Q} \), \( b \in \mathbb{R} \setminus \{ 0 \} \) and \( t \) be a transcendental number. For every \( \epsilon > 0 \), define \( f_\epsilon : X \rightarrow \mathbb{R} \) by \( f_\epsilon(x) = 0 \), if \( |x - a| < \epsilon \) and \( f_\epsilon(x) = b \), if \( |x - a| \geq \epsilon \). Also, define \( f : X \rightarrow \mathbb{R} \) by \( f(x) = 0 \), if \( x \in \mathbb{Q} \) and \( f(x) = t \), if \( x \notin \mathbb{Q} \). Let \( R \) be the algebra over \( \mathbb{Q} \) generated by \( \{ f_\epsilon : \epsilon > 0 \} \cup \{ f, 1 \} \). Evidently, \( R \) is a subring of \( \mathbb{R}^X \), and \( M_a(R) \) equales to \( (f_a) \) which is not a maximal ideal. It is easy to see that \( M_a(R) \) is a maximal fixed ideal and \( M_b(R) = I \), where \( I \) is the ideal generated by \( \{ f_\epsilon : \epsilon > 0 \} \).

Clearly, \( Z(f) \cap Z(g) \neq \emptyset \), for all \( g \in I \). Hence \( I = (I, f) \) is a \( g \)-ideal which strictly contains \( I \). Therefore, \( I \) is not a maximal \( g \)-ideal.

**Proposition 2.4.** The following statements hold for a subring \( R \) of \( \mathbb{R}^X \).

(a) \( M_x(R) \) is a maximal \( g \)-ideal if and only if whenever \( Z \in Z(R) \) and \( x \notin Z \), then \( x \notin \text{cl}_{Z(R)} Z \).

(b) For each \( x \in X \), \( M_x(R) \) is a maximal \( g \)-ideal if and only if every \( Z \in Z(R) \) is clopen under \( Z(R) \)-topology.

**Proof.** (a \( \Rightarrow \)). Let \( f \in R \) and \( x \notin Z(f) \), thus, the ideal \( (M_x(R), f) \) contains an invertible element of \( \mathbb{R}^X \). Hence, there are \( g \in M_x(R) \) and \( h \in R \) such that \( Z(g + fh) = \emptyset \). Consequently, \( x \in Z(g) \) and \( Z(f) \cap Z(g) = \emptyset \).
(a $\iff$). Assume that $f \not\in M_x(R)$. Then there is some $g \in R$ such that $x \in Z(g)$ and $Z(f) \cap Z(g) = Z(f^2 + g^2) = \emptyset$. Hence, $(M_x(R), f)$ contains an invertible element of $\mathbb{R}^X$. Also, clearly, $M_x(R)$ is a $g$-ideal. Thus, $M_x(R)$ is a maximal $g$-ideal.

(b). An easy consequence of (a). $\square$

**Corollary 2.5.** If $M_x(R)$ is a maximal ideal for each $x \in X$, then every $Z \in Z(R)$ is clopen under $Z(R)$-topology.

**Corollary 2.6.** Let $R$ be an invertible subring. Then every $Z \in Z(R)$ is clopen under $Z(R)$-topology if and only if $M_x(R)$ is a maximal ideal for each $x \in X$.

**Proof.** By our hypothesis and Proposition 2.4, this is clear. $\square$

The following lemma is a restatement of the fact that the transcendental degree of $\mathbb{R}$ over $\mathbb{Q}$ is uncountable, see [14].

**Lemma 2.7.** Let $S = \mathbb{Q}[y_1, \ldots, y_n]$ be the ring of $n$-variable polynomials with rational coefficients. Then there exists an uncountable set $X$ of transcendental numbers for which $F(a_1, \cdots, a_n) \neq 0$, for every distinct elements $a_1, \cdots, a_n$ of $X$ and every $F \in S$.

The following example shows that the converse of Corollary 2.5 does not hold, in general.

**Example 2.8.** Let $S$ be the polynomial ring $\mathbb{Q}[y_1, \ldots, y_n]$, where $n \in \mathbb{N}$ and $n > 1$. By Lemma 2.7, there exists an infinite set of transcendental numbers $X$ for which $F(a_1, \cdots, a_n) \neq 0$, for every $a_1, \cdots, a_n \in X$ and every $F \in S$. For each $a \in X$, define the function $f_a : X \to \mathbb{R}$ by $f_a(a) = 0$ and $f_a(x) = x$ for each $x \neq a$. Now, set

$$R = \{F(f_{a_1}, \ldots, f_{a_n}) : F \in S, \ n \in \mathbb{N}, \ a_1, \ldots, a_n \in X\}.$$ 

Hence, $M_x(R) = (f_a)$, for each $a \in X$, which is not a maximal ideal. However, every $Z \in Z(R)$ is clopen under $Z(R)$-topology.

**Proposition 2.9.** If $R$ is a subalgebra of $\mathbb{R}^X$, then $M_x(R)$ is a maximal $g$-ideal and a maximal fixed ideal for every $x \in X$.

**Proof.** It suffices to prove that every element of $Z(R)$ is closed under $Z(R)$-topology. To this aim, suppose that $a \in X$ and $a \not\in Z(f)$, for some $f \in R$. Put $g = f - f(a)$. Clearly, $Z(g) \in Z(R)$, $a \in Z(g)$ and $Z(g) \cap Z(f) = \emptyset$. $\square$

**Corollary 2.10.** If $R$ is an invertible subalgebra of $\mathbb{R}^X$, then $M_x(R)$ is a maximal ideal for each $x \in X$.

The converse of Corollary 2.10 does not hold, in general. For example, let $R$ denote the collection of all single variable polynomials over $\mathbb{R}$. Then, $M_x(R)$ is the maximal ideal $(x - r)$ for each $r \in \mathbb{R}$. However, $f = x^2 + 1$ is invertible in
Suppose that \( x \neq y \) be distinct points of \( X \), so \( M_x(R) \neq M_y(R) \), say \( M_x(R) \not\subseteq M_y(R) \). Hence, there exists \( f \in M_x(R) \setminus M_y(R) \). Thus \( x \in Z(f) \) and \( y \notin Z(f) \). It is clear that the above reasoning is reversible and hence we are done.

(b \( \Rightarrow \)). Suppose that \( x \) and \( y \) are two distinct points of \( X \). Since \( M_x(R) \not\subseteq M_y(R) \), there exists \( f \in M_x(R) \setminus M_y(R) \). Consequently, \( x \in Z(f) \) and \( y \notin Z(f) \).

(b \( \Leftarrow \)). Suppose that \( x \in X \) and \( I \) is a fixed ideal in \( R \) containing \( M_x(R) \). Take \( y \in \bigcap_{f \in I} Z(f) \). Clearly, \( M_y(R) \subseteq I \subseteq M_x(R) \). It suffices to show \( x = y \). Suppose that \( x \neq y \) and seek a contradiction. By our hypothesis, there exists \( f \in R \) such that \( x \in Z(f) \) and \( y \notin Z(f) \). Therefore, \( M_x(R) \not\subseteq M_y(R) \) and this is a contradiction. Now, by part (a), the proof is complete.

(c). For any two distinct points \( x, y \in X \), clearly, \( M_x(R) + M_y(R) \) is not a \( \mathfrak{g} \)-ideal if and only if there exist \( f \in M_x(R) \) and \( g \in M_y(R) \) such that \( Z(f) \cap Z(g) = \emptyset \).

(d \( \Rightarrow \)). By part (a), clearly, \( (X, \tau_{Z(R)}) \) is a \( T_0 \)-space. Now, Suppose that \( f \in R \) and \( x \notin Z(f) \). Since \( M_x(R) \) is a maximal \( \mathfrak{g} \)-ideal, it follows that \( (M_x(R), f) \) has an invertible element of \( R^X \) and so there exists \( g \in M_x(R) \), such that \( Z(g) \cap Z(f) = \emptyset \). Thus, \( Z(f) \) is closed and hence is clopen under \( Z(R) \)-topology.

(d \( \Leftarrow \)). Suppose that \( x \in X \), it suffices to show that \( M_x(R) \) is a maximal \( \mathfrak{g} \)-ideal. Assume that \( I \) is an ideal which properly contains \( M_x(R) \). Hence, there exists \( f \in I \) such that \( x \notin Z(f) \). By our hypothesis, there is \( g \in R \) such that \( x \in Z(g) \) and \( Z(g) \cap Z(f) = \emptyset \). Therefore, \( Z(f^2 + g^2) = \emptyset \) and \( f^2 + g^2 \in I \), hence, \( I \) is not a \( \mathfrak{g} \)-ideal. \( \square \)

It is easy to see that \( M_x(R) \), for each \( x \in X \), is a prime ideal of \( R \) and thus the hull-kernel topology may be defined on the family \( \{M_x(R) : x \in X\} \).
By considering this space, the next statement gives a relation between $Z(R)$-topology on $X$ and points of $X$.

**Proposition 2.12.** Let $R$ be a subring of $\mathbb{R}^X$ and $X$ equipped with the $Coz(R)$-topology. Then the mapping $\Phi : X \to \{M_\alpha(R) : x \in X\}$ defined by $x \mapsto M_\alpha(R)$ is a homeomorphism if and only if $(X, \tau_{Z(R)})$ is a $T_0$-space.

**Proof.** By part (a) of Theorem 2.12, $\Phi$ is a one-one correspondence if and only if $(X, \tau_{Z(R)})$ is a $T_0$-space. Also, if $f \in R$ and $x \in Z(f)$, then $f \in M_\alpha(R)$ which means that basic closed sets of $X$ equipped with the $Coz(R)$-topology are mapped to the basic closed sets in $\{M_\alpha(R) : x \in X\}$ equipped with the hull-kernel topology by the mapping $\Phi$ and therefore, it is a homeomorphism. \(\square\)

### 3. $z_R$-Ideals and $z$-Ideals in Subrings

In this section we introduce $z_R$-ideals in a subring $R$ and via the $Z(R)$-topology and maximal $g$-ideals of $R$, various characterizations of these ideals are given.

**Definition 3.1.** A subset $F$ of $Z(R)$ is called $z_R$-filter on $X$, if

(a) $\emptyset \notin F$.

(b) If $Z_1, Z_2 \in F$, then $Z_1 \cap Z_2 \in F$.

(c) If $Z_1 \in F$, $Z_2 \in Z(R)$ and $Z_1 \subseteq Z_2$, then $Z_2 \in F$.

Moreover, $F$ is called a prime $z_R$-filter, if whenever $Z_1 \cup Z_2 \in F$, then $Z_1 \in F$ or $Z_2 \in F$ for each $Z_1, Z_2 \in Z(R)$. Also, $F$ is called a $z_R$-ultrafilter, if $F$ is maximal among $z_R$-filters on $X$.

The following proposition immediately follows from Definition 3.1.

**Proposition 3.2.** For any subring $R$, the following statements hold.

(a) $I \subseteq R$ is a $g$-ideal in $R$ if and only if $Z_R(I) = \{Z(f) : f \in I\}$ is a $z_R$-filter on $X$.

(b) $F$ is a $z_R$-filter on $X$ if and only if $Z_R^{-1}(F) = \{f \in R : Z(f) \in F\}$ is a $g$-ideal.

(c) $F$ is a prime $z_R$-filter on $X$ if and only if $Z_R^{-1}(F)$ is a prime $g$-ideal.

(d) $A$ is a $z_R$-ultrafilter on $X$ if and only if $Z_R^{-1}(A)$ is a maximal $g$-ideal.

(e) If $M$ is a maximal $g$-ideal in $R$, then $Z_R(M)$ is a $z_R$-ultrafilter on $X$.

It is easy to see that for an ideal $I$ of $R$ we always have $I \subseteq Z_R^{-1}Z_R(I)$ and the inclusion may be proper. We call an ideal $I$ in $R$ a $z_R$-ideal, if $I = Z_R^{-1}Z_R(I)$.

It follows that every $z_R$-ideal is semiprime and arbitrary intersections of $z_R$-ideals are a $z_R$-ideal. Also, the zero ideal, the ideals of the form $M_\alpha(R)$, maximal $g$-ideals and $Z^{-1}(F)$, for each $z_R$-filter $F$, are all $z_R$-ideals of $R$. For each $f \in R$, the intersection of all the maximal ideals, maximal $g$-ideals and maximal fixed ideals of $R$ containing $f$ are denoted by $M_f(R)$, $MG_f(R)$ and $MF_f(R)$, respectively. It is easy to observe that $MG_f(R)$ is a $z_R$-ideal for each $f \in R$.  


Obviously, $MG_f \cap MG_g = MG_{fg}$, $MF_f \cap MF_g = MF_{fg}$, $MG_{f^2+g^2} = MG_{(f,g)}$ and $MF_{f^2+g^2} = MF_{(f,g)}$ for all $f, g \in R$.

**Proposition 3.3.** Let $(X, \tau_{Z(R)})$ be a $T_1$-space. Then the following statements hold.

(a) The following statements are equivalent.
   (1) $g \in MF_f(R)$.
   (2) $MF_g(R) \subseteq MF_f(R)$.
   (3) $Z(f) \subseteq Z(g)$.

(b) $MF_f(R) = \{g \in R : Z(f) \subseteq Z(g)\}$.

(c) An ideal $I$ of $R$ is a $z_R$-ideal if and only if $MF_f(R) \subseteq I$ for every $f \in I$.

**Proof.** (a: 1 $\Rightarrow$ 2). Evident.

(a: 2 $\Rightarrow$ 3). Let $x \in Z(f)$. Then $f \in M_x(R)$ and thus $MF_g(R) \subseteq MF_f(R) \subseteq M_x(R)$. This implies $g \in M_x(R)$ and hence $x \in Z(g)$.

(a: 3 $\Rightarrow$ 1). If $g \notin MF_f(R)$, then there exists $x \in X$ such that $f \in M_x(R)$ and $g \notin M_x(R)$. Therefore, $x \in Z(f) \setminus Z(g)$ and so $Z(f) \subsetneq Z(g)$.

(b) and (c) obviously follow from part (a). \qed

**Lemma 3.4.** Assume that every $Z \in Z(R)$ is clopen under $Z(R)$-topology. Then $MG_f(R) = MF_f(R)$, for every $f \in R$.

**Proof.** Suppose that $f \in R$. By part (b) of Proposition 2.4, $M_x(R)$ is a maximal $g$-ideal for each $x \in X$. Consequently, $MG_f(R) \subseteq MF_f(R)$. Now, assume that $g \notin MG_f(R)$. Hence, there exists a maximal $g$-ideal $M$ in $R$ such that $f \in M$ and $g \notin M$. Thus, there exists $h \in M$ such that $Z(g) \cap Z(h) = \emptyset$. Since $f^2 + h^2 \in M$ and $M$ is a $g$-ideal, there is a point $x \in Z(f^2 + h^2) = Z(f) \cap Z(h)$. Clearly, $g \notin M_x(R)$ and $f \in M_x(R)$. Therefore, $g \notin MF_f(R)$. \qed

Proposition 3.3 and Lemma 3.4 imply the next statement.

**Proposition 3.5.** Let $(X, \tau_{Z(R)})$ be a $T_1$-space and every $Z \in Z(R)$ be a clopen set under $Z(R)$-topology. Then the following statements hold.

(a) The following statements are equivalent.
   (1) $g \in MG_f(R)$.
   (2) $MG_g(R) \subseteq MG_f(R)$.
   (3) $Z(f) \subseteq Z(g)$.

(b) $MG_f(R) = \{g \in R : Z(f) \subseteq Z(g)\}$.

(c) An ideal $I$ of $R$ is a $z_R$-ideal if and only if $MG_f(R) \subseteq I$ for every $f \in I$.

The following corollary follows from Corollary 2.6 and Proposition 3.5.

**Corollary 3.6.** Let $R$ be an invertible subalgebra of $\mathbb{R}^X$. Then the following statements hold.
(a) The following conditions are equivalent:
(1) \( g \in M_f(R) \).
(2) \( M_g(R) \subseteq M_f(R) \).
(3) \( Z(f) \subseteq Z(g) \).

(b) \( M_f(R) = \{ g \in R : Z(f) \subseteq Z(g) \} \).

(c) An ideal \( I \) of \( R \) is \( z_R \)-ideal if and only if \( M_f(R) \subseteq I \) for every \( f \in I \).

It follows from Corollary 3.6 that for an invertible subalgebra \( R \), the notion of \( z_R \)-ideal coincides with the notion of \( z \)-ideal. The next statement extend this fact and shows that this coincidence is equivalent to invertibility of \( R \).

**Theorem 3.7.** Let \( R \) be a subring of \( \mathbb{R}^X \). The following statements are equivalent.

(a) Every maximal ideal in \( R \) is a \( g \)-ideal.

(b) Every maximal \( g \)-ideal of \( R \) is a maximal ideal and if \( J \) is a maximal ideal of \( R \), then every maximal element in the set of \( g \)-ideals contained in \( J \) is a prime ideal.

(c) Every maximal ideal in \( R \) is a \( g \)-ideal.

(d) \( R \) is an invertible subring.

(e) Every \( z \)-ideal of \( R \) is a \( z_R \)-ideal.

Moreover, if \( R \) is a subalgebra and one of (a)-(c) holds, then every \( z_R \)-ideal is a \( z \)-ideal.

**Proof.** (a) \( \Rightarrow \) (b). This is clear.

(b) \( \Rightarrow \) (c). Suppose that \( M \) is a maximal ideal and \( P \) is a maximal element of \( G_M \), where \( G_M \) is the set of all \( g \)-ideals contained in \( M \). Assume that \( J \) is a maximal ideal of \( R \) containing \( P \). Then \( M \cap J = P \). As \( M \cap J \) is prime and both \( M \) and \( J \) are maximal ideal, we have \( M = J \). Hence, \( M \) is a maximal \( g \)-ideal.

(c) \( \Rightarrow \) (d). Suppose that \( Z(f) = \emptyset \) for \( f \in R \) and, on the contrary, \( f \) is a non-unit element of \( R \). Clearly, there exists a maximal ideal \( M \) of \( R \) containing \( f \).

By our hypothesis, \( M \) is a \( g \)-ideal which contradicts with \( Z(f) = \emptyset \).

(d) \( \Rightarrow \) (e). Suppose that \( I \) is a \( z \)-ideal and \( Z(f) \subseteq Z(g) \) where \( f \in I \) and \( g \in R \). Since \( I \) is a \( z \)-ideal, it follows that \( M_f \subseteq I \). It suffices to prove that \( g \in M_f \). To see this, suppose that \( M \) is a maximal ideal containing \( f \). As \( R \) is invertible, \( M \) is a \( g \)-ideal and so it is a maximal \( g \)-ideal. Obviously, \( M \) is a \( z_R \)-ideal and so \( g \in M \).

(e) \( \Rightarrow \) (a). Suppose that \( M \) is a maximal ideal and, on the contrary, \( M \) is not a \( g \)-ideal. Thus, there exists \( f \in M \) such that \( Z(f) = \emptyset \). By (e), \( M \) is a \( z_R \)-ideal and since \( f \in M \), it follows that \( M = R \), which is a contradiction.

Now, suppose that one of (a)-(c) holds, \( R \) is a subalgebra and \( I \) is a \( z_R \)-ideal of \( R \). By our hypothesis, \( M_{F(f)}(R) = M_f(R) \) for every \( f \in R \), and thus we are done. \( \square \)
It is well-known that every minimal prime ideals over a $z$-ideal is also a $z$-ideal, see [10, Theorem 14.7]. The same statement holds for $z_R$-ideals as the following proposition shows.

**Proposition 3.8.** Let $I$ be a $z_R$-ideal of $R$ and $P$ a prime ideal in $R$ minimal over $I$. Then $P$ is a $z_R$-ideal.

*Proof.* Assume that $Z(f) = Z(g)$ and $f \in P$. Thus, there exists $h \notin P$, such that $fh \in I$. Since $Z(fh) = Z(gh)$ and $I$ is a $z_R$-ideal, it follows that $gh \in I \subseteq P$. As $h \notin P$, clearly, this implies that $g \in P$. □

An immediate consequence of Proposition 3.8 is that every minimal prime ideal in a subring $R$ is a $z_R$-ideal. By the following statement, we extend some fundamental statements about $z$-ideals in the literature of $C(X)$ to the subrings of $\mathbb{R}^X$, namely, [10, 2.9, 5.3 and 5.5]. The proofs are left to the reader.

**Proposition 3.9.** Let $R$ be a lattice-ordered subring of $\mathbb{R}^X$ and $I$ be a $z_R$-ideal in $R$. Then the following statements hold.

(a) The following statements are equivalent

1. $I$ is a prime ideal;
2. $I$ contains a prime ideal;
3. if $fg = 0$, then $f \in I$ or $g \in I$;
4. for each $f \in R$, there is a $Z \in Z_R(I)$ on which $f$ does not change sign.

(b) Every prime $g$-ideal of $R$ is contained in a unique maximal $g$-ideal.

(c) If $P$ is a prime ideal of $R$, then $Z_R(P)$ is a prime $z_R$-filter on $X$.

(d) If $P$ is a prime $z_R$-filter on $X$, then $Z_R^{-1}(P)$ is a prime ideal in $R$.

(e) Every $z_R$-ideal of $R$ is absolutely convex.

Thus, if $I$ is an absolutely convex ideal of $R$, then $R/I$ is a lattice ring.

(f) $I(f) \geq 0$ if and only if $f \geq 0$ on some $Z \in Z_R(I)$.

(g) Suppose that there exists $Z \in Z_R(I)$ such that $f(x) > 0$, for every $x \in Z$, then $I(f) > 0$. The converse is true whenever $I$ is a maximal $g$-ideal.

4. $z_R^o$-Ideals and $z^o$-Ideals in Subrings

In this section we generalize the concept of $z^o$-ideals of $C(X)$ to the subrings of $\mathbb{R}^X$ and introduce $z_R^o$-ideal. Coincidence of $z_R^o$-ideals with $z^o$-ideals of $R$ is discussed. Note that, for each element $f$ of a commutative rings $S$, we use $P_f(S)$ to denote the intersection of all the minimal prime ideals in $S$ containing $f$.

**Definition 4.1.** An ideal $I$ of a subring $R$ of $\mathbb{R}^X$ is called a $z_R^o$-ideal, if $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$, where $f \in I$ and $g \in R$, implies $g \in I$.

The following statement investigates some characterizations of $z_R^o$-ideals in subrings.

**Theorem 4.2.** Let $R$ be a subring of $\mathbb{R}^X$ and $I$ be an ideal in $R$. The following statements are equivalent.


(a) $I$ is a $z^0_R$-ideal.
(b) Whenever $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$ where $f \in I$ and $g \in R$, then $g \in I$.
(c) $R \cap P_f(C) \subseteq I$ for each $f \in I$.
(d) Whenever $P_g(C) \cap R \subseteq P_f(C) \cap R$, where $f \in I$ and $g \in R$, then $g \in I$.

Proof. (a$\Rightarrow$b). First note that by [3, Lemma 2.1] we have $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$ if and only if $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ for each $f, g \in C(X)$. Now, let $I$ be a $z^0_R$-ideal in $R$ and $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$ where $f \in I$ and $g \in R$. Thus, by our hypothesis, we have $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ which implies that $g \in I$.

(b$\Rightarrow$c). By [3, Proposition 2.3], we have $P_f(C) = \{ g \in C(X) : \text{Ann}_C(f) \subseteq \text{Ann}_C(g) \}$. Thus the proof is evident.

(c$\Rightarrow$d). Let $P_g(C) \cap R \subseteq P_f(C) \cap R$, where $f \in I$ and $g \in R$. As $f \in I$, by our hypothesis, $P_f(C) \cap R \subseteq I$ and thus $P_g(C) \cap R \subseteq I$ which implies that $g \in I$.

(d$\Rightarrow$a). Let $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ where $f \in I$ and $g \in R$. Therefore, by [3, Lemma 2.1], we have $P_f(C) \subseteq P_g(C)$ and hence $P_f(C) \cap R \subseteq P_g(C) \cap R$. Thus we are done by our hypothesis. □

Lemma 4.3. Let $R$ be a subring of $\mathbb{R}^X$, then for each $f \in R$ we have $P_f(C) \subseteq P_f(R)$.

Proof. Let $g \in P_f(C)$. By [3, Proposition 2.3], we have $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$. Therefore, $\text{Ann}_R(f) = \text{Ann}_C(f) \cap R \subseteq \text{Ann}_C(g) \cap R = \text{Ann}_R(g)$. Thus, by [2, Proposition 1.5] we are done. □

Theorem 4.4. Let $R$ be a subring of $\mathbb{R}^X$. Then every $z^0_R$-ideal in $R$ is a $z^0$-ideal if and only if $P_f(R) = P_f(C)$ for each $f \in R$.

Proof. ($\Rightarrow$). Assume on the contrary that there exists some $f \in R$ such that $P_f(R) \neq P_f(C)$. Thus, using Theorem 4.2 we have $P_f(C) \subseteq P_f(R)$. Again by Theorem 4.2, $P_f(C) \cap R$ is a $z^0_R$-ideal in $R$. Also, it is clear that this ideal is not a $z^0$-ideal, since, $P_f(R) \not\subseteq P_f(C) \cap R$.

($\Leftarrow$). Let $I$ be a $z^0_R$-ideal in $R$ and $f \in I$. By Theorem 4.2, $P_f(C) \cap R \subseteq I$. Thus, by our hypothesis, $P_f(R) \subseteq I$ which means that $I$ is a $z^0$-ideal in $R$. □

From Theorem 4.2 it follows that every $z^0$-ideal in a subring $R$ is a $z^0_R$-ideal. However, the converse of this fact does not hold, in general. The following example gives an example of a subring $R$ which has a $z^0_R$-ideal that is not a $z^0$-ideal.

Example 4.5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}$. It is clear that $f \in C(\mathbb{R})$. Now, let $R = \{ \sum_{i=0}^{n} r_i f^i : r_i \in \mathbb{R}, n = 0, 1, \ldots \}$. It is easy to see that $P_f(R) = R$, however, $P_f(C) \cap R \neq R$. Also, by Theorem 4.2, $P_f(C) \cap R$ is $z^0_R$-ideal and it is clear that this ideal is not a $z^0$-ideal.
The next theorem gives a sufficient conditions on $X$ in order that $z_R^2$-ideals in a subring $R$ coincide with $z^0$-ideals of $R$.

**Theorem 4.6.** Let $R$ be a subring of $\mathbb{R}^X$ and $X$ be equipped with the $Coz(R)$-topology. Then an ideal $I$ in $R$ is a $z^0$-ideal if and only if it is a $z_R^2$-ideal.

**Proof.** Let $I$ be a $z_R^2$-ideal in $R$ and $f \in I$. As $X$ is equipped with the $Coz(R)$-topology, we have $g \in Ann_R(f)$ if and only if $Coz(g) \subseteq int_X Z(f)$ for each $f, g \in R$. Therefore, $P_I(R) = Ann_R Ann_R(f) = \{g \in R : Coz(g) \cap int_X Z(f) = \emptyset\} = \{g \in R : Ann_R(f) \subseteq Ann_R(g)\}$. Hence, $P_I(R) \subseteq I$ which means that $I$ is a $z^0$-ideal in $R$. This completes the proof, since, as former stated, every $z^0$-ideal in $R$ is a $z_R^2$-ideal.

Note that the condition that $X$ is equipped with the $Coz(R)$-topology is a sufficient condition for coincidence of $z_R^2$-ideals with $z^0$-ideals in a given subring $R$. The next example shows that this condition is not necessary.

**Example 4.7.** Let $X = \mathbb{R} \setminus \{0\}$ with the topology inherits from the usual topology on $\mathbb{R}$. Also, let $f : X \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$. It is clear that $f \in C(X)$ and $f^2 = f$. Now, set $R = \{r + sf : r, s \in \mathbb{R}\}$. It is clear that $R$ is a subring of $C(X)$. Also, by a routine reasoning, one can proves that the only ideals of $R$ are the ideals $(0), (f), (1 - f)$ and $R$. Moreover, the minimal prime ideals of $R$ are only the ideals $(f)$ and $(1 - f)$. These imply that every $z_R^2$-ideal is a $z^0$-ideal in $R$. However, clearly, $X$ is not equipped with the $Coz(R)$-topology.

It follows from Theorem 4.6 that for an intermediate subalgebra $A(X)$ of $C(X)$, $z_A^2$-ideals coincide with $z^0$-ideals of $A(X)$. However, the same statement does not true for $z_A$-ideals and $z$-ideals in $A(X)$, in general, see [6, Theorem 2.2]. Moreover, Theorem 3.7 together with Theorem 4.6 imply that in the subalgebras of $C(X)$ which are of the form $\mathbb{R} + I$, where $I$ is a free ideal in $C(X)$, $z_{\mathbb{R}+I}$-ideals coincide with $z$-ideals of $\mathbb{R} + I$ and $z_{\mathbb{R}+I}^2$-ideals coincide with $z^2$-ideals, too. Note that whenever $I$ is a free ideal in $C(X)$, then $\mathbb{R} + I$ determines the topology of $X$.

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**References**