Domination and Signed Domination Number of Cayley Graphs

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Abstract. In this paper, we investigate domination number as well as signed domination numbers of Cayley$(G:S)$ for all cyclic group $G$ of order $n$, where $n \in \{p^m, pq\}$ and $S = \{k < n : \gcd(k, n) = 1\}$. We also introduce some families of connected regular graphs $\Gamma$ such that $\gamma_S(\Gamma) \in \{2, 3, 4, 5\}$.

Keywords: Cayley graph, Cyclic group, Domination number, Signed domination number.

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1. Introduction

By a graph $\Gamma$ we mean a simple graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. A graph is said to be connected if each pair of vertices are joined by a walk. The number of edges of the shortest walk joining $v_i$ and $v_j$ is called the distance between $v_i$ and $v_j$ and denoted by $d(v_i, v_j)$. A graph $\Gamma$ is said to be regular of degree $k$ or, $k$-regular if every vertex has degree $k$. A subset $P$ of vertices of $\Gamma$ is a $k$-packing if $d(x, y) > k$ for all pairs of distinct vertices $x$ and $y$ of $P$ [9].

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Let $G$ be a non-trivial group, $S$ be an inverse closed subset of $G$ which does not contain the identity element of $G$, i.e. $S = S^{-1} = \{s^{-1} : s \in S\}$. The Cayley graph of $G$ denoted by $\text{Cay}(G : S)$, is a graph with vertex set $G$ and two vertices $a$ and $b$ are adjacent if and only if $ab^{-1} \in S$. The Cayley graph $\text{Cay}(G : S)$ is connected if and only if $S$ generates $G$.

A set $D \subseteq V$ of vertices in a graph $\Gamma$ is a dominating set if every vertex $v \in V$ is an element of $D$ or adjacent to an element of $D$. The domination number $\gamma(\Gamma)$ of a graph $\Gamma$ is the minimum cardinality of a dominating set of $\Gamma$.

For a vertex $v \in V(\Gamma)$, the closed neighborhood $N[v]$ of $v$ is the set consisting $v$ and all of its neighbors. For a function $f : V(\Gamma) \to \{-1, 1\}$ and a subset $W$ of $V$ we define $f(W) = \sum_{u \in W} f(u)$. A signed dominating function of $\Gamma$ is a function $f : V(\Gamma) \to \{-1, 1\}$ such that $f(N[v]) > 0$ for all $v \in V(\Gamma)$. The weight of a function $f$ is $\omega(f) = \sum_{v \in V} f(v)$. The signed domination number $\gamma_s(\Gamma)$ is the minimum weight of a signed dominating function of $\Gamma$. A signed dominating function of weight $\gamma_s(\Gamma)$ is called a $\gamma_s(\Gamma)$–function. We denote $f(N[v])$ by $[v]$. Also for $A \subseteq V(\Gamma)$ and signed dominating function $f$, set $\{v \in A : f(v) = -1\}$ is denoted by $A_f^−$.

Finding some kinds of domination numbers of graphs is certainly one of the most important properties in any graph. (See for instance [2, 3, 5, 6, 11, 13])

These motivated us to consider on domination and signed domination number of Cayley graphs of cyclic group of orders $p^n, pq$, where $p$ and $q$ are prime numbers.

2. Cayley Graphs of Order $p^n$

In this section $p$ is a prime number and $B(1, n) = \{k < n : \gcd(k, n) = 1\}$.

**Lemma 2.1.** Let $G$ be a group and $H$ be a proper subgroup of $G$ such that $[G : H] = t$. If $S = G \setminus H$, then $\text{Cay}(G : S)$ is a complete $t$-partite graph.

**Proof.** One can see $G = \langle S \rangle$ and $e \notin S = S^{-1}$. Let $a \in G$. If $x, y \in Ha$, then $x = h_1a, y = h_2a$. Since $xy^{-1} \in H$, $xy \notin E(\text{Cay}(G : S))$. So induced subgraph on every coset of $H$ is empty. Let $Ha$ and $Hb$ two disjoint cosets of $H$ and $x \in Ha, y \in Hb$. Hence, $xy^{-1} \in S$. So $xy \in E(\text{Cay}(G : S))$. Therefore, $\text{Cay}(G : S) = K_{|H|, |H|, \ldots, |H|}$.

**Lemma 2.2.** Let $G$ be a group of order $n$ and $G = \langle S \rangle$, where $S = S^{-1}$ and $0 \notin S$. Then $\gamma(\text{Cay}(G : S)) = 1$ if and only if $S = G \setminus \{0\}$.

**Proof.** The proof is straightforward.
Theorem 2.3. \[13\] Let \(K_{a,b}\) be a complete bipartite graph with \(b \leq a\). Then
\[
\gamma_s(K_{a,b}) = \begin{cases} 
  a + 1 & \text{if } b = 1, \\
  b & \text{if } 2 \leq b \leq 3 \text{ and } a \text{ is even}, \\
  b + 1 & \text{if } 2 \leq b \leq 3 \text{ and } a \text{ is odd}, \\
  4 & \text{if } b \geq 4 \text{ and } b \text{ is even}, \\
  6 & \text{if } b \geq 4 \text{ and } b \text{ is odd}, \\
  5 & \text{if } b \geq 4 \text{ and } a, b \text{ have different parity.}
\end{cases}
\]

Theorem 2.4. Let \(\mathbb{Z}_{2^n} = \langle S \rangle\) and \(S = B(1, 2^n)\). Then
i. \(\text{Cay}(\mathbb{Z}_{2^n} : S) = K_{2^{n-1}, 2^{n-1}}\)
ii. \(\gamma(\text{Cay}(\mathbb{Z}_{2^n} : S)) = 2\).
iii. \(\gamma_S(\text{Cay}(\mathbb{Z}_{2^n} : S)) = \begin{cases} 
  2 & \text{if } n = 1, 2, \\
  4 & \text{if } n \geq 3.
\end{cases}\)

Proof. i. Let \(H = \mathbb{Z}_{2^n} \setminus S\). Then \(H = \{i : 2 \mid i\}\). It is not hard to see that \(H\) is a subgroup of \(\mathbb{Z}_{2^n}\) and \([\mathbb{Z}_{2^n} : H] = 2\). Hence, by Lemma 2.1, \(\text{Cay}(\mathbb{Z}_{2^n} : S) = K_{2^{n-1}, 2^{n-1}}\).
ii. By part i. \(\text{Cay}(\mathbb{Z}_{2^n} : S)\) is a complete bipartite graph. So \(\gamma(\text{Cay}(\mathbb{Z}_{2^n} : S)) = 2\).
iii. The proof is straightforward by Theorem 2.3.

\(\square\)

Corollary 2.5. For any integer \(n > 2\), there is a \(2^{n-1}\)-regular graph \(\Gamma\) with \(2^n\) vertices such that \(\gamma_s(\Gamma) = 4\).

Theorem 2.6. Let \(\mathbb{Z}_{p^n} = \langle S \rangle\) (\(p\) odd prime) and \(S = B(1, p^n)\). Then following statements hold:

i. \(\text{Cay}(\mathbb{Z}_{p^n} : S)\) is a complete \(p\)-partite graph.

ii. \(\gamma(\text{Cay}(\mathbb{Z}_{p^n} : S)) = 2\).

iii. \(\gamma_s(\text{Cay}(\mathbb{Z}_{p^n} : S)) = 3\).

Proof. i. Let \(H = \mathbb{Z}_{p^n} \setminus S\). Then \(H = \{i : p \mid i\}\). \(H\) is a subgroup of \(\mathbb{Z}_{p^n}\) and \(|H| = p^n - \Phi(p^n) = p^{n-1}\). So \([\mathbb{Z}_{p^n} : H] = p\). Hence, by Lemma 2.1, \(\text{Cay}(\mathbb{Z}_{p^n} : S)\) is a complete \(p\)-partite graph of size \(p^{n-1}\).

ii. Since \(\text{Cay}(\mathbb{Z}_{p^n} : S)\) is a complete \(p\)-partite graph, \(D = \{a, b\}\) is a minimal dominating set where \(a, b\) are not in the same partition.

iii. Let \(\Gamma = \text{Cay}(\mathbb{Z}_{p^n} : S)\). Let \(V(\Gamma) = \bigcup_{i=1}^{p} A_i\) where \(A_i = \{v_{ij} : 1 \leq j \leq p^{n-1}\}\). Define \(f : V(\Gamma) \to \{-1, 1\}\)
\[
f(v_{ij}) = \begin{cases} 
  -1 & \text{if } 1 \leq i \leq \left[ \frac{p}{2} \right] \text{ and } 1 \leq j \leq \left[ \frac{p^{n-1}}{2} \right], \\
  -1 & \text{if } \left[ \frac{p}{2} \right] \leq i \leq p \text{ and } 1 \leq j \leq \left[ \frac{p^{n-1}}{2} \right], \\
  1 & \text{otherwise.}
\end{cases}
\]
Let \( v \in \bigcup_{i=1}^{\lfloor \frac{p}{2} \rfloor - 1} A_i \). So \(|\mathcal{N}(v) \cap V_j^-| = \frac{1}{2}(p^n - p^{n-1} - 4)\). So \(f[v] = f(v) + 4 \geq 3\). If \( v \in \bigcup_{i=1}^{p} A_i \), then \(|\mathcal{N}(v) \cap V_j^-| = \frac{1}{2}(p^n - p^{n-1} - 2)\).

So \(f[v] = f(v) + 2 \geq 1\). Hence, \( f \) is a signed dominating function. Since \( |V_j^-| = \frac{1}{2}(p^n - 3)\), \( \omega(f) = 3\). So \( \gamma_S(\Gamma) \leq 3\). On the contrary, suppose \( \gamma_S(\Gamma) < 3\). So there is a \( \gamma_S \)-function \( g \) such that \( \omega(g) < 3\). So \( |V_j^-| > \frac{1}{2}(p^n - 3)\). Let \( |V_j^-| = \frac{1}{2}(p^n - 1)\). If \( A_i \cap V_j^- = \emptyset \) for some \( 1 \leq i \leq p \), then \( g[v] = 1 - p^{n-1} \) for every \( v \in A_i \). Hence, \( A_i \cap V_j^- \neq \emptyset \) for every \( 1 \leq i \leq p \). If \( |A_i \cap V_j^-| \geq \lceil \frac{p^{n-1}}{2} \rceil \) for every \( 1 \leq i \leq p \), then \( |V_j^-| \geq \frac{1}{2}(p^n + p)\). This is impossible. So there is \( j \in \{1, 2, \ldots, p\} \) such that \( |A_j \cap V_j^-| \leq \lceil \frac{p^{n-1}}{2} \rceil \). Let \( u \in A_j \cap V_j^-\). So \( g[u] = \deg(u) + 1 - 2|\mathcal{N}(u) \cap V_j^-| < 0\). This is contradiction. Therefore \( \gamma_S(\Gamma) = 3\).

\[ \square \]

**Corollary 2.7.** For every integer \( n \), there is a \((p^n - p^{n-1})\)-regular graph \( \Gamma \) with \( p^n \) vertices such that \( \gamma_S(\Gamma) = 3\).

3. \textbf{Cayley Graphs of Order} \( pq \)

In this section \( p \) and \( q \) are distinct prime numbers where \( p < q \). Let \( B(1, pq) \) be a generator of \( \mathbb{Z}_{pq} \). For \( 1 \leq i \leq p \) and \( 1 \leq j \leq q \), set

\[ A_i = \{i + kp: 0 \leq k \leq q - 1\} \]

and

\[ B_j = \{j + k'q: 0 \leq k' \leq p - 1\} \]

With these notations in mind we will prove the following results.

**Lemma 3.1.** Let \( \mathbb{Z}_{pq} = \langle S \rangle \) and \( S = B(1, pq) \). Then following statements hold.

i. \( V(\text{Cay}(\mathbb{Z}_{pq}: S)) = \bigcup_{i=1}^{p} A_i \) and \( \text{Cay}(\mathbb{Z}_{pq}: S) \) is a \( p \)-partite graph.

ii. \( V(\text{Cay}(\mathbb{Z}_{pq}: S)) = \bigcup_{j=1}^{q} B_j \) and \( \text{Cay}(\mathbb{Z}_{pq}: S) \) is a \( q \)-partite graph.

iii. Let \( 1 \leq i \leq p \). For any \( x \in A_i \) there is some \( 1 \leq j \leq q \) such that \( x \in B_j \).

iv. \(|A_i \cap B_j| = 1\) for every \( i, j \).

**Proof.**

i. Let \( s \in V(\text{Cay}(\mathbb{Z}_{pq}: S)) \). If \( p \mid s \), then \( s \in A_p \). Otherwise, \( s \in A_i \) where \( s = kp + i \) for some \( 1 \leq k \leq (p - 1) \). Thus \( V(\text{Cay}(\mathbb{Z}_{pq}: S)) = \bigcup_{i=1}^{p} A_i \). Since \( 1 \leq i \neq j \leq p \), \( A_i \cap A_j = \emptyset \). We show that the
induced subgraph on $A_i$ is empty. Let $l + t \in E(Cay(Z_{pq} : S))$. If $l, t \in A_i$ for some $1 \leq s \leq p$, then $l = s + kp, t = s + k'p$. So $p \mid (l - t)$. This is impossible.

ii. The proof is likewise part i.

iii. Let $1 \leq i \leq p$ and let $x \in A_i$. If $x \leq q$, then $x \in B_x$. If not, $x = i + kp > q$ such that $1 \leq k \leq q - 1$. Hence, $x \equiv t \pmod{q}$ where $1 \leq t \leq q$, and so $x \in B_t$.

iv. By Case iii and since $|A_i| = q$ and also for every $j \neq j'$, $B_j \cap B_{j'} = \emptyset$, the result reaches.

$\square$

Theorem 3.3. [6] For any graph $\Gamma$, \[ \frac{n}{\Delta(\Gamma)} \leq \gamma(\Gamma) \leq n - \Delta(\Gamma) \] where $\Delta(\Gamma)$ is the maximum degree of $\Gamma$.

Theorem 3.3. Let $Z_{pq} = \langle S \rangle$ and $S = B(1, pq)$. Then the following is hold.

\[ \gamma(Cay(Z_{pq} : S)) = \begin{cases} 2 & p = 2; \\ 3 & p > 2. \end{cases} \]

Proof. Let $p = 2$. By Lemma 3.1, $D = \{i, i + q\}$ is a dominating set. Since $Cay(Z_{pq} : S)$ is a $(q - 1)$-regular graph, by Theorem 3.2, $\gamma(Cay(Z_{pq} : S)) \geq 2$. Thus $\gamma(Cay(Z_{pq} : S)) = 2$.

Let $p > 2$. We define $D = \{1, 2, s\}$ where $s \in A_1 \setminus N(2)$. Since $1, 2$ are adjacent, $N(1) \cup N(2) = V(Cay(Z_{pq} : S)) \setminus D$. Thus $D$ is a dominating set. As a consequence, $\gamma(Cay(Z_{pq} : S)) \leq 2$. It is enough to show that $\gamma(Cay(Z_{pq} : S)) \neq 2$. Let $D' = \{x, y\}$. We show that $D'$ is not a dominating set. If $x, y \in A_1$ for some $1 \leq i \leq p$, then for every $z \in A_1 \setminus D', z \notin N(D')$. If not, $x \in A_i$ and $y \in A_j$ for some $1 \leq i \neq j \leq p$. If $x, y$ are adjacent, then there is $x' \in A_i \setminus \{x\}$ such that $x' \notin N(y)$. Thus $D'$ is not dominating set. If $x$ and $y$ are not adjacent, then there is $z \in A_2, l \neq i, j$, such that the induced subgraph on $\{x, y, z\}$ is empty. Hence, $D'$ is not a dominating set and the proof is completed.

$\square$

Theorem 3.4. Let $Z_{pq} = \langle S \rangle$ where $p \in \{2, 3, 5\}$ and $S = B(1, pq)$. Then

\[ \gamma_s(Cay(Z_{pq} : S)) = p. \]

Proof. Let $A = \{1, 1 + p, \ldots, 1 + (\frac{q}{2} - 1)p\}$ and $B = \{i + tq : i \in A \text{ and } 1 \leq t \leq p - 1\}$. We define $f : V(Cay(Z_{pq} : S)) \to \{-1, 1\}$ such that

\[ f(x) = \begin{cases} -1 & x \in A \cup B, \\ 1 & \text{otherwise}. \end{cases} \]

Let $v \in V(Cay(Z_{pq} : S))$. If $f(v) = -1$, then

\[ f[v] = -1 + (p - 1)(q - 1) - 2 \left(\left(\frac{q}{2}\right) - 1\right) (p - 1) = 2p - 3. \]
Otherwise,
\[ f[v] = 1 + (p - 1)(q - 1) - 2 \left\lfloor \frac{q}{2} \right\rfloor (p - 1) = 1. \]
Hence, \( f \) is a dominating function. Also
\[ \omega(f) = pq - 2(|A| + |B|) = pq - 2 \left( \left\lfloor \frac{q}{2} \right\rfloor + (p - 1) \left\lceil \frac{q}{2} \right\rceil \right) = p. \]

It is enough to show that \( f \) has the minimal wait. Let, to the contrary, \( g \) be a dominating function and \( \omega(g) < \omega(f) \). So \( |V_g^-| > |V_f^-| \). Without lose of generality, suppose that \( |V_g^-| = p\left\lfloor \frac{q}{2} \right\rfloor + 1 \). Let \( A_i^- = A_i \cap V_g^- \), \( A_i^+ = A_i \setminus A_i^- \) and \( B_j^- = B_j \cap V_g^- \). We will reach the contradiction by three steps.

Step 1. For every \( 1 \leq i \leq p, A_i^- \neq \emptyset \).
On the contrary, let \( A_i^- = \emptyset \) for some \( 1 \leq s \leq p \). Let \( u \in A_s \). Then by Lemma 3.1, \( u \in A_s \cap B_t \) for some \( 1 \leq t \leq q \). So
\[ g[u] = (p - 1)(q - 1) + 1 - 2(|V_g^-| - |B_t^+|) \geq 1. \]
Thus \( |B_t^+| \geq \left\lceil \frac{q}{2} \right\rceil \). Hence, \( |V_g^-| \geq |A_s|\left\lceil \frac{q}{2} \right\rceil \). This implies \( q + (q-p)\left\lceil \frac{q}{2} \right\rceil < 1 \). This is a contradiction. Hence, \( A_i^- \neq \emptyset \).
Similar argument applies for \( B_j \). Therefore, \( B_j^- \neq \emptyset \) for every \( 1 \leq j \leq q \).

Step 2. For every \( 1 \leq i \leq p, |A_i^-| \geq \left\lceil \frac{q}{2} \right\rceil \).
On the contrary, Let \( |A_i^-| < \left\lceil \frac{q}{2} \right\rceil \) for some \( 1 \leq l \leq p \). Without lose of generality suppose that \( |A_i^-| = \left\lfloor \frac{q}{2} \right\rfloor - 1 \). Let \( v \in A_i \). By Lemma 3.1, \( v \in A_i \cap B_k \) for some \( 1 \leq k \leq q \). If \( g(v) = -1 \), then \( g[v] = (p - 1)(q - 1) - 1 - 2(|V_g^-| - |A_i^-| - |B_k^-| + 2) \geq 1 \). Then \( |B_k^- \setminus \{v\}| \geq 4 \). If \( g(v) = 1 \), then \( |B_k^- \setminus \{v\}| \geq 2 \). Hence, \( |V_g^-| \geq 4|A_i^-| + |A_i^+| + 2|A_i^-| \).
As a consequence \( p > 8 \). This is impossible.
Therefore, for every \( 1 \leq i \leq p, |A_i^-| \geq \left\lfloor \frac{q}{2} \right\rceil \) and since \( |V_g^-| = p\left\lfloor \frac{q}{2} \right\rceil + 1 \), we may suppose that \( |A_i^-| = \left\lfloor \frac{q}{2} \right\rceil \) and \( |A_i^-| = \left\lceil \frac{q}{2} \right\rceil \) for \( 2 \leq i \leq p \).

Step 3. For every \( 1 \leq j \leq q, |B_j^-| \geq \left\lceil \frac{q}{2} \right\rceil \).
On the contrary, let \( |B_k^-| < \left\lceil \frac{q}{2} \right\rceil \) for some \( 1 \leq h \leq q \). Suppose that \( |B_k^-| = \left\lfloor \frac{q}{2} \right\rceil \). By Lemma 3.1, \( B_k \cap A_i \neq \emptyset \) for any \( 1 \leq i \leq p \). Let \( z \in B_k \cap A_i \).
\[ g[z] = -1 + (p - 1)(q - 1) - 2(|V_g^-| - |A_i^-| - |B_k^-| + 2) \leq -1 + (p - 1)(q - 1) - 2 \left(p \left\lfloor \frac{q}{2} \right\rceil + 1 - \left\lfloor \frac{q}{2} \right\rceil - \left\lceil \frac{p}{2} \right\rceil + 2 \right) \leq p - 6 \]

Since \( p \in \{2, 3, 5\} \), \( g[z] \leq -1 \). This is a contradiction.
By Step 3, \( |V_g^-| \geq q\left\lfloor \frac{p}{2} \right\rceil \). Hence, \( p\left\lfloor \frac{p}{2} \right\rceil + 1 \geq q\left\lceil \frac{p}{2} \right\rceil \). So \( p + q \leq 2 \). This is impossible. Therefore \( \gamma_s(Cay(G : S)) = \omega(f) = p. \)

Theorem 3.5. Let \( \mathbb{Z}_{pq} = \langle S \rangle \) where \( p \geq 7 \) and \( S = B(1, pq) \). Then
\[ \gamma_s(Cay(\mathbb{Z}_{pq} : S)) = 5. \]
Proof. We define $f: V(Cay(Z_{pq}, S)) \rightarrow \{-1, 1\}$ such that $f(i) = -1$ if and only if $i \in \{1, 2, \ldots, \frac{pq-5}{2}\}$. It is easily seen that $\lfloor \frac{q}{2} \rfloor \leq |A^-_i| \leq \lfloor \frac{q}{2} \rfloor$ for every $1 \leq i \leq p$. Also $\lfloor \frac{q}{2} \rfloor \leq |B^-_j| \leq \lfloor \frac{q}{2} \rfloor$ for any $1 \leq j \leq q$. Let $v \in A_i \cap B_j$, such that $1 \leq t \leq p$ and $1 \leq s \leq q$. In the worst situation, $|A^-_i| = \lfloor \frac{q}{2} \rfloor$ and $|B^-_j| = \lfloor \frac{q}{2} \rfloor$. In this case $1 \leq f[v] \leq 5$. Hence, $f$ is a signed dominating function. Also $\omega(f) = pq - 2|V_f^-| = 5$. Thus $\gamma_S(Cay(Z_{pq}, S)) \leq 5$. What is left is to show that if $g$ is a $\gamma_S$-function, then $\omega(g) \geq 5$. On the contrary, suppose that $g$ be a $\gamma_S$-function and $\omega(g) < \omega(f)$. Hence, $|V_g^-| < |V_f^-|$. There is no loss of generality in assuming $|V_g^-| = \frac{pq-3}{2}$. Let $A^-_i = A_i \cap V_g^-$ and $B^-_j = B_j \cap V_g^-$. In order to reach the contradiction we use two following steps:

Step 1. $A^-_i \neq \emptyset$ for every $1 \leq i \leq p$.

On the contrary, suppose that for some $1 \leq m \leq p$, $A^-_m = \emptyset$. Let $w \in A_m$. So there is $1 \leq t \leq q$ such that $w \in A_m \cap B_t$. Hence, $g[w] = (p-1)(q-1) - 2|V_g^-| - |B^-_t| + 2 \geq 1$. Thus $|B^-_t| \geq \frac{pq-3}{2} - 2$. So $|V_g^-| \geq \frac{pq-3}{2}$. Hence, $pq - 3 \geq g(pq - 4)$. This makes a contradiction.

By similar argument we have $B^-_j \neq \emptyset$ for every $1 \leq j \leq q$.

Step 2. For every $1 \leq i \leq p$, $|A^-_i| \geq \lfloor \frac{q}{2} \rfloor$.

On the contrary, let $|A^-_i| = \lfloor \frac{q}{2} \rfloor - 1$. Let $v \in A_i$. There is $1 \leq t \leq q$ such that $v \in A_i \cap B_t$. If $g(v) = -1$, then $g[v] = (p-1)(q-1) + 1 - 2|V_g^-| - |A^-_i| - |B^-_t| + 2 \geq 1$. Hence, $|B^-_t \setminus \{v\}| \geq \lfloor \frac{q}{2} \rfloor$. If $g(v) = 1$, then $|B^-_t| \geq \lfloor \frac{q}{2} \rfloor$. Therefore, $|V_g^-| \geq |A^-_i|(|\lfloor \frac{q}{2} \rfloor + 1) + |A^-_i| |\lfloor \frac{q}{2} \rfloor|$. This implies that $q \leq 3$. This is a contradiction.

Likewise Step 2, $|B^-_j| \geq \lfloor \frac{q}{2} \rfloor$ for every $1 \leq j \leq q$. Since $|V_g^-| = \frac{pq-3}{2}$, there is $1 \leq k \leq p$ such that $|A^-_k| = \lfloor \frac{q}{2} \rfloor$. On the other hand, suppose that for $1 \leq t \leq q$, $|B^-_t| = \lfloor \frac{q}{2} \rfloor$. Let $u \in A_k \cap B^-_t$. If $s \in \{l_1, \cdots, l_t\}$, then $g[u] = -1 + (p-1)(q-1) - 2|V_g^-| - |A^-_k| - |B^-_s| + 2$

\[= -1 + (p-1)(q-1) - 2 \left( \frac{pq-3}{2} - \left\lfloor \frac{q}{2} \right\rfloor \right) + 2 \left\lfloor \frac{q}{2} \right\rfloor \]

\[= -3. \]

This is a contradiction by $g$ is a signed dominating function. Hence, $s$ is not in $\{l_1, \cdots, l_t\}$. Hence, $|A^-_k| = \lfloor \frac{q}{2} \rfloor$, $q - t \geq \left\lfloor \frac{q}{2} \right\rfloor$ and so $t \leq \left\lfloor \frac{q}{2} \right\rfloor$. As a consequence, $|V_g^-| \geq t\left(\frac{p}{2}\right) + (q-t)\left(\frac{p}{2}\right) \geq \left\lfloor \frac{q}{2} \right\rfloor \left(\frac{p}{2}\right) + \left\lfloor \frac{q}{2} \right\rfloor \left(\frac{p}{2}\right)$.

Since $|V_g^-| = \frac{pq-3}{2}$, this makes a contradiction. Therefore, $\gamma_S(Cay(Z_{pq}, S)) = 5$.

\[\square\]

**Corollary 3.6.** For any $k$-regular graph $\Gamma$ on $n$ vertices $\gamma_S(\Gamma) \geq \frac{n}{k+1}$. Hence, $\gamma_S(\Gamma) \geq 1$. It is easy to check that $\gamma_S(\Gamma) = 1$ if and only if $\Gamma$ is a complete
graph and $n$ is odd. Furthermore, for any prime numbers $p < q$, there is a $(p-1)(q-1)$-regular graph $\Gamma$ with $pq$ vertices such that $\gamma_S(\Gamma) \in \{2, 3, 5\}$.

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