Domination and Signed Domination Number of Cayley Graphs

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Abstract. In this paper, we investigate domination number as well as signed domination numbers of Cayley($G : S$) for all cyclic group $G$ of order $n$, where $n \in \{p^m, pq\}$ and $S = \{k < n : gcd(k, n) = 1\}$. We also introduce some families of connected regular graphs $\Gamma$ such that $\gamma_S(\Gamma) \in \{2, 3, 4, 5\}$.

Keywords: Cayley graph, Cyclic group, Domination number, Signed domination number.

2000 Mathematics subject classification: 05C69, 05C25

1. Introduction

By a graph $\Gamma$ we mean a simple graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. A graph is said to be connected if each pair of vertices are joined by a walk. The number of edges of the shortest walk joining $v_i$ and $v_j$ is called the distance between $v_i$ and $v_j$ and denoted by $d(v_i, v_j)$. A graph $\Gamma$ is said to be regular of degree $k$ or, $k$-regular if every vertex has degree $k$. A subset $P$ of vertices of $\Gamma$ is a $k$-packing if $d(x, y) > k$ for all pairs of distinct vertices $x$ and $y$ of $P$ [9].

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Received 20 April 2016; Accepted 14 January 2017
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Let $G$ be a non-trivial group, $S$ be an inverse closed subset of $G$ which does not contain the identity element of $G$, i.e. $S = S^{-1} = \{s^{-1} : s \in S\}$. The Cayley graph of $G$ denoted by $\text{Cay}(G : S)$, is a graph with vertex set $G$ and two vertices $a$ and $b$ are adjacent if and only if $ab^{-1} \in S$. The Cayley graph $\text{Cay}(G : S)$ is connected if and only if $S$ generates $G$.

A set $D \subseteq V$ of vertices in a graph $\Gamma$ is a dominating set if every vertex $v \in V$ is an element of $D$ or adjacent to an element of $D$. The domination number $\gamma(\Gamma)$ of a graph $\Gamma$ is the minimum cardinality of a dominating set of $\Gamma$.

For a vertex $v \in V(\Gamma)$, the closed neighborhood $N[v]$ of $v$ is the set consisting $v$ and all of its neighbors. For a function $f : V(\Gamma) \rightarrow \{-1, 1\}$ and a subset $W$ of $V$ we define $f(W) = \sum_{u \in W} f(u)$. A signed dominating function of $\Gamma$ is a function $f : V(\Gamma) \rightarrow \{-1, 1\}$ such that $f(N[v]) > 0$ for all $v \in V(\Gamma)$. The weight of a function $f$ is $\omega(f) = \sum_{v \in V} f(v)$. The signed domination number $\gamma_s(\Gamma)$ is the minimum weight of a signed dominating function of $\Gamma$. A signed dominating function of weight $\gamma_s(\Gamma)$ is called a $\gamma_s(\Gamma)$–function. We denote $f(N[v])$ by $f[v]$. Also for $A \subseteq V(\Gamma)$ and signed dominating function $f$, set $\{v \in A : f(v) = -1\}$ is denoted by $A_f^-$.

Finding some kinds of domination numbers of graphs is certainly one of the most important properties in any graph. (See for instance [2, 3, 5, 6, 11, 13])

These motivated us to consider on domination and signed domination number of Cayley graphs of cyclic group of orders $p^n$, $pq$, where $p$ and $q$ are prime numbers.

## 2. Cayley Graphs of Order $p^n$

In this section $p$ is a prime number and $B(1, n) = \{k < n : \gcd(k, n) = 1\}$.

**Lemma 2.1.** Let $G$ be a group and $H$ be a proper subgroup of $G$ such that $[G : H] = t$. If $S = G \setminus H$, then $\text{Cay}(G : S)$ is a complete $t$-partite graph.

**Proof.** One can see $G = \langle S \rangle$ and $e \notin S = S^{-1}$. Let $a \in G$. If $x, y \in Ha$, then $xy^{-1} \in H$. Since $xy^{-1} \in H$, $xy \notin E(\text{Cay}(G : S))$. So induced subgraph on every coset of $H$ is empty. Let $Ha$ and $Hb$ two disjoint cosets of $H$ and $x \in Ha, y \in Hb$. Hence, $xy^{-1} \in S$. So $xy \in E(\text{Cay}(G : S))$. Therefore, $\text{Cay}(G : S) = K_{|H|, |H|, \cdots, |H|}$. □

**Lemma 2.2.** Let $G$ be a group of order $n$ and $G = \langle S \rangle$, where $S = S^{-1}$ and $0 \notin S$. Then $\gamma(\text{Cay}(G : S)) = 1$ if and only if $S = G \setminus \{0\}$.

**Proof.** The proof is straightforward. □
Theorem 2.3. [13] Let $K_{a,b}$ be a complete bipartite graph with $b \leq a$. Then

$$\gamma_s(K_{a,b}) = \begin{cases} a + 1 & \text{if } b = 1, \\ b & \text{if } 2 \leq b \leq 3 \text{ and } a \text{ is even}, \\ b + 1 & \text{if } 2 < b \leq 3 \text{ and } a \text{ is odd}, \\ 4 & \text{if } b \geq 4 \text{ and } a, b \text{ are both even}, \\ 6 & \text{if } b \geq 4 \text{ and } a, b \text{ are both odd}, \\ 5 & \text{if } b \geq 4 \text{ and } a, b \text{ have different parity}. \end{cases}$$

Theorem 2.4. Let $\mathbb{Z}_{2^n} = \langle S \rangle$ and $S = B(1, 2^n)$. Then

i. $\text{Cay}(\mathbb{Z}_{2^n} : S) = K_{2^{n-1}, 2^{n-1}}$

ii. $\gamma(\text{Cay}(\mathbb{Z}_{2^n} : S)) = 2.$

iii. $\gamma_S(\text{Cay}(\mathbb{Z}_{2^n} : S)) = \begin{cases} 2 & \text{if } n = 1, 2, \\ 4 & \text{if } n \geq 3. \end{cases}$

Proof. i. Let $H = \mathbb{Z}_{2^n} \setminus S$. Then $H = \{i : 2 \mid i\}$. It is not hard to see that $H$ is a subgroup of $\mathbb{Z}_{2^n}$ and $[\mathbb{Z}_{2^n} : H] = 2$. Hence, by Lemma 2.1, $\text{Cay}(\mathbb{Z}_{2^n} : S) = K_{2^{n-1}, 2^{n-1}}$.

ii. By part i. $\text{Cay}(\mathbb{Z}_{2^n} : S)$ is a complete bipartite graph. So $\gamma(\text{Cay}(\mathbb{Z}_{2^n} : S)) = 2$.

iii. The proof is straightforward by Theorem 2.3. \qed

Corollary 2.5. For any integer $n > 2$, there is a $2^{n-1} - \text{regular graph } \Gamma$ with $2^n$ vertices such that $\gamma_s(\Gamma) = 4$.

Theorem 2.6. Let $\mathbb{Z}_{p^n} = \langle S \rangle$ ($p$ odd prime) and $S = B(1, p^n)$. Then following statements hold:

i. $\text{Cay}(\mathbb{Z}_{p^n} : S)$ is a complete $p$-partite graph.

ii. $\gamma(\text{Cay}(\mathbb{Z}_{p^n} : S)) = 2.$

iii. $\gamma_S(\text{Cay}(\mathbb{Z}_{p^n} : S)) = 3.$

Proof. i. Let $H = \mathbb{Z}_{p^n} \setminus S$. Then $H = \{i : p \mid i\}$. $H$ is a subgroup of $\mathbb{Z}_{p^n}$ and $|H| = p^n - \Phi(p^n) = p^{n-1}$. So $[\mathbb{Z}_{p^n} : H] = p$. Hence, by Lemma 2.1, $\text{Cay}(\mathbb{Z}_{p^n} : S)$ is a complete $p$-partite graph of size $p^{n-1}$.

ii. Since $\text{Cay}(\mathbb{Z}_{p^n} : S)$ is a complete $p$-partite graph, $D = \{a, b\}$ is a minimal dominating set where $a, b$ are not in the same partition.

iii. Let $\Gamma = \text{Cay}(\mathbb{Z}_{p^n} : S)$. Let $V(\Gamma) = \bigcup_{i=1}^{p} A_i$, where $A_i = \{v_{ij} : 1 \leq j \leq p^{n-1}\}$. Define $f : V(\Gamma) \to \{-1, 1\}$

$$f(v_{ij}) = \begin{cases} -1 & \text{if } 1 \leq i \leq \left\lfloor \frac{j}{2} \right\rfloor - 1 	ext{ and } 1 \leq j \leq \left\lfloor \frac{p^{n-1}}{2} \right\rfloor, \\ -1 & \text{if } \left\lfloor \frac{j}{2} \right\rfloor \leq i \leq p \text{ and } 1 \leq j \leq \left\lfloor \frac{p^{n-1}}{2} \right\rfloor, \\ 1 & \text{otherwise}. \end{cases}$$
Let \( v \in \bigcup_{i=1}^{\lfloor \frac{p}{2} \rfloor - 1} A_i \). So \( |N(v) \cap V_f| = \frac{1}{2}(p^n - p^{n-1} - 4) \). So \( f[v] = f(v) + 4 \geq 3 \). If \( v \in \bigcup_{i=1}^{p} A_i \), then \( |N(v) \cap V_f| = \frac{1}{2}(p^n - p^{n-1} - 2) \). So \( f[v] = f(v) + 2 \geq 1 \). Hence, \( f \) is a signed dominating function.

Since \( |V_f| = \frac{1}{2}(p^n - 3) \), \( \omega(f) = 3 \). So \( \gamma_S(\Gamma) \leq 3 \). On the contrary, suppose \( \gamma_S(\Gamma) < 3 \). So there is a \( \gamma_S \)-function \( g \) such that \( \gamma(g) < 3 \). So \( |V_g| > \frac{1}{2}(p^n - 3) \). Let \( |V_g^-| = \frac{1}{2}(p^n - 1) \). If \( A_i \cap V_g^- = \emptyset \) for some \( 1 \leq i \leq p \), then \( g[v] = 1 - p^{n-1} \) for every \( v \in A_i \). Hence, \( A_i \cap V_g^- \neq \emptyset \) for every \( 1 \leq i \leq p \). If \( |A_i \cap V_g^-| \geq \frac{|p^{n-1}|}{2} \) for every \( 1 \leq i \leq p \), then \( |V_g^-| \geq \frac{1}{2}(p^n + p) \). This is impossible. So there is \( j \in \{1, 2, \ldots, p\} \) such that \( |A_j \cap V_g^-| \leq \frac{|p^{n-1}|}{2} \). Let \( u \in A_j \cap V_g^- \). So \( g[u] = \deg(u) + 1 - 2|N(u) \cap V_g^-| < 0 \). This is contradiction. Therefore \( \gamma_S(\Gamma) = 3 \).

\( \square \)

**Corollary 2.7.** For every integer \( n \), there is a \((p^n - p^{n-1})\)-regular graph \( \Gamma \) with \( p^n \) vertices such that \( \gamma_S(\Gamma) = 3 \).

3. Cayley Graphs of Order \( pq \)

In this section \( p \) and \( q \) are distinct prime numbers where \( p < q \). Let \( B(1, pq) \) be a generator of \( \mathbb{Z}_{pq} \). For \( 1 \leq i \leq p \) and \( 1 \leq j \leq q \), set

\[
A_i = \{ i + kp : 0 \leq k \leq q - 1 \}
\]

and

\[
B_j = \{ j + k'q : 0 \leq k' \leq p - 1 \}.
\]

With these notations in mind we will prove the following results.

**Lemma 3.1.** Let \( \mathbb{Z}_{pq} = \langle S \rangle \) and \( S = B(1, pq) \). Then following statements hold.

i. \( V(\text{Cay}(\mathbb{Z}_{pq} : S)) = \bigcup_{i=1}^{p} A_i \) and \( \text{Cay}(\mathbb{Z}_{pq} : S) \) is a \( p \)-partite graph.

ii. \( V(\text{Cay}(\mathbb{Z}_{pq} : S)) = \bigcup_{j=1}^{q} B_j \) and \( \text{Cay}(\mathbb{Z}_{pq} : S) \) is a \( q \)-partite graph.

iii. Let \( 1 \leq i \leq p \). For any \( x \in A_i \) there is some \( 1 \leq j \leq q \) such that \( x \in B_j \).

iv. \( |A_i \cap B_j| = 1 \) for every \( i, j \).

**Proof.**

i. Let \( s \in V(\text{Cay}(\mathbb{Z}_{pq} : S)) \). If \( p \mid s \), then \( s \in A_p \). Otherwise, \( s \in A_i \) where \( s = kp + i \) for some \( 1 \leq k \leq (p - 1) \). Thus \( V(\text{Cay}(\mathbb{Z}_{pq} : S)) = \bigcup_{i=1}^{p} A_i \). Since \( 1 \leq i \neq j \leq p \), \( A_i \cap A_j = \emptyset \). We show that the
induced subgraph on $A_i$ is empty. Let $l + t \in E(Cay(Z_{pq} : S))$. If $l, t \in A_i$ for some $1 \leq s \leq p$, then $l = s + kp, t = s + k'p$. So $p \mid (l - t)$. This is impossible.

ii. The proof is likewise part i.

iii. Let $1 \leq i \leq p$ and let $x \in A_i$. If $x \leq q$, then $x \in B_x$. If not, $x = i + kp > q$ such that $1 \leq k \leq q - 1$. Hence, $x \equiv t \pmod{q}$ where $1 \leq t \leq q$, and so $x \in B_t$.

iv. By Case iii and since $|A_i| = q$ and also for every $j \neq j'$, $B_j \cap B_{j'} = \emptyset$, the result reaches.

$\square$

**Theorem 3.2.** [6] For any graph $\Gamma$, $\left\lceil \frac{n}{\gamma(\Gamma) + \Delta(\Gamma)} \right\rceil \leq \gamma(\Gamma) \leq n - \Delta(\Gamma)$ where $\Delta(\Gamma)$ is the maximum degree of $\Gamma$.

**Theorem 3.3.** Let $Z_{pq} = \langle S \rangle$ and $S = B(1, pq)$. Then the following is hold.

$$\gamma(Cay(Z_{pq} : S)) = \begin{cases} 2 & p = 2; \\ 3 & p > 2. \end{cases}$$

**Proof.** Let $p = 2$. By Lemma 3.1, $D = \{i, i + q\}$ is a dominating set. Since $Cay(Z_{pq} : S)$ is a $(q - 1)$-regular graph, by Theorem 3.2, $\gamma(Cay(Z_{pq} : S)) \geq 2$. Thus $\gamma(Cay(Z_{pq} : S)) = 2$. Let $p > 2$. We define $D = \{1, 2, s\}$ where $s \in A_1 \setminus N(2)$. Since $1, 2$ are adjacent, $N(1) \cup N(2) = V(Cay(Z_{pq} : S)) \setminus D$. Thus $D$ is a dominating set. As a consequence, $\gamma(Cay(Z_{pq} : S)) \leq 2$. It is enough to show that $\gamma(Cay(Z_{pq} : S)) \neq 2$. Let $D' = \{x, y\}$. We show that $D'$ is not a dominating set. If $x, y \in A_1$ for some $1 \leq i \leq p$, then for every $z \in A_1 \setminus D', z \in N(D')$. If not, $x \in A_i$, and $y \in A_j$ for some $1 \leq i \neq j \leq p$. If $x, y$ are adjacent, then there is $x' \in A_i \setminus \{x\}$ such that $x' \in N(y)$. Thus $D'$ is not dominating set. If $x$ and $y$ are not adjacent, then there is $z \in A_1, l \neq i, j$, such that the induced subgraph on $\{x, y, z\}$ is empty. Hence, $D'$ is not a dominating set and the proof is completed.

$\square$

**Theorem 3.4.** Let $Z_{pq} = \langle S \rangle$ where $p \in \{2, 3, 5\}$ and $S = B(1, pq)$. Then

$$\gamma_s(Cay(Z_{pq} : S)) = p.$$  

**Proof.** Let $A = \{1, 1 + p, \ldots, 1 + \left(\left\lfloor \frac{q}{2} \right\rfloor - 1\right)p\}$ and $B = \{i + tq : i \in A \text{ and } 1 \leq t \leq p - 1\}$. We define $f : V(Cay(Z_{pq} : S)) \rightarrow \{-1, 1\}$ such that

$$f(x) = \begin{cases} -1 & x \in A \cup B, \\ 1 & \text{otherwise}. \end{cases}$$

Let $v \in V(Cay(Z_{pq} : S))$. If $f(v) = -1$, then

$$f[v] = -1 + (p - 1)(q - 1) - 2 \left(\left(\left\lfloor \frac{q}{2} \right\rfloor - 1\right)(p - 1)\right) = 2p - 3.$$
Otherwise, \( f[v] = 1 + (p - 1)(q - 1) - 2\left\lceil \frac{q}{2} \right\rceil (p - 1) = 1. \)

Hence, \( f \) is a dominating function. Also

\[
\omega(f) = pq - 2(|A| + |B|) = pq - 2\left(\left\lceil \frac{q}{2} \right\rceil + (p - 1)\left\lfloor \frac{q}{2} \right\rfloor\right) = p.
\]

It is enough to show that \( f \) has the minimal wait. Let, to the contrary, \( g \) be a dominating function and \( \omega(g) < \omega(f) \). So \( |V_g^-| > |V_f^-| \). Without lose of generality, suppose that \( |V_g^-| = p\left\lceil \frac{q}{2} \right\rceil + 1 \). Let \( A_i^g = A_i \cap V_g^- \), \( A_i^g = A_i \setminus A_i^g \) and \( B_j^- = B_j \setminus V_g^- \). We will reach the contradiction by three steps.

**Step 1.** For every \( 1 \leq i \leq p \), \( A_i^- \neq \emptyset \).

On the contrary, let \( A_i^- = \emptyset \) for some \( 1 \leq s \leq p \). Let \( u \in A_s \). Then by Lemma 3.1, \( u \in A_s \cap B_t \) for some \( 1 \leq t \leq q \). So

\[
g(u) = (p - 1)(q - 1) - 2(|V_g^-| - |B_t^-|) - 1.
\]

Thus \( |B_t^-| \geq \left\lceil \frac{q}{2} \right\rceil \). Hence, \( |V_g^-| \geq |A_s|\left\lceil \frac{q}{2} \right\rceil \). This implies \( g + (q - p)\left\lceil \frac{q}{2} \right\rceil < 1 \). This is a contradiction. Hence, \( A_i^- \neq \emptyset \).

Similar argument applies for \( B_j \). Therefore, \( B_j^- \neq \emptyset \) for every \( 1 \leq j \leq q \).

**Step 2.** For every \( 1 \leq i \leq p \), \( |A_i^-| \geq \left\lceil \frac{q}{2} \right\rceil \).

On the contrary, Let \( |A_i^-| < \left\lceil \frac{q}{2} \right\rceil \) for some \( 1 \leq l \leq p \). Without lose of generality suppose that \( |A_i^-| = \left\lceil \frac{q}{2} \right\rceil - 1 \). Let \( v \in A_l \). By Lemma 3.1, \( v \in A_l \cap B_k \) for some \( 1 \leq k \leq q \). If \( g(v) = -1 \), then \( g[v] = (p - 1)(q - 1) - 2(|V_g^-| - |A_i^-| - |B_k^-|) + 2 \geq 1 \). Then \( |B_k^- \setminus \{v\}| \geq 4 \).

If \( g(v) = 1 \), then \( |B_k^- \setminus \{v\}| \geq 2 \). Hence, \( |V_g^-| \leq 4|A_i^-| + |A_i^-| + 2|A_i^-| \).

As a consequence \( p > 8 \). This is impossible.

Therefore, for every \( 1 \leq i \leq p \), \( |A_i^-| \geq \left\lceil \frac{q}{2} \right\rceil \) and since \( |V_g^-| = p\left\lceil \frac{q}{2} \right\rceil + 1 \), we may suppose that \( |A_i^-| = \left\lceil \frac{q}{2} \right\rceil \) and \( |A_i^-| = \left\lceil \frac{q}{2} \right\rceil \) for \( 2 \leq i \leq p \).

**Step 3.** For every \( 1 \leq j \leq q \), \( |B_j^-| \geq \left\lceil \frac{q}{2} \right\rceil \).

On the contrary, let \( |B_h^-| < \left\lceil \frac{q}{2} \right\rceil \) for some \( 1 \leq h \leq q \). Suppose that \( |B_h^-| = \left\lceil \frac{q}{2} \right\rceil - 1 \). By Lemma 3.1, \( B_h \cap A_i \neq \emptyset \) for any \( 1 \leq i \leq p \). Let \( z \in B_h \cap A_i \). Thus

\[
g[z] = -1 + (p - 1)(q - 1) - 2(|V_g^-| - |A_i^-| - |B_h^-| + 2)
\]

\[
\leq -1 + (p - 1)(q - 1) - 2\left(p\left\lceil \frac{q}{2} \right\rceil + 1 - \left\lceil \frac{q}{2} \right\rceil - \left\lfloor \frac{q}{2} \right\rfloor + 2\right)
\]

\[
\leq -1 + (p - 1)(q - 1) - 2\left(p\left\lceil \frac{q}{2} \right\rceil + 1 - \left\lceil \frac{q}{2} \right\rceil + \left\lfloor \frac{q}{2} \right\rfloor + 2\right)
\]

\[
\leq -6
\]

Since \( p \in \{2, 3, 5\} \), \( g[z] \leq -1 \). This is a contradiction.

By Step 3, \( |V_g^-| \geq q\left\lceil \frac{p}{2} \right\rceil \). Hence, \( p\left\lceil \frac{q}{2} \right\rceil + 1 \geq q\left\lceil \frac{p}{2} \right\rceil \). So \( p + q \leq 2 \). This is impossible. Therefore \( \gamma_s(Cay(G : S)) = \omega(f) = p. \)

\[\square\]

**Theorem 3.5.** Let \( \mathbb{Z}_{pq} = \langle S \rangle \) where \( p \geq 7 \) and \( S = B(1, pq) \). Then

\[
\gamma_s(Cay(\mathbb{Z}_{pq} : S)) = 5.
\]
Proof. We define \( f : V(Cay(Z_{pq} : S)) \to \{-1, 1\} \) such that \( f(i) = -1 \) if and only if \( i \in \{1, 2, \ldots, \frac{pq-5}{2}\} \). It is easily seen that \( \lfloor \frac{q}{2} \rfloor \leq |A^-_i| \leq \lceil \frac{q}{2} \rceil \) for every \( 1 \leq i \leq p \). Also \( \lfloor \frac{q}{2} \rfloor \leq |B^-_j| \leq \lceil \frac{q}{2} \rceil \) for any \( 1 \leq j \leq q \). Let \( v \in A_i \cap B_s \) such that \( 1 \leq t \leq p \) and \( 1 \leq s \leq q \). In the worst situation, \( |A^-_i| = \lfloor \frac{q}{2} \rfloor \) and \( |B^-_s| = \lceil \frac{q}{2} \rceil \). In this case \( 1 \leq |v| \leq 5 \). Hence, \( f \) is a signed dominating function. Also \( \omega(f) = pq - 2|V_f^-| = 5 \). Thus \( \gamma_S(Cay(Z_{pq} : S)) \leq 5 \). What is left is to show that if \( g \) is a \( \gamma_S \)-function, then \( \omega(g) \geq 5 \). On the contrary, suppose that \( g \) be a \( \gamma_S \)-function and \( \omega(g) < \omega(f) \). Hence, \( |V_g^-| < |V_f^-| \). There is no loss of generality in assuming \( |V_g^-| = \frac{pq-3}{2} \). Let \( A^-_i = A_i \cap V_g^- \) and \( B^-_j = B_j \cap V_g^- \). In order to reach the contradiction we use two following steps:

Step 1. \( A^-_i \neq \emptyset \) for every \( 1 \leq i \leq p \).

On the contrary, suppose that for some \( 1 \leq m \leq p \), \( A^-_m = \emptyset \). Let \( w \in A_m \). So there is \( 1 \leq t \leq q \) such that \( w \in A_m \cap B_t \). Hence, \( g[w] = (p-1)(q-1) + 1 - 2(|V_g^-| - |B^-_t|) \geq 1 \). Thus \( |B^-_t| \geq \frac{pq-3}{2} \). So \( |V_g^-| \geq (\frac{pq-3}{2}) \). Hence, \( pq - 3 \geq q(pq - 4) \). This makes a contradiction.

By similar argument we have \( B^-_j \neq \emptyset \) for every \( 1 \leq j \leq q \).

Step 2. For every \( 1 \leq i \leq p \), \( |A^-_i| \geq \lfloor \frac{q}{2} \rfloor \).

On the contrary, let \( |A^-_i| = \lfloor \frac{q}{2} \rfloor - 1 \). Let \( v \in A_i \). There is \( 1 \leq t \leq q \) such that \( v \in A_i \cap B_t \). If \( g(v) = -1 \), then \( g[v] = (p-1)(q-1) + 1 - 2(|V_g^-| - |A^-_i| - |B^-_t| + 2) \geq 1 \). Hence, \( |B^-_t| \geq \lceil \frac{q}{2} \rceil \). If \( g(v) = 1 \), then \( |B^-_t| \geq \lfloor \frac{q}{2} \rfloor \). Therefore, \( |V_g^-| \geq |A^-_i|(|\lfloor \frac{q}{2} \rfloor | + 1) + |A^-_i|(|\lfloor \frac{q}{2} \rfloor | + |A^-_i|(|\lfloor \frac{q}{2} \rfloor |) \geq \frac{pq-3}{2} \). This implies that \( q \leq 3 \). This is a contradiction.

Likewise Step 2, \( |B^-_j| \geq \lfloor \frac{q}{2} \rfloor \) for every \( 1 \leq j \leq q \). Since \( |V_g^-| = \frac{pq-3}{2} \), there is \( 1 \leq k \leq p \) such that \( |A^-_k| = \lceil \frac{q}{2} \rceil \). On the other hand, suppose that for \( 1 \leq t \leq q \), \( |B^-_t| = \lfloor \frac{q}{2} \rfloor \). Let \( u \in A_k \cap B^-_t \). If \( s \in \{l_1, \ldots, l_t\} \), then

\[
g[u] = -1 + (p-1)(q-1) - 2(|V_g^-| - |A^-_k| - |B^-_t| + 2)
= -1 + (p-1)(q-1) - 2\left(\frac{pq-3}{2} - \lfloor \frac{q}{2} \rfloor - \frac{p}{2} + 2\right)
= -3.
\]

This is a contradiction by \( g \) is a signed dominating function. Hence, \( s \) is not in \( \{l_1, \ldots, l_t\} \). Since \( |A^-_k| = \lfloor \frac{q}{2} \rfloor \), \( q - t \geq \lfloor \frac{q}{2} \rfloor \) and so \( t \leq \lceil \frac{q}{2} \rceil \). As a consequence,

\[
|V_g^-| \geq t\left(\frac{p}{2}\right) + (q-t)\left(\frac{q}{2}\right) \geq \frac{q}{2}\left(\frac{p}{2}\right) + \frac{q}{2}\left(\frac{q}{2}\right).
\]

Since \( |V_g^-| = \frac{pq-3}{2} \), this makes a contradiction. Therefore,

\[
\gamma_S(Cay(Z_{pq} : S)) = 5.
\]

\[\square\]

**Corollary 3.6.** For any \( k \)-regular graph \( \Gamma \) on \( n \) vertices \( \gamma_S(\Gamma) \geq \frac{p}{k+1} \). Hence, \( \gamma_S(\Gamma) \geq 1 \). It is easy to check that \( \gamma_S(\Gamma) = 1 \) if and only if \( \Gamma \) is a complete
graph and $n$ is odd. Furthermore, for any prime numbers $p < q$, there is a $(p - 1)(q - 1)-$regular graph $\Gamma$ with $pq$ vertices such that $\gamma_s(\Gamma) \in \{2, 3, 5\}$.

ACKNOWLEDGMENTS

The author is thankful of referees for their valuable comments.

REFERENCES