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On I-Statistical Convergence

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Abstract. In this paper we investigate the notion of I-statistical convergence and introduce I-st limit points and I-st cluster points of real number sequence and also studied some of its basic properties.

Keywords: I-limit point, I-cluster point, I-statistically Convergent.

2000 Mathematics subject classification: 40A35, 40D25.

1. Introduction

In 1951 Fast [6] and Steinhaus [18] introduced the concept of statistical convergence independently and established a relation with summability. Later on it was further investigated from sequence space point of view by Fridy [8], Salat [19] and many others. Some applications of statistical convergence in number theory and mathematical analysis can be found in [1, 2, 13, 14, 21].

The notion of I-convergence is a generalization of the statistical convergence which was introduced by Kostyrko et al. [12]. They used the notion of an ideal I of subsets of the set N to define such a concept. For an extensive view of this article we refer [4, 11, 20].

The idea of I-convergence was further extended to I-statistical convergence by Savas and Das [16]. Later on more investigation in this direction was done

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by Savas and Das [17], Debnath and Debnath [3], Mursaleen et.al [15], Et et al. [5] and many others [9, 10, 22, 23] . In [16], Savas and Das introduced the I-statistical convergence and I-λ-statistical convergence and the relation between them. Also they studied these concept in the notion of $[V, \lambda]$- summability method.

In the present paper we return to the view of I-statistical convergence as a sequential limit concept and we extend this concept in a natural way to define a I-statistical analogue of the set of limit points and cluster points of a real number sequence.

2. Definitions and Preliminaries

Definition 2.1. [8] If K is a subset of the positive integers N, then K_n denotes the set $\{k \in K : k \leq n\}$. The natural density of K is given by $D(K) = lim_{n\to\infty} \frac{|K_n|}{n}$.

Definition 2.2. [8] A sequence (x_n) is said to be statistically convergent to x_0 if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in N : d(x_k, x_0) \geq \varepsilon\}$ has natural density zero. x_0 is called the statistical limit of the sequence (x_n) and we write st-$lim_{n\to\infty} x_n = x_0$.

Definition 2.3. [7] If $\left(x_{k(j)}\right)$ be a subsequence of a sequence $x = (x_n)$ and density of $K = \{k(j) : j \in N\}$ is zero then $\left(x_{k(j)}\right)$ is called a thin subsequence. Otherwise $\left(x_{k(j)}\right)$ is called a non-thin subsequence of x.

x_0 is said to be a statistical limit point of a sequence (x_n), if there exist a non-thin subsequence of (x_n) which conveges to x_0.

Let Λ_x denotes the set of all statistical limit points of the sequence (x_n).

Definition 2.4. [7] x_0 is said to be a statistical cluster point of a sequence $x = (x_n)$, provided that for each $\varepsilon > 0$ the density of the set $\{k \in N : d(x_k, x_0) < \varepsilon\}$ is not equal to 0.

Let Γ_x denotes the set of all statistical cluster points of the sequence (x_n).

Definition 2.5. [12] Let X is a non-empty set. A family of subsets $I \subset P(X)$ is called an ideal on X if and only if

(i) $\emptyset \in I$;
(ii) for each $A, B \in I$ implies $A \cup B \in I$;
(iii) for each $A \in I$ and $B \subset A$ implies $B \in I$.

Definition 2.6. [12] Let X is a non-empty set. A family of subsets $\mathcal{F} \subset P(X)$ is called a filter on X if and only if

(i) $\emptyset \notin \mathcal{F}$;
(ii) for each $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$;
(iii) for each $A \in \mathcal{F}$ and $B \supset A$ implies $B \in \mathcal{F}$.

An ideal I is called non-trivial if $I \neq \emptyset$ and $X \notin I$. The filter $\mathcal{F} = \mathcal{F}(I) = \{X - A : A \in I\}$ is called the filter associated with the ideal I. A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in X if and only if $I \supset \{\{x\} : x \in X\}$

Definition 2.7. [12] Let $I \subset P(N)$ be a non-trivial ideal on N. A sequence (x_n) is said to be I-convergent to x_0 if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in N : d(x_k, x_0) \geq \varepsilon\}$ belongs to I. x_0 is called the I-limit of the sequence (x_n) and we write $I\text{-}lim_{n\to\infty} x_n = x_0$.

Definition 2.8. [12] x_0 is said to be I-limit point of a sequence $x = (x_n)$ provided that there is a subset $K = \{k_1 < k_2 < ...\} \subset N$ such that $K \notin I$ and $lim\, x_{k_i} = x_0$.

Let $I(\Lambda_x)$ denotes the set of all I-limit points of the sequence x.

Definition 2.9. [12] x_0 is said to be I-cluster point of a sequence $x = (x_n)$ provided that for each $\varepsilon > 0$ the set $\{k \in N : d(x_k, x_0) < \varepsilon\} \notin I$.

Let $I(\Gamma_x)$ denotes the set of all I-cluster points of the sequence x.

Definition 2.10. [16] A sequence $x = (x_n)$ is said to be I-statistically convergent to x_0 if for every $\varepsilon > 0$ and every $\delta > 0$,

$\left\{n \in N : \frac{1}{n} |\{k \leq n : d(x_n, x_0) \geq \varepsilon\}| \geq \delta\right\} \in I.$

x_0 is called I-statistical limit of the sequence (x_n) and we write, $I\text{-}st\, lim\, x_n = x_0$.

Throughout the paper we consider I as an admissible ideal.

3. Main Results

Theorem 3.1. *If (x_n) be a sequence such that $I\text{-}st\, lim\, x_n = x_0$, then x_0 determined uniquely.*

Proof. If possible let the sequence (x_n) be I-statistically convergent to two different numbers x_0 and y_0

i.e, for any $\varepsilon > 0$, $\delta > 0$ we have,

$A_1 = \left\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| < \delta\right\} \in \mathcal{F}(I)$

and $A_2 = \left\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, y_0) \geq \varepsilon\}| < \delta\right\} \in \mathcal{F}(I)$

Therefore, $A_1 \cap A_2 \neq \emptyset$, since $A_1 \cap A_2 \in \mathcal{F}(I)$.

Let $m \in A_1 \cap A_2$ and take $\varepsilon = \frac{d(x_0, y_0)}{3} > 0$

so, $\frac{1}{m} |\{k \leq m : d(x_k, x_0) \geq \varepsilon\}| < \delta$

and $\frac{1}{m} |\{k \leq m : d(x_k, y_0) \geq \varepsilon\}| < \delta$

i.e, for maximum $k \leq m$ will satisfy $d(x_k, x_0) < \varepsilon$ and $d(x_k, y_0) < \varepsilon$ for a very small $\delta > 0$.

Thus, we must have

$\{k \leq m : d(x_k, x_0) < \varepsilon\} \cap \{k \leq m : d(x_k, y_0) < \varepsilon\} \neq \emptyset$ a contradiction, as the neighbourhood of x_0 and y_0 are disjoint.

Hence the theorem is proved. □

Theorem 3.2. *For any sequence (x_n), st-lim$x_n = x_0$ implies I-st lim $x_n = x_0$.*

Proof. Let st-lim$x_n = x_0$.

Then for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \leq n : d(x_k, x_0) \geq \varepsilon\}$ has natural density zero.

i.e, $lim_{n\to\infty} \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| = 0$

So for every $\varepsilon > 0$ and $\delta > 0$,

$\left\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, x_0) \geq \varepsilon\}| \geq \delta\right\}$ is a finite set and therefore belongs to I , as I is an admissible ideal.

Hence I-$st\,lim\, x_n = x_0$. □

But the converse is not true.

EXAMPLE 3.3. Let $I = \zeta$ be the class of $A \subset N$ that intersect a finite number of $\triangle_j$'s where $N = \cup_{j=1}^{\infty} \triangle_j$ and $\triangle_i \cap \triangle_j = \emptyset$ for $i \neq j$.

Let $x_n = \frac{1}{n}$ and so $lim_{n\to\infty} d(x_n, 0) = 0$. Put $\epsilon_n = d(x_n, 0)$ for $n \in N$.

Now define a sequence (y_n) by $y_n = x_j$ if $n \in \triangle_j$

Let $\eta > 0$. Choose $\nu \in N$ such that $\epsilon_\nu < \eta$. Then

$A(\eta) = \{n : d(y_n, 0) \geq \eta\} \subset \triangle_1 \cup ..... \cup \triangle_\nu \in \zeta$.

Now, $\{k \leq n : d(y_k, 0) \geq \eta\} \subseteq \{n \in N : d(y_n, 0) \geq \eta\}$

i.e, $\frac{1}{n} |\{k \leq n : d(y_k, 0) \geq \eta\}| \leq |\{n \in N : d(y_n, 0) \geq \eta\}|$

so for any $\delta > 0$,

$\left\{n \in N : \frac{1}{n} |\{k \leq n : d(y_k, 0) \geq \eta\}| \geq \delta\right\} \subseteq \{n \in N : d(y_n, 0) \geq \eta\} \in \zeta$.

Therefore (y_n) is ζ-statistically convergent to 0.

But (y_n) is not a statistically convergent.

Theorem 3.4. *For any sequence (x_n), I-lim$x_n = x_0$ implies I-st lim $x_n = x_0$.*

Proof. The proof is obvious. But the converse is not true. □

EXAMPLE 3.5. If we take $I = I_f$ the sequence (x_n),

where $x_n = \begin{cases} 0, & n = k^2, k \in N \\ 1, & otherwise \end{cases}$

is I-statistically convergent to 1. But (x_n) is not I-convergent.

Theorem 3.6. *If each subsequence of (x_n) is I-statistically convergent to ξ then (x_n) is also I-statistically convergent to ξ.*

Proof. Suppose (x_n) is not I-statistically convergent to ξ, then there exists $\varepsilon > 0$ and $\delta > 0$ such that

$A = \left\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, \xi) \geq \varepsilon\}| \geq \delta\right\} \notin I$. Since I is admissible ideal so A must be an infinite set.

Let $A = \{n_1 < n_2 < ... < n_m < ...\}$. Let $y_m = x_{n_m}$ for $m \in N$. Then $(y_m)_{m\in N}$ is a subsequence of (x_n) which is not I-statistically convergent to ξ, a contradiction. Hence the theorem is proved. □

But the converse is not true. We can easily show this from example 3.5.

Theorem 3.7. *Let (x_n) and (y_n) be two sequences then*

(i) I-$st\,lim\,x_n = x_0$ and $c \in R$ implies I-$st\,lim\,cx_n = cx_0$.

(ii) I-$st\,lim\,x_n = x_0$ and I-$st\,lim\,y_n = y_0$ implies I-$st\,lim\,(x_n + y_n) = x_0 + y_0$.

Proof. (i) If $c = 0$, we have nothing to prove.

So we assume that $c \neq 0$.

Now, $\frac{1}{n}|\{k \leq n : d(cx_k, cx_0) \geq \varepsilon\}| = \frac{1}{n}|\{k \leq n : |c| d(x_k, x_0) \geq \varepsilon\}|$

$$\leq \frac{1}{n}\left|\left\{k \leq n : d(x_k, x_0) \geq \frac{\varepsilon}{|c|}\right\}\right| < \delta$$

Therefore, $\left\{n \in N : \frac{1}{n}|\{k \leq n : d(cx_k, cx_0) \geq \varepsilon\}| < \delta\right\} \in \mathcal{F}(I)$.

i.e, I-$st\,lim\,cx_n = cx_0$.

(ii) We have $A_1 = \left\{n \in N : \frac{1}{n}\left|\left\{k \leq n : d(x_k, x_0) \geq \frac{\varepsilon}{2}\right\}\right| < \frac{\delta}{2}\right\} \in \mathcal{F}(I)$

and $A_2 = \left\{n \in N : \frac{1}{n}\left|\left\{k \leq n : d(y_k, y_0) \geq \frac{\varepsilon}{2}\right\}\right| < \frac{\delta}{2}\right\} \in \mathcal{F}(I)$.

Since $A_1 \cap A_2 \neq \emptyset$, therefore for all $n \in A_1 \cap A_2$ we have,

$\frac{1}{n}|\{k \leq n : d(x_k + y_k, x_0 + y_0) \geq \varepsilon\}|$

$$\leq \frac{1}{n}\left|\left\{k \leq n : d(x_k, x_0) \geq \frac{\varepsilon}{2}\right\}\right| + \frac{1}{n}\left|\left\{k \leq n : d(y_k, y_0) \geq \frac{\varepsilon}{2}\right\}\right| < \delta.$$

i.e, $\left\{n \in N : \frac{1}{n}|\{k \leq n : d(x_k + y_k, x_0 + y_0) \geq \varepsilon\} < \delta\right\} \in \mathcal{F}(I)$.

Hence I-$st\,lim\,(x_n + y_n) = (x_0 + y_0)$. □

Definition 3.8. A sequence $x = (x_n)_{n \in N}$ of elements of X is said to be I^*-statistical convergent to $\xi \in X$ if and only if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \in \mathcal{F}(I)$, such that st-$lim\,d(x_{m_k}, \xi) = 0$.

Theorem 3.9. *If I^*-$st\,lim_{n\to\infty} x_n = \xi$ then I-$st\,lim_{n\to\infty} x_n = \xi$.*

Proof. Let I^*-$st\,lim_{n\to\infty} x_n = \xi$. By assumption there exist a set $H \in I$ such that for $M = N \setminus H = \{m_1 < m_2 < ... < m_k < ...\}$ we have st-$lim\,x_{m_k} = \xi$

i.e, $lim_{n\to\infty}\frac{1}{n}|\{m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon\}| = 0$

so for any $\delta > 0$, $\left\{n \in N : \frac{1}{n}|\{m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon\}| \geq \delta\right\} \in I$ since I is an admissible ideal.

Now, $A(\varepsilon, \delta) = \left\{n \in N : \frac{1}{n}|\{k \leq n : d(x_k, \xi) \geq \varepsilon\}| \geq \delta\right\}$

$$\subset H \cup \left\{n \in N : \frac{1}{n}|\{m_k \leq n : d(x_{m_k}, \xi) \geq \varepsilon\}| \geq \delta\right\} \in I$$

i.e, I-$st\,lim_{n\to\infty} x_n = \xi$. □

But the converse may not be true.

From example 3.3. we have ζ-$st\,lim_{n\to\infty} y_n = 0$.

Suppose that ζ^*-$st\,lim_{n\to\infty} y_n = 0$. Then there exist a set $H \in \zeta$ such that for $M = N \setminus H = \{m_1 < m_2 < ... < m_k < ...\}$ we have st-$lim\,y_{m_k} = 0$. By definition of ζ there exist a $p \in N$ such that $H \subset \triangle_1 \cup ... \cup \triangle_p$. But then $\triangle_{p+1} \subset M$, so for infinitely many $m_k \in \triangle_{p+1}$,

$D\{m_k \in \triangle_{p+1} : d(y_{m_k}, 0) \geq \eta\} = 2^{-(p+1)} > 0$ for $0 < \eta < \frac{1}{p+1}$

i.e, $D\{m_k \in \triangle_{p+1} : d(y_{m_k}, 0) \geq \eta\} \neq 0$, which is a contradicts st-$lim\,y_{m_k} = 0$.

Hence ζ^*-$st\,lim_{n\to\infty} y_n \neq 0$.

Definition 3.10. An element x_0 is said to be an I-statistical limit point of a sequence $x = (x_n)$ provided that for each $\varepsilon > 0$ there is a set $M = \{m_1 < m_2 < ...\} \subset N$ such that $M \notin I$ and *st-lim* $x_{m_k} = x_0$.

I-$S\,(\Lambda_x)$ denotes the set of all I-statistical limit points of the sequence (x_n).

Theorem 3.11. *If (x_n) be a sequence such that I-$st\,lim\,x_n = x_0$ then I-$S\,(\Lambda_x) = \{x_0\}$.*

Proof. Since (x_n) is I-statistically convergent to x_0, so for each $\varepsilon > 0$ and $\delta > 0$ the set,

$A = \left\{n \in N : \frac{1}{n}|\,\{k \leq n : d\,(x_k, x_0) \geq \varepsilon\}\,| \geq \delta\right\} \in I$, where I is an admisible ideal.

Suppose I-$S\,(\Lambda_x)$ contains y_0 different from x_0. i.e, $y_0 \in I$-$S\,(\Lambda_x)$.

So there exist a $M \subset N$ such that $M \notin I$ and *st-lim* $X_{m_k} = y_0$.

Let $B = \left\{n \in M : \frac{1}{n}|\,\{k \leq n : d\,(x_k, y_0) \geq \varepsilon\}\,| \geq \delta\right\}$. So B is a finite set and therefore $B \in I$ and so $B^c = \left\{n \in M : \frac{1}{n}|\,\{k \leq n : d\,(x_k, y_0) \geq \varepsilon\}\,| < \delta\right\} \in \mathcal{F}\,(I)$.

Again let $A_1 = \left\{n \in M : \frac{1}{n}|\,\{k \leq n : d\,(x_k, x_0) \geq \varepsilon\}\,| \geq \delta\right\}$. So $A_1 \subset A \in I$.

i.e, $A_1^c \in \mathcal{F}\,(I)$. Therefore $A_1^c \cap B^c \neq \emptyset$, since $A_1^c \cap B^c \in \mathcal{F}\,(I)$

Let $p \in A_1^c \cap B^c$ and take $\varepsilon = \frac{d(x_0,y_0)}{3} > 0$

so $\frac{1}{p}|\,\{k \leq p : d\,(x_k, x_0) \geq \varepsilon\}\,| < \delta$

and $\frac{1}{p}|\,\{k \leq p : d\,(x_k, y_0) \geq \varepsilon\}\,| < \delta$

i.e, for maximum $k \leq p$ will satisfy $d\,(x_k, x_0) < \varepsilon$ and $d\,(x_k, y_0) < \varepsilon$ for a very small $\delta > 0$.

Thus we must have,

$\{k \leq p : d\,(x_k, x_0) < \varepsilon\} \cap \{k \leq p : d\,(x_k, y_0) < \varepsilon\} \neq \emptyset$ a contradiction, as the neighbourhood of x_0 and y_0 are disjoint.

Hence I-$S\,(\Lambda_x) = \{x_0\}$. □

Definition 3.12. [15] An element x_0 is said to be an I-statistical cluster point of a sequence $x = (x_n)$ if for each $\varepsilon > 0$ and $\delta > 0$

$\left\{n \in N : \frac{1}{n}|\,\{k \leq n : d\,(x_k, x_0) \geq \varepsilon\}\,| < \delta\right\} \notin I$.

I-$S\,(\Gamma_x)$ denotes the set of all I-statistical cluster points of the sequence (x_n).

Theorem 3.13. *For any sequence $x = (x_n)$, I-$S(\Gamma_x)$ is closed.*

Proof. Let y_0 be a limit point of the set I-$S(\Gamma_x)$ then for any $\varepsilon > 0$, I-$S(\Gamma_x) \cap B\,(y_0, \varepsilon) \neq 0$, where $B\,(y_0, \varepsilon) = \{z \in R : d(z, y_0) < \varepsilon\}$

Let $z_0 \in I$-$S(\Gamma_x) \cap B\,(y_0, \varepsilon)$ and choose $\varepsilon_1 > 0$ such that $B\,(z_0, \varepsilon_1) \subseteq B\,(y_0, \varepsilon)$.

Then we have $\{k \leq n : d\,(x_k, z_0) \geq \varepsilon_1\} \supseteq \{k \leq n : d\,(x_k, y_0) \geq \varepsilon\}$

$\Rightarrow \frac{1}{n}|\,\{k \leq n : d\,(x_k, z_0) \geq \varepsilon_1\}\,| \geq \frac{1}{n}|\,\{k \leq n : d\,(x_k, y_0) \geq \varepsilon\}\,|$

Now for any $\delta > 0$,

$\left\{n \in N : \frac{1}{n}|\,\{k \leq n : d\,(x_k, z_0) \geq \varepsilon_1\}\,| < \delta\right\}$

$$\subseteq \left\{n \in N : \frac{1}{n} | \left\{k \leq n : d\left(x_k, y_0\right) \geq \varepsilon\right\} | < \delta\right\}$$

Since $z_0 \in I\text{-}S(\Gamma_x)$ therefore, $\left\{n \in N : \frac{1}{n} | \left\{k \leq n : d\left(x_k, y_0\right) \geq \varepsilon\right\} | < \delta\right\} \notin I$.
i.e, $y_0 \in I\text{-}S(\Gamma_x)$. Hence the theorem is proved. □

Theorem 3.14. *For any sequence $x = (x_n)$, $I\text{-}S\left(\Lambda_x\right) \subseteq I\text{-}S\left(\Gamma_x\right)$.*

Proof. Let $x_0 \in I\text{-}S\left(\Lambda_x\right)$. Then there exist a set $M = \{m_1 < m_2 < ...\} \notin I$ such that, $st\text{-}lim\, x_{m_k} = x_0 \Rightarrow lim_{k\to\infty} \frac{1}{k} | \left\{m_i \leq k : d(x_{m_i}, x_0) \geq \varepsilon\right\} | = 0$.

Take $\delta > 0$, so there exist $k_0 \in N$ such that for $n > k_0$ we have,
$\frac{1}{n} | \left\{m_i \leq n : d(x_{m_i}, x_0) \geq \varepsilon\right\} | < \delta$.

Let $A = \left\{n \in N : \frac{1}{n} | \left\{m_i \leq n : d(x_{m_i}, x_0) \geq \varepsilon\right\} | < \delta\right\}$.

Also, $A \supset M / \left\{m_1 < m_2 < ... < m_{k_0}\right\}$. Since I is an admissible ideal and $M \notin I$, therefore $A \notin I$. So by definition of I-statistical cluster point $x_0 \in I\text{-}S\left(\Gamma_x\right)$.

Hence the theorem is proved. □

Theorem 3.15. *If $x = (x_n)$ and $y = (y_n)$ be two sequences such that $\{n \in N : x_n \neq y_n\} \in I$, then*
(i) $I\text{-}S\left(\Lambda_x\right) = I\text{-}S\left(\Lambda_y\right)$ and (ii) $I\text{-}S\left(\Gamma_x\right) = I\text{-}S\left(\Gamma_y\right)$.

Proof. (i) Let $x_0 \in I\text{-}S\left(\Lambda_x\right)$. So by definition there exist a set
$K = \{k_1 < k_2 < k_3 < \cdots\}$ of N such that $K \notin I$ and $st\text{-}lim\ x_{k_n} = x_0$.
Since $\{n \in K : x_n \neq y_n\} \subset \{n \in N : x_n \neq y_n\} \in I$,
therefore $K' = \{n \in K : x_n = y_n\} \notin I$ and $K' \subseteq K$.
So we have $st\text{-}lim\ y_{k'_n} = x_0$.
This shows that $x_0 \in I\text{-}S\left(\Lambda_y\right)$ and therefore $I\text{-}S\left(\Lambda_x\right) \subseteq I\text{-}S\left(\Lambda_y\right)$.
By symmetry $I\text{-}S\left(\Lambda_y\right) \subseteq I\text{-}S\left(\Lambda_x\right)$.
Hence $I\text{-}S\left(\Lambda_y\right) = I\text{-}S\left(\Lambda_x\right)$.

(ii) Let $x_0 \in I\text{-}S\left(\Gamma_x\right)$. So by definition for each $\varepsilon > 0$ the set,
$A = \left\{n \in N : \frac{1}{n} | \left\{k \leq n : d\left(x_k, x_0\right) \geq \varepsilon\right\} | < \delta\right\} \notin I$.

Let $B = \left\{n \in N : \frac{1}{n} | \left\{k \leq n : d\left(y_k, x_0\right) \geq \varepsilon\right\} < \delta\right\}$. We have to prove that $B \notin I$.

Suppose $B \in I$. So, $B^c = \left\{n \in N : \frac{1}{n} | \left\{k \leq n : d\left(y_k, x_0\right) \geq \varepsilon\right\} \geq \delta\right\} \in \mathcal{F}\left(I\right)$.
By hypothesis the set $C = \{n \in N : x_n = y_n\} \in \mathcal{F}\left(I\right)$.
Therefore $B^c \cap C \in \mathcal{F}\left(I\right)$. Also it is clear that $B^c \cap C \subset A^c \in \mathcal{F}\left(I\right)$,
i.e, $A \in I$, which is a contradiction.
Hence $B \notin I$ and thus the result is proved. □

Theorem 3.16. *If g is a continuous function on X then it preserves I-statistical convergence in X.*

Proof. Let $I\text{-}st\, lim_{n\to\infty} x_n = \xi$.

Since g is continuous, then for each $\varepsilon_1 > 0$, there exist $\varepsilon_2 > 0$ such that if $x \in B\left(\xi, \varepsilon_1\right)$ then $g\left(x\right) \in B\left(g\left(\xi\right), \varepsilon_2\right)$.

Also we have,
$C(\varepsilon_1, \delta) = \left\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, \xi) \geq \varepsilon_1\}| < \delta\right\} \in \mathcal{F}(I)$
Now, $\{k \leq n : d(x_k, \xi) \geq \varepsilon_1\} \supseteq \{k \leq n : d(g(x_k), g(\xi)) \geq \varepsilon_2\}$
so, $\frac{1}{n} |\{k \leq n : d(x_k, \xi) \geq \varepsilon_1\}| \geq \frac{1}{n} |\{k \leq n : d(g(x_k), g(\xi)) \geq \varepsilon_2\}|$
for $\delta > 0$, $\left\{n \in N : \frac{1}{n} |\{k \leq n : d(x_k, \xi) \geq \varepsilon_1\}| < \delta\right\}$

$$\subseteq \left\{n \in N : \frac{1}{n} |\{k \leq n : d(g(x_k), g(\xi)) \geq \varepsilon_2\}| < \delta\right\} \in \mathcal{F}(I)$$

since $C(\varepsilon_1, \delta) \in \mathcal{F}(I)$.
Hence the theorem is proved. □

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