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Szeged Dimension and PI_v Dimension of Composite Graphs

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Abstract. Let G be a simple connected graph. In this paper, Szeged dimension and PI_v dimension of graph G are introduced. It is proved that if G is a graph of Szeged dimension 1 then line graph of G is 2-connected. Trees of Szeged dimension 1 are characterized. The Szeged dimension and PI_v dimension of five composite graphs: sum, corona, composition, disjunction and symmetric difference with strongly regular components are computed. Also explicit formulas of Szeged and PI_v indices for these composite graphs are obtained.

Keywords: Szeged dimension, PI_v dimension, Composite graphs, Strongly regular graph.

2000 Mathematics subject classification: 05C76, 05C12, 05C35.

1. Introduction

All graphs throughout the paper are considered simple connected graphs with at least two vertices. The *distance* between vertices u and v is denoted by $d(u, v)$. The eccentricity of vertex v is denoted by $\varepsilon(v)$ and defined as the largest distance between v and any other vertices u in G. The maximum and minimum eccentricity among all vertices of G are called diameter of G, $\mathrm{diam}(G)$ and radius of G, $\mathrm{rad}(G)$ respectively. The Wiener index [21] is one of the oldest and most thoroughly investigated topological indices. The Wiener index of graph G is defined as sum of distances between all pairs of vertices of G. Generalization of the Wiener index for cyclic graphs, that is known under

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the name of Szeged index was introduced by Ivan Gutman [8]. The Szeged and the Wiener indices are the same for trees. P. V. Khadikar et al. [12, 13] proposed another Szeged-like index called Padmakar-Ivan (PI) index. Since the PI index is as sum edges weight, it is natural to introduce another index called vertex PI index, PI_v which is viewed as vertex-version.

Let $e = uv$ be an edge of graph G. The number of vertices of G lying closer to vertex u than vertex v is denoted by $n_u(e)$. Analogously, $n_v(e)$ is the number of vertices of G lying closer to vertex v than vertex u.The **Szeged index** and the **vertex Padmakar-Ivan index** [14] of G are denoted by $Sz(G)$ and $PI_v(G)$ respectively and defined as:

$$Sz(G) = \sum_{e=uv\in E(G)} n_u(e)n_v(e),$$
$$PI_v(G) = \sum_{e=uv\in E(G)} n_u(e)+n_v(e)$$

Many methods proposed for calculating the Szeged, the PI and the PI_v indices of molecular graphs, composite graphs and topological indices of molecular graphs. For more information see [2, 3, 6, 9, 11, 14, 15, 18, 17, 19, 20, 23, 22]. Study of topological indices under graph operations is interested in graph theory literature. For example see [1, 4, 7, 10, 14, 15, 16].
Let $e = uv \in E(G)$ and $S_G(e) = n_u(e)n_v(e)$ and $P_G(e) = n_u(e) + n_v(e)$. We call the number of different $S_G(e)$ and $P_G(e)$, (simply ($S(e)$ and $P(e)$) **Szeged dimension**, $dim_{Sz}(G)$ and PI_v **dimension** $dim_{PI_v}(G)$ of G respectively. Let $\{S_1, S_2...S_k\}$ and $\{P_1, P_2...P_m\}$ be the different $S(e)$ and $P(e)$ of G. Let G contains t_i edges of weight s_i and r_j edges of weight p_j. Then

$$Sz(G) = \sum_{i=1}^{k} t_iS_i$$
$$PI_v(G) = \sum_{j=1}^{m} r_jP_j$$

The Szeged and PI_v dimensions are interesting because they tell that how complex is the computation of these topological indices of a graph.
The line graph L(G) of graph G has a vertex for each edge of G, and two of these vertices are adjacent if and only if the corresponding edges in G have a common vertex. A cut vertex is any vertex that when removed increases the number of connected components. A graph with no cut-vertex is called 2-connected graph. A bridge is an edge that when removed increase the number of components. A graph G is vertex transitive if for all pairs of vertices u and v, there is an automorphism of G, $\alpha \in Aut(G)$ such that $\alpha(u) = v$. A graph is edge transitive if its line graph is vertex transitive. In this paper, Some non-vertex transitive graphs of the Szeged and PI_v dimensions 1 are

introduced. For such graphs, some structural properties such as 2-connectivity of their line graph is proved. In [16] the Szeged index were studied under two graph operations Sum and Cartesian product. There is not an explicit formula for the Szeged index of sum of two graphs. But in the special case when the components are regular and triangle-free the Szeged index was formulated [16]. In this paper, the Szeged and PI_v dimensions for five composite graphs: sum, corona, composition, disjunction and symmetric difference are investigated. An explicit relation has not been obtained for the Szeged index of these composite graphs. To formulate the Szeged index composite graphs, we consider the components as strongly regular graphs. Also explicit formulas of Szeged and PI_v indices for these composite graphs is obtained. Let G_1 and G_2 be two graphs. number of vertices and number of edges of G_i are denoted by n_i and e_i for $i = 1, 2$ respectively. In the following these five composite graphs are introduce. The **sum** of two graphs G_1 and G_2 is denoted by $G_1 + G_2$ and is defined as the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{u_1u_2 \,|u_1 \in V(G_1), u_2 \in V(G_2)\}.$$

It is easy to see that $diam(G_1 + G_2) \leq 2$. For a vertex u of G_1, $deg_{G_1+G_2}(u) = deg_{G_1}(u) + n_2$. and for a vertex v of G_2, $deg_{G_1+G_2}(v) = deg_{G_2}(v) + n_1$. The **composition** of graphs G_1 and G_2 is denoted by $G_1[G_2]$ and it is the graph with vertex set $V(G_1) \times V(G_2)$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if (u_1 is adjacent with v_1) or ($u_1 = v_1$ and $u_2$and v_2 are adjacent). It is proved that $diam(G_1[G_2]) = max\{2, diam(G_1)\}$ $deg_{G_1[G_2]}((u, v)) = n_2deg_{G_1}(u) + deg_{G_2}(v)$. The **disjunction** $G_1 \vee G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and

$$E(G_1 \vee G_2) = \{(u_1, u_2)(v_1, v_2) \,|u_1v_1 \in E(G_1) \text{ or } u_2v_2 \in E(G_2)\}.$$

The $diam(G_1 \vee G_2)$ is at most 2 and

$$deg_{G_1 \vee G_2}((u, v)) = n_2deg_{G_1}(u) + n_1deg_{G_2}(v) - deg_{G_1}(u)deg_{G_2}(v)$$

. For given graphs $G_1$and G_2, their **symmetric difference** $G_1 \bigoplus G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and edge set

$$E(G_1 \oplus G_2) = \{(u_1, u_2)(v_1, v_2) \,|u_1v_1 \in E(G_1) \text{ or } u_2v_2 \in E(G_2) \text{ not both }\}.$$

$deg_{G_1 \oplus G_2}((u, v)) = n_2deg_{G_1}(u) + n_1deg_{G_2}(v) - 2deg_{G_1}(u)deg_{G_2}(v)$. The disjunction and symmetric difference are both symmetric operations that share a number of common properties. The most remarkable is that their diameter never exceeds 2. The **corona** of two graphs is denoted by $G_1 \circ G_2$ and is obtained by taking one copy of G_1 and n_1 copies of G_2, and joining all vertices of the i-th copy of G_2 to the i-th vertex of G_1 for $i = 1, 2, \cdots n_1$. Let G_1 and G_2 be two simple connected graphs. The number of vertices and edges of graph G_i is denoted by n_i and e_i respectively. Then $|V(G_1 \circ G_2)| = n_1 + n_1n_2$ and

$|E(G_1 \circ G_2)| = e_1 + n_1e_2 + n_1n_2$. Unlike join and disjunction and symmetric difference, composition and corona are non-commutative operations. For a graph G_1 with at least 2 vertices, $diam(G_1 \circ G_2) = 2 + diam(G_1)$.

Interesting classes of graphs can be also obtained by specializing the first component in the corona product. For example, for a graph G, the graph $K_2 \circ G$ is called its bottleneck graph.

2. Main Results

It easy to see that complete, cycle, star, bipartite complete graphs are of Szeged and PI_v dimension 1. Examples of such graphs are all edge-transitive graphs. It is proved that an edge-transitive graph is of Szeged and PI_v dimension 1. We show that also if G is a graph of Szeged dimension (PI_v dimension) 1 then the line graph of G, $L(G)$ is 2-connected.

Theorem 2.1. *Let G be an edge-transitive graph. Then*

$$dim_{Sz}(G) = dim_{PI_v}(G) = 1$$

.

Proof. Let $l_1 = xy$ and $l_2 = uv$ be two edges of G. Since G is edge-transitive there is an automorphism $\alpha \in Aut(G)$ such that $\{\alpha(x), \alpha(y)\} = \{\alpha(u), \alpha(v)\}$. Suppose $\alpha(x) = u$ and $\alpha(y) = v$. Suppose $\Gamma_k(w)$ be the set of vertices with distance k from w. It is easy to see that for edge $l_1 = xy$,

$$n_x(l_1) = \sum_{k}^{\varepsilon(x)} |\Gamma_k(x) \cap \Gamma_{k+1}(y)|,$$

$$n_y(l_1) = \sum_{k}^{\varepsilon(y)} |\Gamma_k(y) \cap \Gamma_{k+1}(x)|.$$

Since α preserves distances, we have $\varepsilon(x) = \varepsilon(u)$ and $\varepsilon(y) = \varepsilon(v)$. Also $|\Gamma_k(x)| = |\Gamma_k(u)|$ and $|\Gamma_k(y)| = |\Gamma_k(v)|$ and $|\Gamma_k(x) \cap \Gamma_{k+1}(y)| = |\Gamma_k(u) \cap \Gamma_{k+1}(v)|$. Then $n_x(l_1) = n_u(l_2)$ and $n_y(l_1) = n_v(l_2)$. Hence for any pair of edges l_1 and l_2 we have $S(l_1) = n_x(l_1)n_y(l_1) = n_u(l_2)n_v(l_2) = S(l_2)$ and similarly $P(l_1) = P(l_2)$. □

Corollary 2.2. *Let G be an edge transitive graph and $e \in E(G)$. Then $Sz(G) = |E(G)|S(e)$ and $PI_v(G) = |E(G)|P(e)$.*

There are some graphs that are vertex transitive but with the dimensions of more than 1. For example see the Fig. 1. The approach using Corollary 2.2 was applied in [5]. Note that all edges of an edge transitive graph G are in a same orbit of $Aut(L(G))$. But $P(e) = P(f)$ does not imply that e and f are in the same orbit.

Figure 2 shows a non vertex transitive and non edge transitive graph of the Szeged dimension and the PI_v dimension 1.

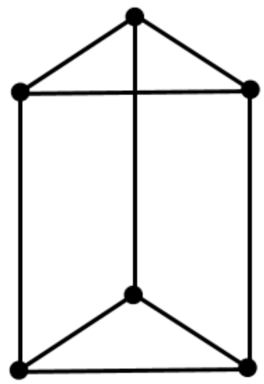

FIGURE 1. A vertex transitive graph of the $dim_{Sz}, dim_{PI_v} > 1$

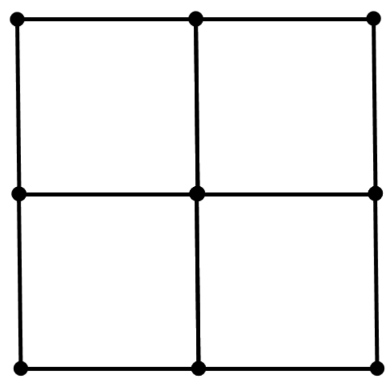

FIGURE 2. A non-vertex transitive and non-edge transitive graph of the $dim_{Sz} = 1 = dim_{PI_v}$

In the next sections, we construct more such graphs. A structural property of graphs of the Szeged dimension 1 is that their line graphs are 2-connected.

Theorem 2.3. *Let G be a graph of Szeged dimension 1 with at least 3 edges. Then $L(G)$ is 2-connected.*

Proof. Suppose on the contrary that e is a cut vertex of $L(G)$. Then e is a bridge in G. Let $G - e = G_1 \cup G_2$. Since e is a cut vertex of $L(G)$, the components $G_i, i = 1, 2$ contains at least one edge e_i that are adjacent to edge e. Let $|V(G_i)| = n_i, i = 1, 2$. Without loss generality suppose that $n_1 \geq n_2$. It is clear that $S(e) = n_1 n_2$ and $S(e_2) \leq (n_1 + 1)(n_2 - 1) < S(e)$. Hence $S(e_2) \neq S(e)$, a contradiction. □

Since all trees are of PI_v dimension 1, then the above theorem is not applicable for PI_v dimension. In the following, it is proved that the only class of trees of Szeged dimension 1 are stars.

Theorem 2.4. *Let T be a tree. Then $dim_{Sz}(T) = 1$ if and only if T is a star graph.*

Proof. Let T has n vertices and e be an edge of T has a pendant vertex (vertex of degree 1) then $S(e) = n-1$. We show that all edges of T has a pendant vertex. In the contrary, let $f = xy$ be an edge of T without pendent vertices. Then $S(f) = k(n-k)$ where $k, n-k \geq 2$. Since $dim_{Sz}(T) = 1$ then $k(n-k) = n-1$ that implies $k = 1$ or $n = k-1$ that is a contradiction. It is clear that the only class of trees which all edges has pendant vertex are stars. □

A k-regular graph with n vertices is called **strongly regular graph** if there exist positive integers λ and μ such that every adjacent pair of vertices has λ common neighbors, and every non-adjacent pair has μ common neighbors. Such strongly graphs is denoted by $srg(n, k, \lambda, \mu)$. Let G be a $srg(n, k, \lambda, \mu)$ and $uv \in E(G)$. The vertices closer to u than v are $k - \lambda - 1$ vertices of neighbors of u. Then for edge $e = uv$ we have $P(e) = (k-\lambda)^2$ and $2(k-\lambda)$ and $dim_{Sz}(G) = 1 = dim_{PI_v}(G)$. The Szeged and PI_v index of G is obtained as: $Sz(G) = \frac{nk}{2}(k-\lambda)^2$ and $PI_v(G) = nk(k-\lambda)$.

3. The Szeged and PI_v Dimensions of Composite Graphs

In this section,The Szeged and PI_v dimensions of five composite graphs: sum, corona, composition, disjunction and symmetric difference with strongly regular components is computed. Also explicit formulas of the Szeged and the PI_v indices for these composite graphs is obtained. In this section, the notations n_i and e_i are the number of vertices and number of edges of G_i respectively and considered graphs are non empty graphs.

3.1. **Sum.** We start with the sum of graphs. In [16] the Szeged index of sum graphs with regular and triangle-free components was computed. Here we compute the Szeged index for the family of strongly regular graphs.

Theorem 3.1. *Let $G_1 = srg(n_1, k_1, \lambda_1, \mu_1)$ and $G_2 = srg(n_2, k_2, \lambda_2, \mu_2)$ be two strongly regular graphs. Then $dim_{Sz}(G_1 + G_2) \leq 3$ and $dim_{PI_v}(G_1 + G_2) \leq 3$. Also*

$$Sz(G_1 + G_2) = e_1(k_1 - \lambda_1)^2 + e_2(k_2 - \lambda_2)^2 + n_1n_2(n_1 - k_1)(n_2 - k_2),$$

$$PI_v(G_1 + G_2) = 2e_1(k_1 - \lambda_1) + 2e_2(k_2 - \lambda_2) + n_1n_2(n_1 + n_2 - k_1 - k_2),$$

Proof. Let $e = xy$ be an edge of $G_1 + G_2$. To compute the $S(e)$ or $P(e)$, 3 cases must be considered.
case 1. $e \in E(G_1)$.

Since the $diam(G_1 + G_2) \leq 2$ then $n_x(e)$ is the number of vertices that are adjacent to x but non adjacent to y. Hence $n_x(e) = k_1 - \lambda_1$ and $S(e) = (k_1 - \lambda_1)^2$, $P(e) = 2(k_1 - \lambda_1)$.

case 2. $e \in E(G_2)$

Similarly we have, $S(e) = (k_2 - \lambda_2)^2$, $P(e) = 2(k_2 - \lambda_2)$.

case 3. $e = xy, x \in V(G_1)$ and $y \in V(G_2)$

In this case, the non adjacent vertices of y (x) in G_2 (G_1) are closer to x (y) than y (x). Hence $S(e) = (n_2 - k_2)(n_1 - k_1)$ and $S(e) = (n_1 + n_2 - k_1 - k_2)$. Now by summing the weight of edges, the Szeged and the PI_v of $G_1 + G_2$ is obtained. □

3.2. **Corona.**

Theorem 3.2. *Let G_1 and G_2 are simple connected graphs and G_2 be strongly regular graph with $G_2 = srg(n_2, k_2, \lambda_2, \mu_2)$. Then*

$$dim_{Sz}(G_1 \circ G_2) \leq dim_{Sz}(G_1) + 2$$

and

$$dim_{PI_v}(G_1 \circ G_2) \leq dim_{PI_v}(G_1) + 2$$

Also we have

$$Sz(G_1 \circ G_2) = (n_2 + 1)^2 Sz(G_1) + n_1 e_2 (k_2 - \lambda_2)^2 + n_1 n_2 (n_1 + n_1 n_2 - k_2 - 1),$$

$$PI_v(G_1 \circ G_2) = 2(n_2 + 1) PI_v(G_1) + 2n_1 e_2 (k_2 - \lambda_2) + n_1 n_2 (n_1 + n_1 n_2 - k_2 - 1).$$

Proof. Let denote the copy of G_2 associated a vertex $x \in V(G_1)$ by $G_{2,x}$. We partition the edges of $G = G_1 \circ G_2$ to 3 sets.

case 1. $e = xy \in E(G_1)$.

For each vertex of G_1 that is closer to x than y, there is $n_2 + 1$ vertices in G closer to x than y. Then $S_G(e) = (n_2+1)^2 S_{G_1}(e)$ and $P_G(e) = 2(n_2+1)P_{G_1}(e)$. Note that there are e_1 such edges.

case 2. $e = xy \in E(G_{2,z}), z \in V(G_1)$.

There are $n_1 e_2$ such edges. In this case $S(e) = (k_2 - \lambda_2)^2$ and $P(e) = 2(k_2 - \lambda_2)$

case 3. $e = xy$, $x \in V(G_1)$ and $y \in V(G_2)$.

All vertices of G except y with its neighbors are closer to x than y. Therefore $n_x(e) = n_1 + n_1 n_2 - k_2 - 1$ and $n_y(e) = 1$. There are $n_1 n_2$ such edges. The Szeged and PI_v indices of $G_1 \circ G_2$ is computed by summing the weight of edges. □

3.3. **Composition.**

Theorem 3.3. *Let $G_1 = srg(n_1, k_1, \lambda_1, \mu_1)$ and $G_2 = srg(n_2, k_2, \lambda_2, \mu_2)$ be two strongly regular graphs. Then $dim_{Sz}(G_1[G_2]) \leq 3$ and $dim_{PI_v}(G_1[G_2]) \leq 3$. Also*

$$\begin{aligned} Sz(G_1[G_2]) &= e_1 n_2 (n_2 k_1 - \lambda_1 n_2 - k_2)^2 + e_1 n_2 (n_2 - 1)(n_2 k_1 + k_2 - \lambda_1 n_2)^2 \\ &\quad + n_1 e_2 (k_2 - \lambda_2)^2, \end{aligned}$$

$$
\begin{aligned}
PI_v(G_1[G_2]) &= 2e_1n_2(n_2k_1-\lambda_1n_2-k_2)+2e_1n_2(n_2-1)(n_2k_1+k_2-\lambda_1n_2)\\
&\quad +2n_1e_2(k_2-\lambda_2)
\end{aligned}
$$

Proof. Let $G = G_1[G_2]$. Then G is a regular graph of degree $k = n_2k_1 + k_2$. Since the diameter of G is at most 2 and G is a k- regular graph then the weight of an edge is $(k-c)^2$ for the Szeged index and $2(k-c)$ for the PI_v index where c is the number of vertices adjacent to the ends of the edge. We partition the edges of $G_1[G_2]$ to 3 sets. The first set is $A_1 = \{(x,u)(y,v)|xy \in E(G_1), u = v\}$. Note that $|A_1| = e_1n_2$. Now we find the common neighbors of the end vertices of edge (x,u)(y,v) that is as:
$|N(x,y)\cap N(u,v)| = |\{(x,w),(y,w)|w \in N(u)\} \cup \{(z,w)|z \in N(x)\cap N(y), w \in V(G_2)\}|$
$= 2k_2 + \lambda_1n_2$. Consider the next set as: $A_2 = \{(x,u)(y,v)|xy \in E(G_1), u \neq v\}$. For A_2, we have $|A_2| = e_1n_2(n_2-1)$ and $|N(x,y)\cap N(u,v)| = |\{(z,w)|z \in N(x)\cap N(y), w \in V(G_2)\}|$
$= \lambda_1n_2$. The last set $A_3 = \{(x,u)(y,v)|x = yuv \in E(G_2)\}$. Similarly we get $|A_3| = n_1e_2$ and $|N(x,y)\cap N(u,v)| = |\{(x,w)|w \in N(u)\cap N(v)\} \cup \{(y,w)|y \in N(x), w \in V(G_2)\}|$. The $Sz(G)$ and $PI_v(G)$ is computed by summing weight of the edge of these three sets. □

3.4. **Disjunction and Symmetric difference.** The operations disjunction and symmetric difference are two commutative graph operations.

Theorem 3.4. *Let $G_1 = srg(n_1,k_1,\lambda_1,\mu_1)$ and $G_2 = srg(n_2,k_2,\lambda_2,\mu_2)$ be two strongly regular graphs. Then $dim_{Sz}(G_1 \vee G_2) \leq 4$ and $dim_{PI_v}(G_1[G_2]) \leq 4$. The Szeged and the PI_V indices of $G = G_1 \vee G_2$ is obtained as:*

$$
\begin{aligned}
Sz(G) &= e_1n_2\left((n_1-k_1)k_2+n_2(k_1-\lambda_1)-2k_2\right)^2+e_2n_1((n_2-k_2)k_1\\
&\quad +e_1n_2(n_2-1)(n_1k_2+n_2k_1-k_1k_2-\lambda_1n_2-\lambda_2n_1+\lambda_1\lambda_2)^2\\
&\quad e_2n_1(n_1-1)(n_2k_1+n_1k_2-k_1k_2-\lambda_2n_1-\lambda_1n_2+\lambda_1\lambda_2)^2\\
&\quad +n_1(k_2-\lambda_2)-2k_1)^2\\
PI_v &= 2e_1n_2\left((n_1-k_1)k_2+n_2(k_1-\lambda_1)-2k_2\right)+2e_2n_1((n_2-k_2)k_1\\
&\quad +2e_1n_2(n_2-1)(n_1k_2+n_2k_1-k_1k_2-\lambda_1n_2-\lambda_2n_1+\lambda_1\lambda_2)\\
&\quad 2e_2n_1(n_1-1)(n_2k_1+n_1k_2-k_1k_2-\lambda_2n_1-\lambda_1n_2+\lambda_1\lambda_2)\\
&\quad +n_1(k_2-\lambda_2)-2k_1)
\end{aligned}
$$

Proof. We partition the edges of $G_1 \vee G_2$ to 4 sets $A_1 ... A_4$ as:

$$\begin{aligned} A_1 &= \{(x,u)(y,v) | xy \in E(G_1), u = v\} \\ A_2 &= \{(x,u)(y,v) | xy \in E(G_1), u \neq v\} \\ A_3 &= \{(x,u)(y,v) | x = y, uv \in E(G_2)\} \\ A_4 &= \{(x,u)(y,v) | x \neq y, uv \in E(G_2)\} \end{aligned}$$

The diameter of $G = G_1 \vee G_2$ is at most 2 and G is a regular graph of degree $n_2k_1 + n_1k_2 - k_1k_2$. Then to compute the edges it is enough to find the number of common neighbors of the end vertices of the edge. The size of A_1 is e_1n_2 and $|N((x,y) \cap N(u,v)| = \lambda_1 n_2 + 2k_2$. Then for $e \in A_1$ we have $S(e) = (n_2k_1 + n_1k_2 - k_1k_2 - \lambda_1n_2 - 2k_2)^2$ and $P(e) = 2(n_2k_1 + n_1k_2 - k_1k_2 - \lambda_1n_2 - 2k_2)$. Similarly for the other set we obtain the following results:
$|A_2| = e_1n_2(n_2 - 1)$, for $e \in A_2$

$$S(e) = (n_2k_1 + n_1k_2 - k_1k_2 - \lambda_1n_2 - \lambda_2n_1 + \lambda_1\lambda_2)^2$$

$$P(e) = 2(n_2k_1 + n_1k_2 - k_1k_2 - \lambda_1n_2 - \lambda_2n_1 + \lambda_1\lambda_2)$$

$|A_3| = e_2n_1$, for $e \in A_3$

$$S(e) = (n_1k_2 + n_2k_1 - k_1k_2 - \lambda_2n_1 - 2k_1)^2$$

$$P(e) = 2(n_1k_2 + n_2k_1 - k_1k_2 - \lambda_2n_1 - 2k_1)$$

$|A_4| = e_2n_1(n_1 - 1)$, for $e \in A_4$

$$S(e) = (n_1k_2 + n_2k_1 - k_1k_2 - \lambda_2n_1 - \lambda_1n_2 - 2K_2)^2$$

$$P(e) = 2(n_1k_2 + n_2k_1 - k_1k_2 - \lambda_2n_1 - \lambda_1n_2 - 2K_2).$$

□

Analogously the Szeged and the PI_v indices for symmetric difference of two strongly regular graph $G_1 = srg(n_1, k_1, \lambda_1, \mu_1)$ and $G_2 = srg(n_2, k_2, \lambda_2, \mu_2)$ is computed as follow. Note that $G_1 \oplus G_2$ is also a regular graph of degree $n_1k_2 + n_2k_1 - 2k_1k_2$.

Theorem 3.5. *Let $G_1 = srg(n_1, k_1, \lambda_1, \mu_1)$ and $G_2 = srg(n_2, k_2, \lambda_2, \mu_2)$ be two strongly regular graphs. Then $dim_{Sz}(G_1 \oplus G_2) \leq 4$ and $dim_{PI_v}(G_1 \oplus G_2) \leq 4$.*

The Szeged and the PI_V indices of $G = G_1 \oplus G_2$ is obtained as:

$$\begin{aligned}
Sz(G) &= e_1n_2(k_2(n_1+\lambda_1)+n_2(k_1-\lambda_1)-2k_1k_2)^2+e_2n_1(k_1(n_2+\lambda_2) \\
&\quad ++e_1(n_2(n_2-1)-e_2^2)(n_1k_2+n_2k_1-2k_1k_2-\lambda_1(2k_2-\lambda_2))^2 \\
&\quad +e_2(n_1(n_1-1)-e_1^2)(n_2k_1+n_1k_2-2k_1k_2-\lambda_2(2k_1-\lambda_1))^2 \\
&\quad n_1(k_2-\lambda_2)-2k_1k_2)^2 \\
PI_v &= 2e_1n_2(k_2(n_1+\lambda_1)+n_2(k_1-\lambda_1)-2k_1k_2)+2e_2n_1(k_1(n_2+\lambda_2) \\
&\quad +2e_1(n_2(n_2-1)-e_2^2)(n_1k_2+n_2k_1-2k_1k_2-\lambda_1(2k_2-\lambda_2)) \\
&\quad +2e_2(n_1(n_1-1)-e_1^2)(n_2k_1+n_1k_2-2k_1k_2-\lambda_2(2k_1-\lambda_1)) \\
&\quad +n_1(k_2-\lambda_2)-2k_1k_2)
\end{aligned}$$

The proof follows much along the same lines as for the disjunction,so we omit the details. The theorem of this section can be used also for graphs that two adjacent vertices have the same number of neighbors.

4. Examples

In this section,we apply some of the derived results to give explicit formulas for Szeged and PI_v indices of some classes of graphs such as suspension, wheel, closed fence and bottleneck graphs of graph G. We start with suspension.

Corollary 4.1. *Let G be strongly regular graph $srg(n, k, \lambda, \mu)$. Then*

$$Sz(K_1 + G) = \frac{nk}{2}(k-\lambda)^2 + n(n-k)$$

$$PI_v(K_1 + G) = 2(k-\lambda) + n(n-k)$$

Now we give the formulas of Sz and PI_v for wheel graphs.

Corollary 4.2.

$$Sz(K_1 + C_n) = n^2 + 2n = PI_v(K_1 + C_n).$$

Next example is closed fence graph.

Corollary 4.3. *Let $n \geq 4$. Then*

$$Sz(C_n[K_2]) = 69n$$

$$PI_v(C_n[K_2]) = 34n$$

For bottleneck graph $K_2 \circ G$ we have:

Corollary 4.4.

$$Sz(K_2 \circ G) = (n+1)^2 + nk(k-\lambda)^2 + 2n(2n+1-k)$$

$$PI_v(K_2 \circ G) = 4(n+1) + 2nk(k-\lambda) + 2n(2n+1-k)$$

Concluding Remarks

The Szeged dimension and the PI_v dimensions of G are the simplest measure to determine how complex is the computation of these indices on G. Also by Szeged dimension some structural properties of graphs are determined. For instance a graph of Szeged dimension 1 has a 2-connected line graph. For a tree T is proved that $dim_{Sz}(T) = 1$ if and only if T is a star graph. It would be an interesting problem study of graphs (trees) of Szeged dimension k, $k \geq 2$.

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Appendix

The following GAP program is presented to compute the Szeged dimension of an arbitrary graph. Input of the program is the set of adjacent vertices of each vertex that is denoted by N. For example,the set N for the graph P_4 is as $N := [[2], [1, 3], [2, 4], [3]]$.

```
SzegedDimension:=function(N)
local n, D, i, j, S, s, t, x, Eg, e, n1, C, SzegedDimension;
n:=Size(N);
D:=[];
for i in [1..n] do
S:=[];
for j in [1..n] do S[j]:=0; od;
        D[i]:=[];
        D[i][1]:=N[i];
        for j in N[i] do
           S[j]:=1;
        od;
        S[i]:=1;
       s:=1;
       t:=1;
       while s<>0 do
           D[i][t+1]:=[];
           for j in D[i][t] do
               for x in N[j] do
                if S[x]=0 then
                  Add(D[i][t+1],x);
                 fi;
                S[x]:=1;
              od;
```

```
          od;
       if D[i][t+1]=[] then
        s:=0;
       fi;
       t:=t+1;
      od;
od;
Eg:=[];
for i in [1..n] do
    for j in N[i] do
        if j>i then Add(Eg,[i,j]); fi;
    od;
od;
SzegedDimension:=[];
n1:=[];
C:=[];
for e in [1..Size(Eg)] do
   i:=Eg[e][1];
   j:=Eg[e][2];
   n1[e]:=1;
   C[e]:=0;
   C[e]:= C[e]+ Size(Intersection(N[i],N[j]));
   for t in [2..Size(D[i])-1] do
      n1[e]:= n1[e]+ Size(Intersection(D[i][t] , D[j][t-1]));
      C[e]:= C[e]+ Size(Intersection(D[i][t], D[j][t]));
   od;
  AddSet( SzegedDimension,(n1[e])*(n - n1[e] - C[e]) );
od;
return Size(SzegedDimension);
end;
```