

## Some Algebraic and Combinatorial Properties of the Complete $T$ -Partite Graphs

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**ABSTRACT.** In this paper, we characterize the shellable complete  $t$ -partite graphs. It is also shown that for these types of graphs the concepts vertex decomposable, shellable and sequentially Cohen-Macaulay are equivalent. Furthermore, we give a combinatorial condition for the Cohen-Macaulay complete  $t$ -partite graphs.

**Keywords:** Cohen-Macaulay, shellable, Vertex decomposable, Edge ideal.

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### 1. INTRODUCTION

Let  $G$  be a finite simple graph on  $n$  vertices. Let  $V_G$  and  $E_G$  denote, respectively, the vertex set and the edge set of  $G$ . An independent set in  $G$  is a subset of  $V_G$  which none of elements are adjacent and the independence complex  $\Delta_G$  of a graph  $G$  is defined by

$$\Delta_G = \{A \subseteq V_G : A \text{ is an independent set in } G\}.$$

We denote by  $MIS(G)$  the set of all maximal independent sets in  $G$  (the set of facets of  $\Delta_G$ ); see [16]. In this paper, we obtain a lower bound for  $|MIS(G)|$ . Recently, researchers studied the algebraic properties of a commutative ring by

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its associated combinatorial structure, like for instance zero divisor graph; see [1, 8, 13]. Now let  $R = k[x_1, \dots, x_n]$  be the polynomial ring over a field  $k$  in  $n$  variable  $x_1, \dots, x_n$ . Identifying  $i \in V_G$  with the variable  $x_i$  in  $R$ , the edge ideal  $I(G)$  of  $G$  will be defined as the monomial ideal generated by all of monomials  $x_i x_j$  such that  $\{x_i, x_j\} \in E_G$ .

In recent years, researchers tried to identify Cohen-Macaulay graphs in terms of their combinatorial properties. Estrada and Villarreal in [3] showed that Cohen-Macaulayness and shellability of a bipartite graph  $G$  are the same. Herzog and Hibi in [6] proved that a bipartite graph  $G$  is Cohen-Macaulay if and only if  $|V_1| = |V_2|$  and there is an order on vertices of  $V_1$  and  $V_2$  as  $x_1, \dots, x_n$  and  $y_1, \dots, y_n$ , respectively, such that:

- i)  $x_i \sim y_i$  for  $i = 1, \dots, n$ ,
- ii) if  $x_i \sim y_j$ , then  $i \leq j$ ,
- iii) if  $x_i \sim y_j$  and  $x_j \sim y_k$ , then  $x_i \sim y_k$ .

Here, we present a necessary and sufficient condition under which shellability of a complete  $t$ -partite graph is equivalent to Cohen-Macaulayness.

It is known that any vertex decomposable graph is shellable (hence sequentially Cohen-Macaulay), but the converse is not valid in general. So, it is interesting to know a family of graphs in which the property of shellability, vertex decomposability and sequentially Cohen-Macaulayness are the same. Francisco and Van Tuyl in [4] showed that  $n$ -cycles for  $n = 3, 5$  belong to this family of graphs. F. Mohammadi and D. Kiani in [10] proved that in  $\theta_{n_1, \dots, n_k}$  for  $\{n_1, \dots, n_k\} \neq \{2, 5\}$ , vertex decomposability, shellability and sequentially Cohen-Macaulayness are coincide. Van Tuyl in [14] showed that in bipartite graphs, three concepts are equivalent. In this paper, attempts have been particularly made to introduce another member of this family, that is, complete  $t$ -partite graphs.

Herzog and Hibi, in [6], showed that a bipartite graph without isolated vertices  $G$  is unmixed if and only if there exists a bipartition  $V_1 = \{x_1, \dots, x_g\}$  and  $V_2 = \{y_1, \dots, y_g\}$  of  $V_G$  such that:

- i)  $\{x_i, y_i\} \in E_G$  for all  $i$ , and
- ii) if  $\{x_i, y_j\}$  and  $\{x_j, y_k\}$  are in  $E_G$  and  $i, j, k$  are distinct, then  $\{x_i, y_k\} \in E_G$ .

In the current paper, among other results, we provide a condition for identifying all the unmixed complete  $t$ -partite graphs.

## 2. MAIN RESULTS

A graph  $G$  is  $t$ -partite if its vertex set can be partitioned into disjoint independent subsets  $V_1, \dots, V_t$ . Moreover, in this paper, we consider  $t$  as the smallest number that has this property. The graph  $G$  is called *complete  $t$ -partite graph* if its vertex set can be partitioned into disjoint independent subsets  $V_1, \dots, V_t$  such that for all  $u$  and  $v$  in different partition sets,  $uv \in E_G$ . A

$k$ -coloring of a graph  $G$  is a labeling  $f : V(G) \rightarrow S$  where  $|S| = k$ . The labels are considered as colors and the set of vertices of one given color form a color class. A  $k$ -coloring is said to be proper if adjacent vertices have different labels. A graph is  $k$ -colorable if it has a proper  $k$ -coloring. The chromatic number of graph  $G$ ,  $\chi(G)$ , is the least  $k$  such that  $G$  is  $k$ -colorable [11].

The following proposition gives a lower bound for cardinality of the set of all maximal independent sets in  $G$ ,  $|MIS(G)|$ , in terms of chromatic number.

**Proposition 2.1.** *Let  $G$  be a graph. If  $\chi(G) = t$ , then  $|MIS(G)| \geq t$ .*

*Proof.* Since  $\chi(G) = t$ , there exist  $t$  color classes for  $G$ . Hence, the graph  $G$  can be considered as a  $t$ -partite graph. Suppose that  $V_i$  is the set of elements of  $i$ -th color class. Then  $V_i$  is an independent set of  $G$  and there exists a maximal independent set  $F_i$  in  $G$  such that  $V_i \subseteq F_i$  for all  $1 \leq i \leq t$ . Thus, there exists at least  $t$  maximal independent set for  $G$ .  $\square$

**Remark 2.2.** Using the definition of a complete  $t$ -partite graph, it follows that  $|MIS(G)| = t$  for any complete  $t$ -partite graph  $G$ .

**Definition 2.3.** A simplicial complex  $\Delta$  is called *shellable* if the facets (maximal faces) of  $\Delta$  can be ordered  $F_1, \dots, F_s$  such that for all  $1 \leq i < j \leq s$ , there exist some  $v \in F_j \setminus F_i$  and some  $l \in \{1, \dots, j-1\}$  with  $F_j \setminus F_l = \{v\}$ . We call  $F_1, \dots, F_s$  to be a *shelling* of a shellable complex,  $\Delta$ , when the facets are ordered as in the definition.

Now by the next theorem, all shellable complete  $t$ -partite graphs can be classified.

**Theorem 2.4.** *Let  $G$  be a complete  $t$ -partite graph.  $G$  is shellable if and only if  $G$  is  $t$ -colorable such that exactly one of color classes has arbitrary elements and other classes have only one element.*

*Proof.*  $\Leftarrow$ ) Assume that  $V_G = \{x_1, \dots, x_n\}$  is the set of vertices. To prove that  $G$  is shellable, we have to find a shelling  $F_1, \dots, F_t$  for  $\Delta_G$ . Since proper  $t$ -vertex coloring gives a partition of  $V_G$  into  $t$  color classes, we suppose that the set of elements of  $i$ -th color class is  $V_i$ . By assumption,  $V_1 = \{x_1, \dots, x_m\}$  where  $m = n - t + 1$  and  $V_i = \{x_{m+i-1}\}$  for all  $2 \leq i \leq t$ . We know that each  $V_i$  is an independent set. Now, if  $x_{m+i-1} \in V_1$  for  $2 \leq i \leq t$ , then we can replace  $V_1$  by  $V_1 \cup V_i$  and obtain  $(t-1)$ -partition for  $G$  that is a contradiction. Therefore,  $V_1$  is a maximal independent set of  $G$  and hence a facet of  $\Delta_G$ . By the same argument, each  $V_i$  is a maximal independent set. We put  $F_i = V_i$ . Thus, we find an ordering on the facets of  $\Delta_G$  as follows:

$$F_1 = \{x_1, \dots, x_m\}, F_2 = \{x_{m+1}\}, \dots, F_t = \{x_{m+t-1}\}.$$

Since  $F_i \setminus F_1 = \{x_{m+i-1}\}$  for all  $2 \leq i \leq t$ ,  $F_1, \dots, F_t$  is a shelling of  $\Delta_G$ .

$\Rightarrow$ ) Suppose that  $V_1, \dots, V_t$  is a partition of  $V_G$ . According to Remark 2.2,

we have  $|MIS(G)| = t$ , so  $\Delta_G$  has exactly  $t$  facets. On the other hand,  $G$  is shellable and we can consider  $F_1, \dots, F_t$  as a shelling of  $\Delta_G$ . Since  $V_i$ 's are maximal independent sets, we obtain  $\{V_1, \dots, V_t\} = \{F_1, \dots, F_t\}$ . Without loss of generality, put  $F_i = V_i$  for all  $1 \leq i \leq t$ . There exists  $x_2 \in F_2 \setminus F_1$  such that  $F_2 \setminus F_1 = \{x_2\}$  because  $F_1, \dots, F_t$  is a shelling. Thus  $F_2 = (F_2 \setminus F_1) \cup (F_2 \cap F_1) = \{x_2\}$ .

Now, suppose by induction that  $F_1 = \{x_1, \dots, x_m\}$ ,  $F_2 = \{x_{m+1}\}$ ,  $\dots$ ,  $F_i = \{x_{m+i-1}\}$ . Since  $F_1, \dots, F_t$  is a shelling of  $\Delta_G$ , there exists  $x_{i+1} \in F_{i+1} \setminus F_1$  and  $l \in \{1, \dots, i\}$  such that  $F_{i+1} \setminus F_l = \{x_{i+1}\}$ , then  $F_{i+1} = (F_{i+1} \setminus F_l) \cup (F_{i+1} \cap F_l) = \{x_{i+1}\}$ . Hence, one of the color classes has arbitrary elements and the other classes have exactly one element.  $\square$

**Definition 2.5.** A simplicial complex  $\Delta$  is recursively defined to be *vertex decomposable* if it is either a simplex, or else has some vertex  $v$  so that

- i) both  $\Delta \setminus v$  and  $link_{\Delta}^v$  are vertex decomposable and
- ii) no face of  $link_{\Delta}^v$  is a facet of  $\Delta \setminus v$ , where

$$link_{\Delta}^F = \{G : G \cap F = \emptyset, G \cup F \in \Delta\}.$$

A graph  $G$  is called vertex decomposable if the simplicial complex  $\Delta_G$  is vertex decomposable.

The following theorem is one of the main results of this paper which characterizes all vertex decomposable complete  $t$ -partite graphs.

**Theorem 2.6.** *Let  $G$  be a complete  $t$ -partite graph. Then,  $G$  is vertex decomposable if and only if  $G$  is  $t$ -colorable such that exactly one of the color classes has arbitrary elements and the other classes have only one element.*

*Proof.*  $\Rightarrow$ ) Assume that for any proper  $t$ -vertex coloring of  $G$ , there exists at least two classes with at least two elements. By Theorem 2.4,  $G$  is not shellable and hence is not vertex decomposable, by ([17], Corollary 7).

$\Leftarrow$ ) By assumption, it follows that  $G$  is a chordal graph, so  $G$  is vertex decomposable by ([17], Corollary 7).  $\square$

**Definition 2.7.** A subset  $C \subset V_G$  is a *minimal vertex cover* of  $G$  if:

- i) every edge of  $G$  is incident with one vertex in  $C$ , and
- ii) there is no proper subset of  $C$  with the first property.

If  $C$  satisfies only condition (i), it is called a vertex cover of  $G$ . A graph  $G$  is said to be *unmixed* if all the minimal vertex covers of  $G$  have the same number of elements.

By the next theorem, we present a combinatorial characterization of all the unmixed complete  $t$ -partite graphs.

**Theorem 2.8.** *Let  $G$  be a complete  $t$ -partite graph.  $G$  is unmixed if and only if  $G$  is  $t$ -colorable such that all color classes have the same number of elements.*

*Proof.* Since any minimal vertex cover of a complete  $t$ -partite graph  $G$  contains all the elements of  $(t - 1)$  classes, then each selected  $(t - 1)$  of color classes has the same number of elements if and only if all the classes have the same cardinality.  $\square$

**Definition 2.9.** ([15], Definition 3.3.8) A pure  $d$ -dimensional complex  $\Delta$  is called *strongly connected* if each pair of facets  $F, G$  can be connected by a sequence of facets  $F = F_0, F_1, \dots, F_s = G$  such that  $\dim(F_i \cap F_{i-1}) = d - 1$  for  $1 \leq i \leq s$ .

**Lemma 2.10.** ([2], Proposition 11.7) *Every Cohen-Macaulay complex is strongly connected.*

**Lemma 2.11.** ([15], Corollary 3.3.7) *A Cohen-Macaulay simplicial complex  $\Delta$  is pure, that is, all its maximal faces have the same dimension.*

The following theorem is an effective combinatorial criterion for Cohen-Macaulayness of the complete  $t$ -partite graphs.

**Theorem 2.12.** *Let  $G$  be a complete  $t$ -partite graph.  $G$  is Cohen-Macaulay graph if and only if  $G$  is  $t$ -colorable such that all color classes have exactly one element.*

*Proof.*  $\Leftarrow$ ) By assumption, we have  $G = K_t$ , so  $G$  is a chordal graph. By [7],  $G$  is Cohen-Macaulay if and only if  $G$  is unmixed. Thus, the assertion follows from Theorem 2.8.

$\Rightarrow$ ) Suppose that for any proper  $t$ -vertex coloring of  $G$ , there exists at least two classes with at least two elements. Since  $G$  is Cohen-Macaulay,  $\Delta_G$  is pure, by Lemma 2.11. Therefore, all the color classes have the same number of elements. Assume that their size is  $d + 1$ , hence all facets of  $\Delta_G$  are of dimension  $d$  and then  $\dim(\Delta_G) = d$ . We will show that it is not possible that  $d + 1 \geq 2$ . For any two facets  $F$  and  $E$ , ( $F \neq E$ ), we have  $F \cap E = \emptyset$ , then  $\dim(F \cap E) = -1 \neq d - 1$ , because of  $d + 1 \geq 2$ . Therefore,  $\Delta_G$  is not strongly connected, a contradiction to Lemma 2.10. It follows that all the color classes have exactly one element.  $\square$

Now, we give a special condition for complete  $t$ -partite graphs under which shellability is equal to Cohen-Macaulayness.

**Corollary 2.13.** *Let  $G$  be a complete  $t$ -partite graph. The property of being Cohen-Macaulay for  $G$  is equivalent to being shellable if and only if  $G$  is  $t$ -colorable such that all color classes have exactly one element.*

**Definition 2.14.** Let  $k$  be a field and  $R = k[x_1, \dots, x_n]$  be the polynomial ring over  $k$ . A graded  $R$ -module  $M$  is called *sequentially Cohen-Macaulay* (over  $k$ ) if there exists finite filtration of graded  $R$ -modules

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots \subset M_r = M$$

such that each  $M_i/M_{i-1}$  is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \dots < \dim(M_r/M_{r-1}).$$

We call a graph  $G$  sequentially Cohen-Macaulay over the field  $k$  if  $R/I(G)$  is sequentially Cohen-Macaulay.

Suppose that  $I$  is a homogeneous ideal of  $R$ . The ideal generated by all homogeneous degree  $d$  elements of  $I$  is denoted by  $(I_d)$ . The concept of componentwise linear was introduced by Herzog and Hibi [5]. A homogeneous ideal  $I$  is componentwise linear if  $(I_d)$  has a linear resolution for all  $d$ . Let  $I$  be a square-free monomial ideal of  $R$  and  $I_{[d]}$  be the ideal generated by the square-free monomials of degree  $d$  of  $I$ . Herzog and Hibi ([5], Proposition 1.5) have shown that the square-free monomial ideal  $I$  is componentwise linear if and only if  $I_{[d]}$  has a linear resolution for all  $d$ . In [5], it is also shown that:

**Theorem 2.15.** *Let  $I$  be a square-free monomial ideal in a polynomial ring. Then  $I^\vee$  is componentwise linear if and only if  $R/I$  is sequentially Cohen-Macaulay.*

In [12], Stanley showed that shellability implies the sequentially Cohen-Macaulayness.

**Theorem 2.16.** *Let  $\Delta$  be a simplicial complex, and suppose that  $R/I_\Delta$  is the associated Stanley Reisner ring. If  $\Delta$  is shellable, then  $R/I_\Delta$  is sequentially Cohen-Macaulay.*

In the following theorem, we show that being sequentially Cohen-Macaulay of the complete  $t$ -partite graph is really a combinatorial property.

**Theorem 2.17.** *Let  $G$  be a complete  $t$ -partite graph.  $G$  is sequentially Cohen-Macaulay if and only if  $G$  is  $t$ -colorable such that exactly one of color classes has arbitrary elements and other classes have only one element.*

*Proof.*  $\Leftarrow$ ) It follows from Theorems 2.4 and 2.16.

$\Rightarrow$ ) Assume that  $V_G = \{x_1, \dots, x_n\}$  and  $V_1, \dots, V_t$  is a partition of  $V_G$  where  $V_i$  is the set of elements in  $i$ -th color class. We proceed by contradiction. One may consider the following cases:

**Case(1) :** Suppose that there exist at least two parts  $V_i$  and  $V_j$  with  $|V_i| \geq 2$  and  $|V_j| \geq 3$  and  $r$  is the maximum cardinality of parts of  $G$  which is at least 3. Let  $J = I(G)_{[d]}^\vee$  where  $d = n - r + 1$ . Using Theorem 2.15, to show that  $G$  is not sequentially Cohen-Macaulay, it suffices to prove that  $J$  does not have a linear resolution.

We use simplicial homology to compute the Betti numbers of  $J$ . A square-free vector is a vector that its entries are in  $\{0, 1\}$ . For a monomial ideal  $I$  and

a degree  $b \in \mathbb{N}^n$ , define

$$K^b(I) = \{\text{square free vectors } c \in \{0, 1\}^n \text{ such that } \frac{x^b}{x^c} \in I\}$$

to be the upper Koszul simplicial complex of  $I$  in degree  $b$  ([9], Definition 1.33). For a vector  $b \in \mathbb{N}^n$ , the Betti numbers of  $I$  in degree  $b$  can be expressed as  $\beta_{i,b}(I) = \dim_k H_{i-1}^\sim(K^b(I), k)$  ([9], Theorem 1.34). The sum of  $\beta_{i,b}(I)$  over all square-free vectors  $b$  of degree  $j$  is equal to  $\beta_{i,j}(I)$ .

To prove that  $J$  does not have a linear resolution, we will show that  $\beta_{1,n}(J) \neq 0$ . We associate to the monomial  $m = x_1 \dots x_n$  a unique square-free vector  $b = (1, \dots, 1)$ . We have a chain complex

$$\dots \rightarrow C_2(K^b(J)) \xrightarrow{\partial_2} C_1(K^b(J)) \xrightarrow{\partial_1} C_0(K^b(J)) \xrightarrow{\partial_0} C_{-1}(K^b(J)) \rightarrow 0.$$

The  $s$ -dimensional faces  $[x_{i_0}, \dots, x_{i_s}]$  of  $K^b(J)$  are the basis of  $C_s(K^b(J))$  and

$$\partial_s([x_{i_0}, \dots, x_{i_s}]) = \sum_{t=0}^s (-1)^t [x_{i_0}, \dots, \hat{x}_{i_t}, \dots, x_{i_s}].$$

The above chain complex is introduced in ([4], Proposition 4.1). To obtain  $\beta_{1,n}(J)$ , we need to compute  $\dim_k H_0^\sim(K^b(M), k) = \dim_k(\ker \partial_0 / \text{im} \partial_1)$ . If we can find an element in  $\ker \partial_0$  that is not in  $\text{im} \partial_1$ , we have shown that  $\beta_{1,n}(J) > 0$ . We suppose that  $x_1, x_2, x_3$  belong to the part with maximum cardinality. Put  $f = [x_1]$ . Then,  $\partial_0(f) = 0$  and hence  $f$  is in the  $\ker$  of  $\partial_0$ . We claim that  $f$  is not in the image of  $\partial_1$ . To prove, assume that  $\partial_1([x_l, x_s]) = [x_1]$ . Then,  $[x_s] - [x_l] = [x_1]$ , hence  $[x_s] - [x_l] - [x_1] = 0$  that is a contradiction because  $[x_l], [x_s], [x_1]$  are linear independent.

**Case(2)** : Assume that for any  $1 \leq i \leq t$ , we have  $|V_i| = 2$ . The proof of this case is similar to that of case 1. It suffices to consider  $J = I(G)_{[d]}^\vee$  where  $d = n - 2$ .

**Case(3)** : Suppose that there exist at least two parts  $V_i$  and  $V_j$  with  $|V_i| = |V_j| = 2$  and at least one part  $V_r$  with  $|V_r| = 1$ . One can apply the same argument to case 2.  $\square$

**Corollary 2.18.** *Let  $G$  be a complete  $t$ -partite graph. The followings are equivalent:*

- (1)  $G$  is shellable.
- (2)  $G$  is vertex decomposable.
- (3)  $G$  is sequentially Cohen-Macaulay.

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