

## Serre Subcategories and Local Cohomology Modules with Respect to a Pair of Ideals

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**ABSTRACT.** This paper is concerned with the relation between local cohomology modules defined by a pair of ideals and the Serre subcategories of the category of modules. We characterize the membership of local cohomology modules in a certain Serre subcategory from lower range or upper range.

**Keywords:** Local cohomology modules, pair of ideals, Serre subcategory.

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### 1. INTRODUCTION

Throughout this paper,  $R$  is denoted a commutative Noetherian ring,  $I$  and  $J$  are denoted two ideals of  $R$ , and  $M$  is an  $R$ -module. We refer the reader to [2] and [4] for any unexplained terminology.

As a generalization of the ordinary local cohomology modules, Takahashi, Yoshino and Yoshizawa [16] introduced the local cohomology modules with respect to a pair of ideals  $(I, J)$ . To be more precise, let  $W(I, J) = \{\mathfrak{p} \in \text{Spec}(R) \mid I^n \subseteq \mathfrak{p} + J \text{ for some positive integer } n\}$ . Then for an  $R$ -module  $M$ , the  $(I, J)$ -torsion submodule  $\Gamma_{I,J}(M)$  of  $M$ , which consists of all elements  $x$  of  $M$  with  $\text{Supp}Rx \subseteq W(I, J)$ , is considered. It is known,  $\Gamma_{I,J}$  is a left exact additive functor from the category of all  $R$ -modules and  $R$ -homomorphism to itself. For all integer  $i$ , the  $i$ -th local cohomology functor  $H_{I,J}^i$  with respect

to  $(I, J)$  is defined to be the  $i$ -th right derived functor of  $\Gamma_{I,J}$ . The  $i$ -th local cohomology module of  $M$  with respect to  $(I, J)$  is denoted by  $H_{I,J}^i(M)$ . When  $J = 0$ , then  $H_{I,J}^i$  coincides with the usual local cohomology functor  $H_I^i$  with the support in the closed subset  $V(I)$ .

The study of this generalized local cohomology modules was continued by many authors (see for example [5], [6], [9] and [10], [14]).

Recall that a class of  $R$ -modules is a Serre subcategory of the category of  $R$ -modules when it is closed under taking submodules, quotients and extensions. Always,  $S$  stands for a Serre subcategory of the category of  $R$ -modules.

Using the generalized local cohomology modules, we can define  $T_{I,J}^s(M)$  (resp.  $T_s^{I,J}(M)$ ) of the  $R$ -module  $M$  relative to a pair  $(I, J)$  of ideals of  $R$  by

$$T_{I,J}^s(M) = \inf\{i \in \mathbb{N} \mid H_{I,J}^i(M) \text{ is not in } S\},$$

$$\text{(resp. } T_s^{I,J}(M) = \text{Sup}\{i \in \mathbb{N} \mid H_{I,J}^i(M) \text{ is not in } S\})$$

with the usual convention that the infimum (resp. Supremum) of the empty set of integers interpreted as  $+\infty$  (resp.  $-\infty$ ).

Our objective in this paper is to investigate the notions  $T_{I,J}^s(M)$  and  $T_s^{I,J}(M)$ . we prove the following,

**Theorem 1.1.** *Let  $S$  be a Melkersson subcategory with respect to  $(I, J)$ . Suppose  $M$  is Weakly Laskerian module. Then  $T_{I,J}^s(M) = \inf\{T_{\mathfrak{a}}^s(M) \mid \mathfrak{a} \in \widetilde{W}(I, J)\}$ , where  $T_{\mathfrak{a}}^s(M)$  is the least non-negative integer  $i$  such that  $H_{\mathfrak{a}}^i(M)$  is not in  $S$ .*

**Theorem 1.2.** *Let  $S$  be a Serre subcategory and let  $M$  be a Weakly Laskerian module. Then  $T_s^{I,J}(M) = \text{Sup}\{T_s^{I,J}(R/\mathfrak{p}) \mid \mathfrak{p} \in \text{Supp}(M)\}$ .*

One can see that the subcategories of finitely generated  $R$ -modules, minimax  $R$ -modules, minimax and  $(I, J)$ -cofinite  $R$ -modules, weakly Laskerian  $R$ -modules, and Matlis reflexive  $R$ -modules are examples of Serre subcategory. So, this paper recovers some results regarding the local cohomology  $R$ -modules that have appeared in different papers (see for instance [3], [5] and [13]).

## 2. THE RESULTS

This section is started with the following definition.

**Definition 2.1.** ( see [1, Definition 3.1]) A Serre subcategory of the category of  $R$ -modules is said to be a Melkersson subcategory with respect to the ideal  $\mathfrak{a}$ , if for any  $\mathfrak{a}$ -torsion  $R$ -module  $X$ ,  $(0 :_X \mathfrak{a})$  is in  $S$  implies that  $X$  is in  $S$ . Examples are given by the class of Artinian modules, minimax and  $\mathfrak{a}$ -cofinite modules.

Also, we say that  $S$  is a Melkersson subcategory with respect to the pair of ideals  $(I, J)$ , if for an  $(I, J)$ -torsion  $R$ -module  $X$ ,  $(0 :_X I)$  is in  $S$  implies that  $X$  is in  $S$ . Obviously, if  $S$  is a Melkersson subcategory with respect to  $(I, J)$ ,

then  $S$  is a Melkersson subcategory with respect to all ideals of  $\widetilde{W}(I, J)$ , where  $\widetilde{W}(I, J)$  denote the set of ideals  $\mathfrak{a}$  of  $R$  such that  $I^n \subseteq \mathfrak{a} + J$  for some integer  $n$ .

**Proposition 2.2.** *Let  $S$  be a Melkersson subcategory with respect to the ideal  $I$  and  $t$  an integer. Let  $T$  be an  $R$ -module such that  $\text{Ext}_R^i(R/I, T)$  is in  $S$  for all  $i < t$ . Then  $H_I^i(T)$  is in  $S$  for all  $i < t$ . Particularly, for an  $R$ -module  $M$ , the module  $H_I^i(H_{I,J}^j(M))$  is in  $S$  for all  $i$  and  $j < T_{I,J}^s(M)$ .*

*Proof.* We prove the theorem by induction on  $i$ . It is straightforward to see that the result is true when  $i = 0$ . Suppose that  $0 < i$  and that the result has been proved for  $i - 1$ . It easily follows from the exact sequence

$$0 \longrightarrow \Gamma_I(T) \longrightarrow T \longrightarrow T/\Gamma_I(T) \longrightarrow 0,$$

that  $\text{Ext}_R^i(R/I, T)$  is in  $S$  if and only if  $\text{Ext}_R^i(R/I, T/\Gamma_I(T))$  is in  $S$ . Also, by [2, Corollary 2.1.7],  $H_I^i(M) \cong H_I^i(M/\Gamma_I(M))$  for all  $i > 0$ . Therefore we assume that  $\Gamma_I(T) = 0$ . Now, we apply Melkersson's technic [12], so let  $E$  be an injective envelope of  $T$ . Then  $\Gamma_I(E) = \text{Hom}(R/I, E) = 0$ . Put  $L = E/T$  and consider the exact sequence

$$0 \longrightarrow T \longrightarrow E \longrightarrow L \longrightarrow 0.$$

We obtain isomorphisms;  $H_I^i(T) \cong H_I^{i-1}(L)$  and  $\text{Ext}_R^i(R/I, T) \cong \text{Ext}_R^{i-1}(R/I, L)$  for all  $i > 0$ . Use the induction hypothesis applied to  $L$ , and conclude that the  $H_I^i(T)$  is in  $S$  for all  $i < t$ . It therefore follows, in view of the definition of  $T_{I,J}^s(M)$ , that  $H_I^i(H_{I,J}^j(M))$  is in  $S$  for all  $j < T_{I,J}^s(M)$ .  $\square$

**EXAMPLE 2.3.** In Theorem 2.2, the assumption  $S$  is Melkersson subcategory is necessary. To see this, let  $(R, m)$  be a local ring, and let  $M$  be a non-zero, finitely generated  $R$ -module of dimension  $n > 0$ . Then  $H_m^n(M)$  is not finitely generated (see [2, Corollary 7.3.3]).

The next theorem recovers the Theorem 2.5 of [3] and Theorem 2.2 of [17].

**Theorem 2.4.** *Let  $M$  be in  $S$  and  $j \leq T_{I,J}^s(M) = t$ . Then*

- (i)  $\text{Ext}_R^i(R/I, H_{I,J}^j(M))$  is in  $S$  for all  $i = 0, 1$ .
- (ii)  $H_I^i(H_{I,J}^j(M))$  is in  $S$  for all  $i = 0, 1$ , if  $S$  is a Melkersson subcategory with respect to  $I$ .

*Proof.* (i) Consider the functors  $F(-) = \text{Hom}_R(R/I, -)$  and  $G(-) = \Gamma_{I,J}(-)$ . Then one has  $FG(-) = \text{Hom}(R/I, -)$ . So, by [14, Theorem 11.38], there is a Grothendieck's spectral sequence

$$E_2^{p,q} = \text{Ext}_R^p(R/I, H_{I,J}^q(M)) \xrightarrow{p} \text{Ext}_R^{p+q}(R/I, M).$$

By using an argument similar to the proof of [8, Theorem 2.2], we obtain that  $\text{Ext}_R^i(R/I, H_{I,J}^j(M))$  is in  $S$  for all  $i = 0, 1$ . This completes the proof.

(ii) Using [15, Theorem 11.38] there exists a Grothendieck's spectral sequence

$$E_2^{p,q} = H_I^p(H_{I,J}^q(M)) \xrightarrow{p} H_I^{p+q}(M).$$

Also, there is a bound filtration  $0 = \varphi^{t+1}H^t \subseteq \varphi^t H^t \subseteq \dots \subseteq \varphi^0 H^t = H_I^t(M)$  such that  $E_\infty^{i,t-i} \cong \frac{\varphi^i H^t}{\varphi^{i+1} H^t}$  for all  $0 \leq i \leq t$ . By the hypotheses with proposition 2.2  $H_I^i(M)$  is in  $S$  for all  $i$  and hence  $E_\infty^{p,q}$  is in  $S$  for all  $p, q$ . Note that  $E_\infty^{p,q} = E_r^{p,q}$  for large  $r$  and each  $p, q$ . It follows that there is an integer  $\ell \geq 2$  such that  $E_r^{p,q}$  is in  $S$  for all  $r \geq \ell$ . We now argue by descending induction on  $\ell$ . Now, assume that  $2 < \ell < r$  and that the claim holds for  $\ell$ . Since  $E_r^{p,q}$  is in a subquotient of  $E_2^{p,q}$  for all  $p, q \in \mathbb{N}_0$ , the hypotheses give  $E_r^{p+r,t-r+1}$  is in  $S$  for all  $r \geq 2$ . In addition,  $E_\ell^{p,t} = \frac{\ker d_{\ell-1}^{p,t}}{\text{im} d_{\ell-1}^{p-\ell+1,t+\ell-2}}$  and  $\text{im} d_{\ell-1}^{p-\ell+1,t+\ell-2} = 0$  for  $p = 0, 1$ , it follows that  $\ker d_{\ell-1}^{p,t}$  is in  $S$  for all  $\ell > 2$  and  $p = 0, 1$ . Let  $r \geq 2$  and  $p = 0, 1$ , we consider the sequence

$$0 \longrightarrow \ker d_r^{p,t} \longrightarrow E_r^{p,t} \longrightarrow E_r^{p+r,t-r+1}.$$

Since both  $\ker d_{\ell-1}^{p,t}$  and  $E_{\ell-1}^{p+r,t-r+1}$  are in  $S$ , it follows that  $E_{\ell-1}^{p,t}$  is in  $S$  for  $p = 0, 1$ . This completes the inductive step.  $\square$

**Proposition 2.5.** *Let  $M$  be in  $S$  such that  $\dim(M/JM) \leq 1$ . Then  $\text{Ext}_R^i(R/I, H_{I,J}^j(M))$  is in  $S$*

*Proof.* Consider the following spectral sequence

$$E_2^{p,q} := \text{Ext}_R^p(R/I, H_{I,J}^q(M)) \xrightarrow{p} \text{Ext}_R^{p+q}(R/I, M) = H^{p+q}.$$

In view of [16, Theorem 4.3]  $E_2^{p,q} = 0$  unless  $q = 0, 1$ . It follows that the exact sequence

$$H^{p+1} \longrightarrow E_2^{p+1,0} \longrightarrow E_2^{p-1,1} \longrightarrow H^p \longrightarrow E_2^{p,0} \longrightarrow E_2^{p-2,1} \longrightarrow H^{p-1}$$

which in turn yields the exact sequence

$$\begin{aligned} \text{Ext}_R^{p+1}(R/I, M) &\longrightarrow \text{Ext}_R^{p+1}(R/I, \Gamma_{I,J}(M)) \longrightarrow \text{Ext}_R^{p-1}(R/I, H_{I,J}^1(M)) \longrightarrow \\ \text{Ext}_R^p(R/I, M) &\longrightarrow \text{Ext}_R^p(R/I, \Gamma_{I,J}(M)) \longrightarrow \text{Ext}_R^{p-2}(R/I, H_{I,J}^1(M)). \end{aligned}$$

Since, by our assumption, the  $R$ -modules  $\text{Ext}_R^i(R/I, \Gamma_{I,J}(M))$  and  $\text{Ext}_R^i(R/I, M)$  are in  $S$  for all  $i$  and hence the result follows.  $\square$

**Corollary 2.6.** *Let  $S$  be a Melkersson subcategory of the category of  $R$ -modules. Let  $M$  be in  $S$  and  $\dim(M/JM) \leq 1$ . Then  $H_I^i(H_{I,J}^j(M))$  is in  $S$  for all  $i$  and  $j$ .*

*Proof.* The result follows from the part (ii) of Theorem (2.4) and argument similar to the proof of Proposition 2.5.

Recall that an  $R$ -module  $M$  is weakly Laskerian if any quotient of  $M$  has a finitely many associated prime ideals. This holds, by employing a method of proof which is similar to that used in [2, Lemma 2.1.1],  $M$  is a  $\mathfrak{a}$ -torsion-free if

and only if  $\mathfrak{a}$  contains a non-zero-divisor on  $M$ . Clearly, any finitely generated module and any minimax module are weakly Laskerian modules.

In addition, by using an argument similar to the proof of [11, Theorem 6.4], there exists a chain  $0 = M_0 \subset M_1 \cdots \subset M_n = M$  of submodules of  $M$  such that for each  $i$  we have  $M_i/M_{i-1} = \mathfrak{p}_i$  with  $\mathfrak{p}_i \in \text{Supp}(M)$ .  $\square$

**Theorem 2.7.** *Let  $\mathfrak{a} \in \widetilde{W}(I, J)$  and  $M$  be a Weakly Laskerian module. Then  $\text{Ext}_R^i(R/\mathfrak{a}, M)$  is in  $S$  for all  $0 \leq i < T_{I,J}^s(M)$ .*

*Proof.* It follows by using induction on  $T_{I,J}^s(M)$ .  $\square$

**Theorem 2.8.** *Let  $\mathfrak{a} \in \widetilde{W}(I, J)$  and  $S$  be a Melkersson subcategory with respect to the ideal  $\mathfrak{a}$ . Suppose  $M$  is Weakly Laskerian module. Then  $H_{\mathfrak{a}}^i(M) \in S$  for all  $i < T_{I,J}^s(M)$ .*

*Proof.* By using the induction on  $t = T_{I,J}^s(M)$ , the theorem is proved. It is straightforward to see that the result is true when  $t = 1$ . Suppose that  $t > 1$ , and the result holds for the case  $t - 1$ . Since  $H_{I,J}^i(M) \cong H_{I,J}^i(M/\Gamma_{I,J}(M))$  for all  $i > 0$ , we may replace  $M$  by  $M/\Gamma_{I,J}(M)$  and hence assume that there is an element  $x \in \mathfrak{a}$ , such that  $x$  is a non-zero divisor on  $M$ . The exact sequence  $0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$  induces two exact sequences

$$\begin{aligned} \rightarrow H_{I,J}^{i-1}(M/xM) \rightarrow H_{I,J}^i(M) \xrightarrow{x} H_{I,J}^i(M) \rightarrow H_{I,J}^i(M/xM) \quad \text{and} \\ \rightarrow H_{\mathfrak{a}}^{i-1}(M/xM) \rightarrow H_{\mathfrak{a}}^i(M) \xrightarrow{x} H_{\mathfrak{a}}^i(M) \rightarrow H_{\mathfrak{a}}^i(M/xM) \quad (*) \end{aligned}$$

of local cohomology modules. The induction hypothesis and the above sequences yield that the  $R$ -modules  $H_{\mathfrak{a}}^i(M)$  and  $H_{\mathfrak{a}}^i(M/xM)$  are in  $S$  for all  $i < t - 1$ . It suffices to show that  $H_{\mathfrak{a}}^{t-1}(M)$  is in  $S$ . Now, the exactness of  $(*)$ , in conjunction with the fact  $(0 :_{H_{\mathfrak{a}}^{t-1}(M)} \mathfrak{a}) \subseteq (0 :_{H_{\mathfrak{a}}^{t-1}(M)} x)$  and our hypotheses, show that  $H_{\mathfrak{a}}^{t-1}(M)$  is in  $S$ , this proves our claim.  $\square$

**Theorem 2.9.** *Let  $S$  be a Melkersson subcategory with respect to  $(I, J)$ . Suppose  $M$  is Weakly Laskerian module. Then  $T_{I,J}^s(M) = \inf\{T_{\mathfrak{a}}^s(M) \mid \mathfrak{a} \in \widetilde{W}(I, J)\}$ , where  $T_{\mathfrak{a}}^s(M)$  is the least non-negative integer  $i$  such that  $H_{\mathfrak{a}}^i(M)$  is not in  $S$ .*

*Proof.* It is enough, in view Definition 2.1 and Theorem 2.8, to show that,  $T_{I,J}^s(M) > i$  if  $i < T_{\mathfrak{a}}^s(M)$  for all  $\mathfrak{a} \in \widetilde{W}(I, J)$ . To do this, let  $\mathfrak{a}$  be an arbitrary ideal in  $\widetilde{W}(I, J)$ . We prove this by induction on  $i$ . It is straightforward to see that the result is true when  $i = 0$ . Suppose that  $0 < i$  and that the result has been proved for  $i - 1$ . It follows from [16, Proposition 1.4] that  $H_{I,J}^i(M) \cong H_{I,J}^i(M/\Gamma_{I,J}(M))$  for all  $i \geq 1$ . Hence, by replacing  $M$  with  $M/\Gamma_{I,J}(M)$ , we may assume that there exists an element  $x \in I$  which is a non-zero divisor on  $M$ . Now, we may consider the exact sequence  $0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$  to obtain the exact sequences

$$H_{\mathfrak{a}}^{i-1}(M/xM) \rightarrow H_{\mathfrak{a}}^i(M) \xrightarrow{x} H_{\mathfrak{a}}^i(M) \rightarrow H_{\mathfrak{a}}^i(M/xM) \quad \text{and}$$

$$H_{I,J}^{i-1}(M/xM) \longrightarrow H_{I,J}^i(M) \xrightarrow{x} H_{I,J}^i(M) \longrightarrow H_{I,J}^i(M/xM).$$

Now, one can use the above exact sequences in conjunction with the inductive hypothesis to see that  $(0 :_{H_{I,J}^i(M)} x)$  is in  $S$ . Since  $(0 :_{H_{I,J}^i(M)} I) \subseteq (0 :_{H_{I,J}^i(M)} x)$ , and hence the result follows.

This shows that the study of generalized local cohomology in a Melkersson subcategory in the upper range depends on the ideals of  $\widetilde{W}(I, J)$ .  $\square$

**Theorem 2.10.** *Let  $M$  be Weakly Laskerian module and let  $r$  be a non-negative integer such that  $H_{I,J}^r(R/\mathfrak{p}) \in S$  for all  $\mathfrak{p} \in \text{Supp}(M)$ . Then  $H_{I,J}^r(M) \in S$ .*

*Proof.* Clearly, there exists a filtration of the submodules of  $M$

$$0 \subseteq M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_\ell = M$$

such that for each  $1 \leq j \leq \ell$ , then  $M_j/M_{j-1} \cong R/\mathfrak{p}_j$ , where  $\mathfrak{p}_j \in \text{Supp}_R M$ . We use induction on  $\ell$ . When  $\ell = 1$ ,  $H_{I,J}^r(R/\mathfrak{p}) = H_{I,J}^r(M)$  is in  $S$ , where we put  $\mathfrak{p} = \mathfrak{p}_j$ . Now Suppose that  $\ell > 1$  and the result has been proved for  $\ell - 1$ . The exact sequence

$$0 \longrightarrow M_{\ell-1} \longrightarrow M_\ell \longrightarrow M_\ell/M_{\ell-1} \longrightarrow 0$$

induces the long exact sequence

$$H_{I,J}^r(M_{\ell-1}) \longrightarrow H_{I,J}^r(M_\ell) \longrightarrow H_{I,J}^r(M_\ell/M_{\ell-1}).$$

It follows that  $H_{I,J}^r(M_\ell)$  is in  $S$ . This completes the proof.  $\square$

**Definition 2.11.** ( see [17, Definition 2.1]) An  $R$ -module  $T$  is called  $(I, J)$ -cofinite if  $\text{Supp} T \subseteq W(I, J)$  and  $\text{Ext}_R^i(R/I, T)$  is a finite  $R$ -modules, for every  $i \geq 0$ . Whence according to [9, Lemma 2], the class of  $(I, J)$ -cofinite minimax (Artinian) modules is closed under taking submodules, quotients and extensions, it is a Serre subcategory of the category of  $R$ -modules.

The following result is an application of the Theorem 2.10.

**Corollary 2.12.** *Let  $M$  be a finitely generated  $R$ -module with  $\dim M = d$ . Then  $H_{I,J}^d(M)$  is Artinian and  $(I, J)$ -cofinite.*

*Proof.* Let  $S$  be the class of  $(I, J)$ -cofinite Artinian modules. It is enough, in view Theorem 2.10, to show that  $R$ -module  $H_{I,J}^d(R/\mathfrak{p})$  is Artinian and  $(I, J)$ -cofinite for all  $\mathfrak{p} \in \text{Supp} M$ . If  $J \subseteq \mathfrak{p}$ , then  $R/\mathfrak{p}$  is  $J$ -torsion and then  $H_{I,J}^d(R/\mathfrak{p}) \cong H_I^d(R/\mathfrak{p})$ . Since  $\dim R/\mathfrak{p} \leq d$ , then, in view of [12, Proposition 5.1],  $H_I^d(R/\mathfrak{p})$  is Artinian and  $I$ -cofinite. If  $J \not\subseteq \mathfrak{p}$ , then  $\dim(R/\mathfrak{p})/J(R/\mathfrak{p}) < \dim(R/\mathfrak{p}) \leq d$  and so  $H_{I,J}^d(R/\mathfrak{p}) = 0$  by [16, Theorem 4.3]. The proof is completed.  $\square$

**Theorem 2.13.** *Let  $M$  be a Weakly Laskerian module. Then  $H_{I,J}^i(R/\mathfrak{p})$  is in  $S$  for all  $i > T_s^{I,J}(M)$  and  $\mathfrak{p} \in \text{Supp}(M)$ .*

*Proof.* We use descending induction on  $i$ . Now, assume that  $i > T_s^{I,J}(M)$  and that the claim holds for  $i + 1$ . We want to show that  $H_{I,J}^i(R/\mathfrak{p})$  is in  $S$  for all  $\mathfrak{p} \in \text{Supp}(M)$ . Suppose the contrary. We set:  $A = \{\mathfrak{p} \mid \mathfrak{p} \in \text{Supp}M, H_{I,J}^i(R/\mathfrak{p}) \text{ is not in } S\}$ . Clearly  $A \neq \emptyset$ ; it follows that the set  $A$  has a maximal element, let  $\mathfrak{p}$  be one such. Since  $\mathfrak{p} \in \text{Supp}(M)$ , there is a non-zero map  $f : M \rightarrow R/\mathfrak{p}$ . The exact sequence  $0 \rightarrow \text{Ker}f \rightarrow M \rightarrow \text{Im}f \rightarrow 0$ , yields the exact sequence

$$H_{I,J}^i(M) \rightarrow H_{I,J}^i(\text{Im}f) \rightarrow H_{I,J}^{i+1}(\text{Ker}f).$$

Since  $\text{Supp}(\text{Ker}f) \subset \text{Supp}(M)$ , it follows from the inductive hypothesis that the  $R$ -module  $H_{I,J}^{i+1}(R/\mathfrak{p})$  is in  $S$  for all  $\mathfrak{p} \in \text{Supp}(\text{ker } f)$ , so, that, in view of the Theorem 2.10, and the above exact sequence, the  $R$ -module  $H_{I,J}^i(\text{Im}f)$  is in  $S$ . There is a filtration

$$0 = N_t \subset N_{t-1} \subset N_{t-2} \subset \dots \subset N_0 = \text{Coker}f$$

of submodules of  $\text{Coker}f$ , such that for each  $0 \leq i \leq t$ ,  $N_{i-1}/N_i \cong R/\mathfrak{q}_i$  where  $\mathfrak{q}_i \in \text{Supp}(\text{Coker}f)$ . Then by maximality of  $\mathfrak{p}$ ,  $H_{I,J}^i(R/\mathfrak{q}_i)$  is in  $S$ . Next the exact sequence  $0 \rightarrow \text{Im}f \rightarrow R/\mathfrak{p} \rightarrow \text{Coker}f \rightarrow 0$ , yields the exact sequence

$$H_{I,J}^i(\text{Im}f) \rightarrow H_{I,J}^i(R/\mathfrak{p}) \rightarrow H_{I,J}^i(\text{Coker}f).$$

It follows that  $H_{I,J}^i(R/\mathfrak{p})$  is in  $S$ , which is a contradiction. □

**Lemma 2.14.** *If  $N$  and  $M$  are Weakly Laskerian modules such that  $\text{Supp}(N) \subseteq \text{Supp}(M)$ , then  $T_s^{I,J}(N) \leq T_s^{I,J}(M)$ . In particular, if  $\text{Supp}(N) = \text{Supp}(M)$  then  $T_s^{I,J}(N) = T_s^{I,J}(M)$ .*

*Proof.* It is enough to show that  $H_{I,J}^i(N)$  is in  $S$  for all finite  $R$ -module  $N$  with  $\text{Supp}N \subseteq \text{Supp}M$  and for all  $i > T_s^{I,J}(M)$ . In view of the previous theorem,  $H_{I,J}^i(R/\mathfrak{p})$  is in  $S$  for all  $\mathfrak{p} \in \text{Supp}(M)$ . Now, since  $\text{Supp}(N) \subseteq \text{Supp}(M)$ , the result follows by Theorem 2.10. □

As an immediate result of Theorems (2.13) and (2.10), we have the following Corollary. This shows that the study of generalized local cohomology of Weakly Laskerian module  $M$  in a Serre subcategory in the lower range depends just on the support of module  $M$ .

**Corollary 2.15.** *Let  $M$  be a Weakly Laskerian module. Then*

$$T_s^{I,J}(M) = \text{Sup}\{T_s^{I,J}(R/\mathfrak{p}) \mid \mathfrak{p} \in \text{Supp}(M)\}.$$

**Theorem 2.16.** *Let  $T_s^{I,J}(M) > 0$  and  $\mathfrak{a} \in \widetilde{W}(I, J)$ . If  $M$  has finite krull dimension, then  $H_{I,J}^i(M)/\mathfrak{a}H_{I,J}^i(M)$  is in  $S$  for all  $i \geq T_s^{I,J}(M) = t$ .*

*Proof.* When  $i > T_s^{I,J}(M)$ , the result is clearly, it is enough to show that  $H_{I,J}^i(M)/\mathfrak{a}H_{I,J}^i(M)$  is in  $S$ . We proceed by induction on  $\dim M = n$ . If  $n = 0$ , then  $M$  is  $\mathfrak{m}$ -torsion and there is nothing to prove. So let  $n > 0$  and

suppose that the result has been proved for any finitely generated module  $N$  with  $\dim(N) = n - 1$ . Since  $H_{I,J}^i(M) \cong H_{I,J}^i(M/\Gamma_{I,J}(M))$  for  $i > 0$ , we can assume that  $M$  is  $(I, J)$ -torsion. Thus, there is an element  $x \in \mathfrak{a}$ , such that  $x$  is a non-zero divisor on  $M$ . Now, one can complete the proof by using an argument similar to the proof of [7, Theorem 3.3].  $\square$

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