

Serre Subcategories and Local Cohomology Modules with Respect to a Pair of Ideals

Fatemeh Dehghani-Zadeh

Department of Mathematics, Islamic Azad University,
Yazd Branch, Yazd, Iran.

E-mail: fdzadeh@gmail.com, dehghanizadeh@iauyazd.ac.ir

ABSTRACT. This paper is concerned with the relation between local cohomology modules defined by a pair of ideals and the Serre subcategories of the category of modules. We characterize the membership of local cohomology modules in a certain Serre subcategory from lower range or upper range.

Keywords: Local cohomology modules, pair of ideals, Serre subcategory.

2000 Mathematics subject classification: 13D45, 13E05, 13E99.

1. INTRODUCTION

Throughout this paper, R is denoted a commutative Noetherian ring, I and J are denoted two ideals of R , and M is an R -module. We refer the reader to [2] and [4] for any unexplained terminology.

As a generalization of the ordinary local cohomology modules, Takahashi, Yoshino and Yoshizawa [16] introduced the local cohomology modules with respect to a pair of ideals (I, J) . To be more precise, let $W(I, J) = \{\mathfrak{p} \in \text{Spec}(R) \mid I^n \subseteq \mathfrak{p} + J \text{ for some positive integer } n\}$. Then for an R -module M , the (I, J) -torsion submodule $\Gamma_{I,J}(M)$ of M , which consists of all elements x of M with $\text{Supp}Rx \subseteq W(I, J)$, is considered. It is known, $\Gamma_{I,J}$ is a left exact additive functor from the category of all R -modules and R -homomorphism to itself. For all integer i , the i -th local cohomology functor $H_{I,J}^i$ with respect

to (I, J) is defined to be the i -th right derived functor of $\Gamma_{I,J}$. The i -th local cohomology module of M with respect to (I, J) is denoted by $H_{I,J}^i(M)$. When $J = 0$, then $H_{I,J}^i$ coincides with the usual local cohomology functor H_I^i with the support in the closed subset $V(I)$.

The study of this generalized local cohomology modules was continued by many authors (see for example [5], [6], [9] and [10], [14]).

Recall that a class of R -modules is a Serre subcategory of the category of R -modules when it is closed under taking submodules, quotients and extensions. Always, S stands for a Serre subcategory of the category of R -modules.

Using the generalized local cohomology modules, we can define $T_{I,J}^s(M)$ (resp. $T_s^{I,J}(M)$) of the R -module M relative to a pair (I, J) of ideals of R by

$$T_{I,J}^s(M) = \inf\{i \in \mathbb{N} \mid H_{I,J}^i(M) \text{ is not in } S\},$$

$$\text{(resp. } T_s^{I,J}(M) = \text{Sup}\{i \in \mathbb{N} \mid H_{I,J}^i(M) \text{ is not in } S\})$$

with the usual convention that the infimum (resp. Supremum) of the empty set of integers interpreted as $+\infty$ (resp. $-\infty$).

Our objective in this paper is to investigate the notions $T_{I,J}^s(M)$ and $T_s^{I,J}(M)$. we prove the following,

Theorem 1.1. *Let S be a Melkersson subcategory with respect to (I, J) . Suppose M is Weakly Laskerian module. Then $T_{I,J}^s(M) = \inf\{T_{\mathfrak{a}}^s(M) \mid \mathfrak{a} \in \widetilde{W}(I, J)\}$, where $T_{\mathfrak{a}}^s(M)$ is the least non-negative integer i such that $H_{\mathfrak{a}}^i(M)$ is not in S .*

Theorem 1.2. *Let S be a Serre subcategory and let M be a Weakly Laskerian module. Then $T_s^{I,J}(M) = \text{Sup}\{T_s^{I,J}(R/\mathfrak{p}) \mid \mathfrak{p} \in \text{Supp}(M)\}$.*

One can see that the subcategories of finitely generated R -modules, minimax R -modules, minimax and (I, J) -cofinite R -modules, weakly Laskerian R -modules, and Matlis reflexive R -modules are examples of Serre subcategory. So, this paper recovers some results regarding the local cohomology R -modules that have appeared in different papers (see for instance [3], [5] and [13]).

2. THE RESULTS

This section is started with the following definition.

Definition 2.1. (see [1, Definition 3.1]) A Serre subcategory of the category of R -modules is said to be a Melkersson subcategory with respect to the ideal \mathfrak{a} , if for any \mathfrak{a} -torsion R -module X , $(0 :_X \mathfrak{a})$ is in S implies that X is in S . Examples are given by the class of Artinian modules, minimax and \mathfrak{a} -cofinite modules.

Also, we say that S is a Melkersson subcategory with respect to the pair of ideals (I, J) , if for an (I, J) -torsion R -module X , $(0 :_X I)$ is in S implies that X is in S . Obviously, if S is a Melkersson subcategory with respect to (I, J) ,

then S is a Melkersson subcategory with respect to all ideals of $\widetilde{W}(I, J)$, where $\widetilde{W}(I, J)$ denote the set of ideals \mathfrak{a} of R such that $I^n \subseteq \mathfrak{a} + J$ for some integer n .

Proposition 2.2. *Let S be a Melkersson subcategory with respect to the ideal I and t an integer. Let T be an R -module such that $\text{Ext}_R^i(R/I, T)$ is in S for all $i < t$. Then $H_I^i(T)$ is in S for all $i < t$. Particularly, for an R -module M , the module $H_I^i(H_{I,J}^j(M))$ is in S for all i and $j < T_{I,J}^s(M)$.*

Proof. We prove the theorem by induction on i . It is straightforward to see that the result is true when $i = 0$. Suppose that $0 < i$ and that the result has been proved for $i - 1$. It easily follows from the exact sequence

$$0 \longrightarrow \Gamma_I(T) \longrightarrow T \longrightarrow T/\Gamma_I(T) \longrightarrow 0,$$

that $\text{Ext}_R^i(R/I, T)$ is in S if and only if $\text{Ext}_R^i(R/I, T/\Gamma_I(T))$ is in S . Also, by [2, Corollary 2.1.7], $H_I^i(M) \cong H_I^i(M/\Gamma_I(M))$ for all $i > 0$. Therefore we assume that $\Gamma_I(T) = 0$. Now, we apply Melkersson's technic [12], so let E be an injective envelope of T . Then $\Gamma_I(E) = \text{Hom}(R/I, E) = 0$. Put $L = E/T$ and consider the exact sequence

$$0 \longrightarrow T \longrightarrow E \longrightarrow L \longrightarrow 0.$$

We obtain isomorphisms; $H_I^i(T) \cong H_I^{i-1}(L)$ and $\text{Ext}_R^i(R/I, T) \cong \text{Ext}_R^{i-1}(R/I, L)$ for all $i > 0$. Use the induction hypothesis applied to L , and conclude that the $H_I^i(T)$ is in S for all $i < t$. It therefore follows, in view of the definition of $T_{I,J}^s(M)$, that $H_I^i(H_{I,J}^j(M))$ is in S for all $j < T_{I,J}^s(M)$. \square

EXAMPLE 2.3. In Theorem 2.2, the assumption S is Melkersson subcategory is necessary. To see this, let (R, m) be a local ring, and let M be a non-zero, finitely generated R -module of dimension $n > 0$. Then $H_m^n(M)$ is not finitely generated (see [2, Corollary 7.3.3]).

The next theorem recovers the Theorem 2.5 of [3] and Theorem 2.2 of [17].

Theorem 2.4. *Let M be in S and $j \leq T_{I,J}^s(M) = t$. Then*

- (i) $\text{Ext}_R^i(R/I, H_{I,J}^j(M))$ is in S for all $i = 0, 1$.
- (ii) $H_I^i(H_{I,J}^j(M))$ is in S for all $i = 0, 1$, if S is a Melkersson subcategory with respect to I .

Proof. (i) Consider the functors $F(-) = \text{Hom}_R(R/I, -)$ and $G(-) = \Gamma_{I,J}(-)$. Then one has $FG(-) = \text{Hom}(R/I, -)$. So, by [14, Theorem 11.38], there is a Grothendieck's spectral sequence

$$E_2^{p,q} = \text{Ext}_R^p(R/I, H_{I,J}^q(M)) \xrightarrow{p} \text{Ext}_R^{p+q}(R/I, M).$$

By using an argument similar to the proof of [8, Theorem 2.2], we obtain that $\text{Ext}_R^i(R/I, H_{I,J}^j(M))$ is in S for all $i = 0, 1$. This completes the proof.

(ii) Using [15, Theorem 11.38] there exists a Grothendieck's spectral sequence

$$E_2^{p,q} = H_I^p(H_{I,J}^q(M)) \xrightarrow{p} H_I^{p+q}(M).$$

Also, there is a bound filtration $0 = \varphi^{t+1}H^t \subseteq \varphi^t H^t \subseteq \dots \subseteq \varphi^0 H^t = H_I^t(M)$ such that $E_\infty^{i,t-i} \cong \frac{\varphi^i H^t}{\varphi^{i+1} H^t}$ for all $0 \leq i \leq t$. By the hypotheses with proposition 2.2 $H_I^i(M)$ is in S for all i and hence $E_\infty^{p,q}$ is in S for all p, q . Note that $E_\infty^{p,q} = E_r^{p,q}$ for large r and each p, q . It follows that there is an integer $\ell \geq 2$ such that $E_r^{p,q}$ is in S for all $r \geq \ell$. We now argue by descending induction on ℓ . Now, assume that $2 < \ell < r$ and that the claim holds for ℓ . Since $E_r^{p,q}$ is in a subquotient of $E_2^{p,q}$ for all $p, q \in \mathbb{N}_0$, the hypotheses give $E_r^{p+r,t-r+1}$ is in S for all $r \geq 2$. In addition, $E_\ell^{p,t} = \frac{\ker d_{\ell-1}^{p,t}}{\text{im} d_{\ell-1}^{p-\ell+1,t+\ell-2}}$ and $\text{im} d_{\ell-1}^{p-\ell+1,t+\ell-2} = 0$ for $p = 0, 1$, it follows that $\ker d_{\ell-1}^{p,t}$ is in S for all $\ell > 2$ and $p = 0, 1$. Let $r \geq 2$ and $p = 0, 1$, we consider the sequence

$$0 \longrightarrow \ker d_r^{p,t} \longrightarrow E_r^{p,t} \longrightarrow E_r^{p+r,t-r+1}.$$

Since both $\ker d_{\ell-1}^{p,t}$ and $E_{\ell-1}^{p+r,t-r+1}$ are in S , it follows that $E_{\ell-1}^{p,t}$ is in S for $p = 0, 1$. This completes the inductive step. \square

Proposition 2.5. *Let M be in S such that $\dim(M/JM) \leq 1$. Then $\text{Ext}_R^i(R/I, H_{I,J}^j(M))$ is in S*

Proof. Consider the following spectral sequence

$$E_2^{p,q} := \text{Ext}_R^p(R/I, H_{I,J}^q(M)) \xrightarrow{p} \text{Ext}_R^{p+q}(R/I, M) = H^{p+q}.$$

In view of [16, Theorem 4.3] $E_2^{p,q} = 0$ unless $q = 0, 1$. It follows that the exact sequence

$$H^{p+1} \longrightarrow E_2^{p+1,0} \longrightarrow E_2^{p-1,1} \longrightarrow H^p \longrightarrow E_2^{p,0} \longrightarrow E_2^{p-2,1} \longrightarrow H^{p-1}$$

which in turn yields the exact sequence

$$\begin{aligned} \text{Ext}_R^{p+1}(R/I, M) &\longrightarrow \text{Ext}_R^{p+1}(R/I, \Gamma_{I,J}(M)) \longrightarrow \text{Ext}_R^{p-1}(R/I, H_{I,J}^1(M)) \longrightarrow \\ \text{Ext}_R^p(R/I, M) &\longrightarrow \text{Ext}_R^p(R/I, \Gamma_{I,J}(M)) \longrightarrow \text{Ext}_R^{p-2}(R/I, H_{I,J}^1(M)). \end{aligned}$$

Since, by our assumption, the R -modules $\text{Ext}_R^i(R/I, \Gamma_{I,J}(M))$ and $\text{Ext}_R^i(R/I, M)$ are in S for all i and hence the result follows. \square

Corollary 2.6. *Let S be a Melkersson subcategory of the category of R -modules. Let M be in S and $\dim(M/JM) \leq 1$. Then $H_I^i(H_{I,J}^j(M))$ is in S for all i and j .*

Proof. The result follows from the part (ii) of Theorem (2.4) and argument similar to the proof of Proposition 2.5.

Recall that an R -module M is weakly Laskerian if any quotient of M has a finitely many associated prime ideals. This holds, by employing a method of proof which is similar to that used in [2, Lemma 2.1.1], M is a \mathfrak{a} -torsion-free if

and only if \mathfrak{a} contains a non-zero-divisor on M . Clearly, any finitely generated module and any minimax module are weakly Laskerian modules.

In addition, by using an argument similar to the proof of [11, Theorem 6.4], there exists a chain $0 = M_0 \subset M_1 \cdots \subset M_n = M$ of submodules of M such that for each i we have $M_i/M_{i-1} = \mathfrak{p}_i$ with $\mathfrak{p}_i \in \text{Supp}(M)$. \square

Theorem 2.7. *Let $\mathfrak{a} \in \widetilde{W}(I, J)$ and M be a Weakly Laskerian module. Then $\text{Ext}_R^i(R/\mathfrak{a}, M)$ is in S for all $0 \leq i < T_{I,J}^s(M)$.*

Proof. It follows by using induction on $T_{I,J}^s(M)$. \square

Theorem 2.8. *Let $\mathfrak{a} \in \widetilde{W}(I, J)$ and S be a Melkersson subcategory with respect to the ideal \mathfrak{a} . Suppose M is Weakly Laskerian module. Then $H_{\mathfrak{a}}^i(M) \in S$ for all $i < T_{I,J}^s(M)$.*

Proof. By using the induction on $t = T_{I,J}^s(M)$, the theorem is proved. It is straightforward to see that the result is true when $t = 1$. Suppose that $t > 1$, and the result holds for the case $t - 1$. Since $H_{I,J}^i(M) \cong H_{I,J}^i(M/\Gamma_{I,J}(M))$ for all $i > 0$, we may replace M by $M/\Gamma_{I,J}(M)$ and hence assume that there is an element $x \in \mathfrak{a}$, such that x is a non-zero divisor on M . The exact sequence $0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$ induces two exact sequences

$$\begin{aligned} &\rightarrow H_{I,J}^{i-1}(M/xM) \rightarrow H_{I,J}^i(M) \xrightarrow{x} H_{I,J}^i(M) \rightarrow H_{I,J}^i(M/xM) \quad \text{and} \\ &\rightarrow H_{\mathfrak{a}}^{i-1}(M/xM) \rightarrow H_{\mathfrak{a}}^i(M) \xrightarrow{x} H_{\mathfrak{a}}^i(M) \rightarrow H_{\mathfrak{a}}^i(M/xM) \quad (*) \end{aligned}$$

of local cohomology modules. The induction hypothesis and the above sequences yield that the R -modules $H_{\mathfrak{a}}^i(M)$ and $H_{\mathfrak{a}}^i(M/xM)$ are in S for all $i < t - 1$. It suffices to show that $H_{\mathfrak{a}}^{t-1}(M)$ is in S . Now, the exactness of $(*)$, in conjunction with the fact $(0 :_{H_{\mathfrak{a}}^{t-1}(M)} \mathfrak{a}) \subseteq (0 :_{H_{\mathfrak{a}}^{t-1}(M)} x)$ and our hypotheses, show that $H_{\mathfrak{a}}^{t-1}(M)$ is in S , this proves our claim. \square

Theorem 2.9. *Let S be a Melkersson subcategory with respect to (I, J) . Suppose M is Weakly Laskerian module. Then $T_{I,J}^s(M) = \inf\{T_{\mathfrak{a}}^s(M) \mid \mathfrak{a} \in \widetilde{W}(I, J)\}$, where $T_{\mathfrak{a}}^s(M)$ is the least non-negative integer i such that $H_{\mathfrak{a}}^i(M)$ is not in S .*

Proof. It is enough, in view Definition 2.1 and Theorem 2.8, to show that, $T_{I,J}^s(M) > i$ if $i < T_{\mathfrak{a}}^s(M)$ for all $\mathfrak{a} \in \widetilde{W}(I, J)$. To do this, let \mathfrak{a} be an arbitrary ideal in $\widetilde{W}(I, J)$. We prove this by induction on i . It is straightforward to see that the result is true when $i = 0$. Suppose that $0 < i$ and that the result has been proved for $i - 1$. It follows from [16, Proposition 1.4] that $H_{I,J}^i(M) \cong H_{I,J}^i(M/\Gamma_{I,J}(M))$ for all $i \geq 1$. Hence, by replacing M with $M/\Gamma_{I,J}(M)$, we may assume that there exists an element $x \in I$ which is a non-zero divisor on M . Now, we may consider the exact sequence $0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$ to obtain the exact sequences

$$H_{\mathfrak{a}}^{i-1}(M/xM) \rightarrow H_{\mathfrak{a}}^i(M) \xrightarrow{x} H_{\mathfrak{a}}^i(M) \rightarrow H_{\mathfrak{a}}^i(M/xM) \quad \text{and}$$

$$H_{I,J}^{i-1}(M/xM) \longrightarrow H_{I,J}^i(M) \xrightarrow{x} H_{I,J}^i(M) \longrightarrow H_{I,J}^i(M/xM).$$

Now, one can use the above exact sequences in conjunction with the inductive hypothesis to see that $(0 :_{H_{I,J}^i(M)} x)$ is in S . Since $(0 :_{H_{I,J}^i(M)} I) \subseteq (0 :_{H_{I,J}^i(M)} x)$, and hence the result follows.

This shows that the study of generalized local cohomology in a Melkersson subcategory in the upper range depends on the ideals of $\widetilde{W}(I, J)$. \square

Theorem 2.10. *Let M be Weakly Laskerian module and let r be a non-negative integer such that $H_{I,J}^r(R/\mathfrak{p}) \in S$ for all $\mathfrak{p} \in \text{Supp}(M)$. Then $H_{I,J}^r(M) \in S$.*

Proof. Clearly, there exists a filtration of the submodules of M

$$0 \subseteq M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_\ell = M$$

such that for each $1 \leq j \leq \ell$, then $M_j/M_{j-1} \cong R/\mathfrak{p}_j$, where $\mathfrak{p}_j \in \text{Supp}_R M$. We use induction on ℓ . When $\ell = 1$, $H_{I,J}^r(R/\mathfrak{p}) = H_{I,J}^r(M)$ is in S , where we put $\mathfrak{p} = \mathfrak{p}_j$. Now Suppose that $\ell > 1$ and the result has been proved for $\ell - 1$. The exact sequence

$$0 \longrightarrow M_{\ell-1} \longrightarrow M_\ell \longrightarrow M_\ell/M_{\ell-1} \longrightarrow 0$$

induces the long exact sequence

$$H_{I,J}^r(M_{\ell-1}) \longrightarrow H_{I,J}^r(M_\ell) \longrightarrow H_{I,J}^r(M_\ell/M_{\ell-1}).$$

It follows that $H_{I,J}^r(M_\ell)$ is in S . This completes the proof. \square

Definition 2.11. (see [17, Definition 2.1]) An R -module T is called (I, J) -cofinite if $\text{Supp} T \subseteq W(I, J)$ and $\text{Ext}_R^i(R/I, T)$ is a finite R -modules, for every $i \geq 0$. Whence according to [9, Lemma 2], the class of (I, J) -cofinite minimax (Artinian) modules is closed under taking submodules, quotients and extensions, it is a Serre subcategory of the category of R -modules.

The following result is an application of the Theorem 2.10.

Corollary 2.12. *Let M be a finitely generated R -module with $\dim M = d$. Then $H_{I,J}^d(M)$ is Artinian and (I, J) -cofinite.*

Proof. Let S be the class of (I, J) -cofinite Artinian modules. It is enough, in view Theorem 2.10, to show that R -module $H_{I,J}^d(R/\mathfrak{p})$ is Artinian and (I, J) -cofinite for all $\mathfrak{p} \in \text{Supp} M$. If $J \subseteq \mathfrak{p}$, then R/\mathfrak{p} is J -torsion and then $H_{I,J}^d(R/\mathfrak{p}) \cong H_I^d(R/\mathfrak{p})$. Since $\dim R/\mathfrak{p} \leq d$, then, in view of [12, Proposition 5.1], $H_I^d(R/\mathfrak{p})$ is Artinian and I -cofinite. If $J \not\subseteq \mathfrak{p}$, then $\dim(R/\mathfrak{p})/J(R/\mathfrak{p}) < \dim(R/\mathfrak{p}) \leq d$ and so $H_{I,J}^d(R/\mathfrak{p}) = 0$ by [16, Theorem 4.3]. The proof is completed. \square

Theorem 2.13. *Let M be a Weakly Laskerian module. Then $H_{I,J}^i(R/\mathfrak{p})$ is in S for all $i > T_s^{I,J}(M)$ and $\mathfrak{p} \in \text{Supp}(M)$.*

Proof. We use descending induction on i . Now, assume that $i > T_s^{I,J}(M)$ and that the claim holds for $i + 1$. We want to show that $H_{I,J}^i(R/\mathfrak{p})$ is in S for all $\mathfrak{p} \in \text{Supp}(M)$. Suppose the contrary. We set: $A = \{\mathfrak{p} \mid \mathfrak{p} \in \text{Supp}M, H_{I,J}^i(R/\mathfrak{p}) \text{ is not in } S\}$. Clearly $A \neq \emptyset$; it follows that the set A has a maximal element, let \mathfrak{p} be one such. Since $\mathfrak{p} \in \text{Supp}(M)$, there is a non-zero map $f : M \rightarrow R/\mathfrak{p}$. The exact sequence $0 \rightarrow \text{Ker}f \rightarrow M \rightarrow \text{Im}f \rightarrow 0$, yields the exact sequence

$$H_{I,J}^i(M) \rightarrow H_{I,J}^i(\text{Im}f) \rightarrow H_{I,J}^{i+1}(\text{Ker}f).$$

Since $\text{Supp}(\text{Ker}f) \subset \text{Supp}(M)$, it follows from the inductive hypothesis that the R -module $H_{I,J}^{i+1}(R/\mathfrak{p})$ is in S for all $\mathfrak{p} \in \text{Supp}(\text{ker } f)$, so, that, in view of the Theorem 2.10, and the above exact sequence, the R -module $H_{I,J}^i(\text{Im}f)$ is in S . There is a filtration

$$0 = N_t \subset N_{t-1} \subset N_{t-2} \subset \dots \subset N_0 = \text{Coker}f$$

of submodules of $\text{Coker}f$, such that for each $0 \leq i \leq t$, $N_{i-1}/N_i \cong R/\mathfrak{q}_i$ where $\mathfrak{q}_i \in \text{Supp}(\text{Coker}f)$. Then by maximality of \mathfrak{p} , $H_{I,J}^i(R/\mathfrak{q}_i)$ is in S . Next the exact sequence $0 \rightarrow \text{Im}f \rightarrow R/\mathfrak{p} \rightarrow \text{Coker}f \rightarrow 0$, yields the exact sequence

$$H_{I,J}^i(\text{Im}f) \rightarrow H_{I,J}^i(R/\mathfrak{p}) \rightarrow H_{I,J}^i(\text{Coker}f).$$

It follows that $H_{I,J}^i(R/\mathfrak{p})$ is in S , which is a contradiction. □

Lemma 2.14. *If N and M are Weakly Laskerian modules such that $\text{Supp}(N) \subseteq \text{Supp}(M)$, then $T_s^{I,J}(N) \leq T_s^{I,J}(M)$. In particular, if $\text{Supp}(N) = \text{Supp}(M)$ then $T_s^{I,J}(N) = T_s^{I,J}(M)$.*

Proof. It is enough to show that $H_{I,J}^i(N)$ is in S for all finite R -module N with $\text{Supp}N \subseteq \text{Supp}M$ and for all $i > T_s^{I,J}(M)$. In view of the previous theorem, $H_{I,J}^i(R/\mathfrak{p})$ is in S for all $\mathfrak{p} \in \text{Supp}(M)$. Now, since $\text{Supp}(N) \subseteq \text{Supp}(M)$, the result follows by Theorem 2.10. □

As an immediate result of Theorems (2.13) and (2.10), we have the following Corollary. This shows that the study of generalized local cohomology of Weakly Laskerian module M in a Serre subcategory in the lower range depends just on the support of module M .

Corollary 2.15. *Let M be a Weakly Laskerian module. Then*

$$T_s^{I,J}(M) = \text{Sup}\{T_s^{I,J}(R/\mathfrak{p}) \mid \mathfrak{p} \in \text{Supp}(M)\}.$$

Theorem 2.16. *Let $T_s^{I,J}(M) > 0$ and $\mathfrak{a} \in \widetilde{W}(I, J)$. If M has finite krull dimension, then $H_{I,J}^i(M)/\mathfrak{a}H_{I,J}^i(M)$ is in S for all $i \geq T_s^{I,J}(M) = t$.*

Proof. When $i > T_s^{I,J}(M)$, the result is clearly, it is enough to show that $H_{I,J}^i(M)/\mathfrak{a}H_{I,J}^i(M)$ is in S . We proceed by induction on $\dim M = n$. If $n = 0$, then M is \mathfrak{m} -torsion and there is nothing to prove. So let $n > 0$ and

suppose that the result has been proved for any finitely generated module N with $\dim(N) = n - 1$. Since $H_{I,J}^i(M) \cong H_{I,J}^i(M/\Gamma_{I,J}(M))$ for $i > 0$, we can assume that M is (I, J) -torsion. Thus, there is an element $x \in \mathfrak{a}$, such that x is a non-zero divisor on M . Now, one can complete the proof by using an argument similar to the proof of [7, Theorem 3.3]. \square

ACKNOWLEDGMENTS

I would like to thank the referee for his/her careful reading of the paper.

REFERENCES

1. M. Aghapournahr, A. J. Taherizadeh, A. Vahidi, Extension functors of local cohomology modules, *Bulletin of the Iranian Mathematical Society*, **37**(3), (2011), 117-134.
2. M. P. Brodmann, Sharp. R. Y, *Local cohomology: An Algebraic introduction with geometric applications*, Cambridge studies in Advanced Mathematics, 60, Cambridge University Press, Cambridge, 1998.
3. K. Borna Lorestani, P. Sahandi, S. Yassemi, Artinian local cohomology modules, *Canadian Mathematical Bulletin*, **50**(4), (2007), 598-602.
4. W. Bruns, J. Herzog, *Cohen-Macaulay Rings*, Cambridge studies in Advanced Mathematics, 39, Cambridge University Press, Cambridge, 1993.
5. L. Chu, Q. Wang, Some results on local cohomology modules defined by a pair of ideals, *Kyoto Journal of Mathematics*, **49**(1), (2009), 193-200.
6. L. Chu, Top local cohomology modules with respect to a pair of ideals, *Proceedings of the American Mathematical Society*, **139**(3), (2011), 777-782.
7. F. Dehghani-Zadeh, On the finiteness properties of generalized local cohomology modules, *International Electronic Journal of Algebra*, **10**, (2011), 113-122.
8. F. Dehghani-Zadeh, Finiteness properties generalized local cohomology with respect to an ideal containing the irrelevant ideal, *Journal of the Korean Mathematical Society*, **49**(6), (2012), 1215-1227.
9. F. Dehghani-Zadeh, Results of generalized local cohomology with respect to a pair of ideals, *International Journal of Pure and Applied Mathematics*, **99** (4), (2015), 471-476.
10. A. Mafi, A. Pour Eshmanan Talemi, Results of certain local cohomology modules, *Bulletin of the Korean Mathematical Society*, **51**(3), (2014), 653-657.
11. H. Matsumura, *Commutative Ring Theory*, Cambridge University Press 1986.
12. L. Melkersson, Modules cofinite with respect to an ideal, *Journal of Algebra*, **285**(2), (2005), 619-668.
13. Sh. Payrovi, M. Lotfi Parsa, Artinianness of Local Cohomology Modules Defined by a pair of Ideals, *Bulletin of the Malaysian Mathematical Sciences Society*, **35**(4), (2012), 877-883.
14. A. Pour Eshmanan Talemi, A. Tehranian, Local cohomology with respect to a cohomologically complete intersection pair of ideals, *Iranian Journal of Mathematical Sciences and Informatics*, **9**(2), (2014), 7-13.
15. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, orlando, 1979.
16. R. Takahashi, Y. Yoshino, T. Yoshinawa, Local cohomology based on a nonclosed support defined by a pair of ideals, *Journal of Pure and Applied Algebra*, **213**(4), (2009), 582-600.
17. A. Tehranian, A. Pour Eshmanan Talemi, Cofiniteness of local cohomology based on a non-closed support defined by a pair of ideals, *Bulletin of the Iranian Mathematical Society*, **36**(2), (2010), 145-155.