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On Ricci identities for submanifolds in the 2-osculator bundle

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Abstract. It is the purpose of the present paper to outline an introduction in theory of embeddings in the 2-osculator bundle. First, we recall the notion of 2-osculator bundle ([9], [2], [4]) and the notion of submanifolds in the 2-osculator bundle ([9]). A moving frame is constructed. The induced connections and the relative covariant derivation are discussed in the fourth and fifth section ([15], [16]). The Ricci identities for the deflection tensors are presented in the seventh section.

Keywords: nonlinear connection, linear connection, induced linear connection, d-torsions and d-curvatures.

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1. Introduction

Generally, the geometries of higher order defined as the study of the category of bundles of jets $\left(J_0^k M, \pi^k, M\right)$ is based on a direct approach of the properties of objects and morphisms in this category, without local coordinates.

But, many mathematical models from Lagrangian Mechanics, Theoretical Physics and Variational Calculus used multivariate Lagrangians of higher order accelerations.

From here one can see the reason of construction of the geometry of the total space of the bundle of higher accelerations (or the osculator bundle of higher order) in local coordinates.

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As far we know the general theory of submanifolds (in particular the Finsler submanifolds [5]) is far from being settled ([11], [5], [6], [7]). In [10] and [11] R.Miron and M. Anastasiei give the theory of subspaces in generalized Lagrange spaces. Also, in [8] and [9] R. Miron presented the theory of subspaces in higher order Finsler and Lagrange spaces respectively.

This article is inspired by the original construction of the higher order geometry given by R. Miron and Gh. Atanasiu ([9], [12], [13], [14]) and the new aspects give by Gh. Atanasiu in [1] and [2].

If $\check{M}$ is an immersed manifold in manifold M, a nonlinear connection on Osc^2M induce a nonlinear connection $\check{N}$ on $Osc^2\check{M}$. We take the canonical N-linear metric connection D on the manifold Osc^2M. This allows obtain of the induced tangent and normal connections and the introduction of the relative covariant derivation in the algebra of d-tensor fields ([15]). If in [9] R. Miron use the canonical metrical N-linear connection of the space $\mathrm{L}^{(2)n}$ having *three* coeficients $\left(F^i_{jk}, \underset{1}{C}{}^i_{jk}, \underset{2}{C}{}^i_{jk}\right)$, in this article we take the canonical metrical N-linear connection of the manifold Osc^2M having *nine* coeficients $\left(\underset{(i0)}{L}{}^a_{bc}, \underset{(i1)}{C}{}^a_{bc}, \underset{(i2)}{C}{}^a_{bc}\right), (i = 0, 1, 2)$,([15], [16]).

In this paper we present the Ricci identities for the Liouville d-vector fields $z^{(1)\alpha}$ and $z^{(2)\alpha}$ on the submanifold $Osc^2\check{M}$. For the Liouville d-vector fields $z^{(1)a}$ and $z^{(2)a}$ on the manifold Osc^2M the problem was solved by professor Atanasiu Gh. in [1] and [2].

2. The 2-osculator bundle $\left(Osc^2M, \pi^2, M\right)$

Let M be a real differentiable manifold of dimension n. A point of M will be denoted by x and its local coordinate system by $(U, \varphi), \varphi(x) = (x^a)$. The indices $a, b, ...$run over the set {1,2,...,n} and Einstein convention of summarizing is adopted all over this work.

Let us consider two curves $\rho, \sigma : I \to M,$ having images in a domain of local chart $U \subset M$. We say that ρ and σ have a "contact of order 2" in a point $x_0 \in U$ if: $\rho(0) = \sigma(0) = x_0$, $(0 \in I)$, and for any function $f \in \mathcal{F}(U)$:

$$\frac{d^\beta}{dt^\beta}(f\circ\rho)(t)\,|_{t=0} = \frac{d^\beta}{dt^\beta}(f\circ\sigma)(t)\,|_{t=0}, (\beta = 1, 2) \tag{2.1}$$

The relation "contact of order 2" is an equivalence relation on the set of smooth curves in M, which pass through the point x_0. Let $[\rho]_{x_0}$ be a class of equivalence relation. It will be called a "2-osculator space" at the point $x_0 \in M$. The set of 2-osculator spaces at the point $x_0 \in M$ will be denoted by $Osc^2_{x_0}M$, and we put

$$Osc^2M = \underset{x_0\in M}{\cup} Osc^2_{x_0}M$$

One considers the mapping $\pi^2 : Osc^2M \to M$ define by $\pi^2\left([\rho]_{x_0}\right) = x_0$. Obviously, π^2 is a surjection.

The set Osc^2M is endowed with a natural differentiable structure, induced by that of the manifold M, so that π^2 is a differentiable maping. It will be descrieb bellow.

The curve $\rho : I \to M$ $(\phi Im\rho \subset U)$ is analytically represented in the local chart (U,φ) by $x_0 = x_0^a\left(= x^a\left(0\right)\right)$. Taking the function f from (2.1), succesively equal to the coordinate functions x^a, then a representative of the class $[\rho]_{x_0}$ is given by

$$x^{*a}\left(t\right) = x^a\left(0\right) + t\frac{dx^a}{dt}\left(0\right) + \frac{1}{2}t^2\frac{d^2x^a}{dt^2}\left(0\right),\ t \in \left(-\varepsilon,\varepsilon\right) \subset I.$$

The previous polynomials are determined by the coefficients

$$x_0^a = x^a\left(0\right), y^{(1)a} = \frac{dx^a}{dt}\left(0\right), y^{(2)a} = \frac{1}{2}\frac{d^2x^a}{dt^2}\left(0\right) \tag{2.2}$$

Hence, the pair $\left(\left(\pi^2\right)^{-1}\left(U\right),\Phi\right)$, with $\Phi\left([\rho]_{x_0}\right) = \left(x_0^a, y^{(1)a}, y^{(2)a}\right) \in R^{3n}$, $\forall[\rho]_{x_0} \in \left(\pi^2\right)^{-1}\left(U\right)$ is a local chart on Osc^2M. Thus a differentiable atlas $\mathcal{A}_M$ of the diferentiable structure on the manifold M determines a differentiable atlas A_{Osc^2M} on Osc^2M and therefore the triple $\left(Osc^2M,\pi^2,M\right)$ is a differentiable bundle. We will identified the 2-osculator bundle $\left(Osc^2M,\pi^2,M\right)$ with 2-tangent bundle $\left(T^2M,\pi^2,M\right)$.

By (2.2), a transformation of local coordinates $\left(x^a, y^{(1)a}, y^{(2)a}\right) \to \left(\tilde{x}^a, \tilde{y}^{(1)a}, \tilde{y}^{(2)a}\right)$ on the manifold Osc^2M is given by

$$\begin{cases} \tilde{x}^a = \tilde{x}^a\left(x^1,...,x^n\right),\ \det\left(\dfrac{\partial\tilde{x}^a}{\partial x^b}\right) \neq 0 \\[2ex] \tilde{y}^{(1)a} = \dfrac{\partial\tilde{x}^a}{\partial x^b}y^{(1)b} \\[2ex] 2\tilde{y}^{(2)a} = \dfrac{\partial\tilde{y}^{(1)a}}{\partial x^b}y^{(1)b} + 2\dfrac{\partial\tilde{y}^{(1)a}}{\partial y^{(1)b}}y^{(2)b} \end{cases} \tag{2.3}$$

One can see that Osc^2M is of dimension 3n.

Let us consider the 2-**tangent structure** $\mathbb{J}$ on Osc^2M

$$\mathbb{J}\left(\frac{\partial}{\partial x^a}\right) = \frac{\partial}{\partial y^{(1)a}},\quad \mathbb{J}\left(\frac{\partial}{\partial y^{(1)a}}\right) = \frac{\partial}{\partial y^{(2)a}},\quad \mathbb{J}\left(\frac{\partial}{\partial y^{(2)a}}\right) = 0,$$

where $\left(\dfrac{\partial}{\partial x^a}\mid_u, \dfrac{\partial}{\partial y^{(1)a}}\mid_u, \dfrac{\partial}{\partial y^{(2)a}}\mid_u\right)$ is the natural basis of the tangent space T_uOsc^2M $,u \in Osc^2M$.If N is a nonlinear connection on Osc^2M, then $N_0 = N$,$\mathbb{J}\left(N_0\right) = N_1$ are two distributions geometrically defined on Osc^2M, all of dimension n. Let us consider the distribution V_2 on Osc^2M locally generated by the vector fields $\left\{\dfrac{\partial}{\partial y^{(2)a}}\right\}$. Consequently, the tangent bundle of Osc^2M at the point $u \in Osc^2M$ is given by a direct sum of the vector space:

$$T_u Osc^2 M = N_0(u) \oplus N_1(u) \oplus V_2(u), \forall u \in Osc^2 M. \tag{2.4}$$

We consider $\left\{ \frac{\delta}{\delta x^a}, \frac{\delta}{\delta y^{1(a)}}, \frac{\delta}{\delta y^{(2)(a)}} \right\}$ an adapted basis to the decomposition (2.4) and its dual basis denoted by $\left(dx^a, \delta y^{(1)a}, \delta y^{(2)a}\right)$, where

$$\begin{cases} dx^a = & dx^a \\ \delta y^{(1)a} = & dy^{(1)a} + \underset{(1)}{M}{}^a{}_b dx^b \\ \delta y^{(2)a} = dy^{(2)a} + \underset{(1)}{M}{}^a{}_b \delta y^b + \underset{(2)}{M}{}^a{}_b \delta y^{(2)b}. \end{cases} \tag{2.5}$$

Definition 2.1. A linear connection D on $Osc^2 M$ is called **N-linear connection** if it preserves by parallelism the horizontal and vertical distribution N_0, N_1 and V_2 on $Osc^2 M$.

Any N-linear connection D can be represented by a unique system of functions $D\Gamma(N) = \left(\underset{(i0)}{L}{}^a_{bd}, \underset{(i1)}{C}{}^a_{bd}, \underset{(i2)}{C}{}^a_{bd} \right), (i = 0, 1, 2)$. These functions are called **the coefficients** of the N-linear connection D.

If on the manifold $Osc^2 M$ a N-linear connection D is given, then there exists an h_i-, v_{1i}- and v_{2i}-**covariant derivatives** in local adapted basis $(i = 0, 1, 2)$.

Any d-tensor T, of type (r, s) can be represented in the adapted basis and its dual basis in the form

$$T = T^{a_1 \dots a_r}_{b_1 \dots b_s} \delta_{a_1} \otimes \dots \otimes \dot{\partial}_{2a_r} \otimes dx^{b_1} \otimes \dots \otimes \delta y^{(2)b_s}$$

and we have

$$\begin{aligned} & T^{a_1 \dots a_r}_{b_1 \dots b_s|id} = \delta_a T^{a_1 \dots a_r}_{b_1 \dots b_s} + \underset{(i0)}{L}{}^{a_1}_{cd} T^{ca_2 \dots a_r}_{b_1 \dots b_s} + \dots + \\ & + \underset{(i0)}{L}{}^{a_r}_{cd} T^{a_1 \dots a_{r-1}c}_{b_1 \dots b_s} - \underset{(i0)}{L}{}^{c}_{b_1 d} T^{a_1 \dots a_r}_{cb_2 \dots b_s} - \dots - \underset{(i0)}{L}{}^{c}_{b_s d} T^{a_1 \dots a_r}_{cb_2 \dots b_{s-1}c} \end{aligned}$$

$$\begin{aligned} & T^{a_1 \dots a_r}_{b_1 \dots b_s} \overset{(1)}{|}_{id} = \delta_{1a} T^{a_1 \dots a_r}_{b_1 \dots b_s} + \underset{(i1)}{C}{}^{a_1}_{cd} T^{ca_2 \dots a_r}_{b_1 \dots b_s} + \dots + \\ & + \underset{(i1)}{C}{}^{a_r}_{cd} T^{a_1 \dots a_{r-1}c}_{b_1 \dots b_s} - \underset{(i1)}{C}{}^{c}_{b_1 d} T^{a_1 \dots a_r}_{cb_2 \dots b_s} - \dots - \underset{(i1)}{C}{}^{c}_{b_s d} T^{a_1 \dots a_r}_{cb_2 \dots b_{s-1}c} \end{aligned}$$

$$\begin{aligned} & T^{a_1 \dots a_r}_{b_1 \dots b_s} \overset{(2)}{|}_{id} = \delta_{2a} T^{a_1 \dots a_r}_{b_1 \dots b_s} + \underset{(i2)}{C}{}^{a_1}_{cd} T^{ca_2 \dots a_r}_{b_1 \dots b_s} + \dots + \\ & + \underset{(i2)}{C}{}^{a_r}_{cd} T^{a_1 \dots a_{r-1}c}_{b_1 \dots b_s} - \underset{(i2)}{C}{}^{c}_{b_1 d} T^{a_1 \dots a_r}_{cb_2 \dots b_s} - \dots - \underset{(i2)}{C}{}^{c}_{b_s d} T^{a_1 \dots a_r}_{cb_2 \dots b_{s-1}c} \end{aligned}$$

$$\left(\delta_{1a} = \frac{\delta}{\delta y^{(1)a}}, \delta_{2a} = \frac{\delta}{\delta y^{(2)a}}; i = 0, 1, 2 \right).$$

The operators "$|_{id}$" ,"$\overset{(1)}{|}_{id}$" and "$\overset{(2)}{|}_{id}$" are called the h$_i$-,v$_{1i}$- and v$_{2i}$-**covariant derivatives** with respect to $D\Gamma(N)$.

Definition 2.2. A **metric structure** on the manifold Osc^2M is a symmetric covariant tensor field $\mathbb{G}$ of the type (0,2) which is non degenerate at each point $u \in Osc^2M$ and of constant signature on Osc^2M.

Locally, a metric structure looks as follows:

$$\mathbb{G} = \underset{(0)}{g}_{ab}dx^a \otimes dx^b + \underset{(1)}{g}_{ab}\delta y^{(1)a} \otimes \delta y^{(1)b} + \underset{(2)}{g}_{ab}\delta y^{(2)a} \otimes \delta y^{(2)b},$$

where

$$rank\left\|\underset{(i)}{g}_{ab}\right\| = n, (i = 0, 1, 2).$$

Definition 2.3. An N-linear connection D on Osc^2M endowed with a structure metric $\mathbb{G}$ is said to be a **N-linear metric connection** if $D_X\mathbb{G} = 0$ for every $X \in X\left(Osc^2M\right)$.

3. Submanifolds in the 2-osculator bundle

Let M be a C^∞ real, n-dimensional manifold and let $\check{M}$ be a real, m-dimensional manifold, immersed in M via the immersion $i : \check{M} \to M$. Localy, i can be given in the form

$$x^a = x^a\left(u^1, ..., u^m\right), \qquad rank\left\|\frac{\partial x^a}{\partial u^\alpha}\right\| = m \tag{3.1}$$

The indices $a, b, c, ...$.run over the set $\{1, ..., n\}$ and $\alpha, \beta, \gamma, ...$ run on the set $\{1, ..., m\}$. We assume $1 < m < n$. If i is an embedding, then we identify $\check{M}$ with $i\left(\check{M}\right)$ and say that $\check{M}$ is a submanifold of the manifold M. Therefore (3.1) will be called the parametric equations of the submanifold M in the manifold M.

The embedding $i : \check{M} \to M$ determines an immersion $Osc^2 i : Osc^2\check{M} \to Osc^2M$, defined by the covariant functor $Osc^2 : Man \to Man$.([9])

The mapping $Osc^2 i : Osc^2\check{M} \to Osc^2M$ has the parametric equations:

$$\begin{cases} x^a = x^a\left(u^1, ..., u^m\right), rank\left\|\dfrac{\partial x^a}{\partial u^\alpha}\right\| = m \\ \\ y^{(1)a} = \dfrac{\partial x^a}{\partial u^\alpha}v^{(1)\alpha} \\ \\ 2y^{(2)a} = \dfrac{\partial y^{(1)a}}{\partial u^\alpha}v^{(1)\alpha} + 2\dfrac{\partial y^{(1)a}}{\partial v^{(1)\alpha}}v^{(2)\alpha}, \end{cases} \tag{3.2}$$

where

$$\begin{cases} \dfrac{\partial x^a}{\partial u^\alpha} = \dfrac{\partial y^{(1)a}}{\partial v^{(1)\alpha}} = \dfrac{\partial y^{(2)a}}{\partial v^{(2)\alpha}} \\ \\ \dfrac{\partial y^{(1)a}}{\partial u^\alpha} = \dfrac{\partial y^{(2)a}}{\partial v^{(1)\alpha}}. \end{cases} \tag{3.3}$$

The Jacobian matrix of 3.2 is $J\left(Osc^2 i\right)$ and its rank is $3m$. So, $Osc^2 i$ is an immersion. The differential i_* of the mapping $Osc^2 i : Osc^2 \check{M} \to Osc^2 M$ maps the cotangent space $T^*\left(Osc^2 M\right)$ at a point of $Osc^2 M$, into the cotangent space $T^*\left(Osc^2 \check{M}\right)$ at a point of $Osc^2 \check{M}$ by the rule:

$$\begin{aligned} dx^a &= \frac{\partial x^a}{\partial u^\alpha} du^\alpha \\ dy^{(1)a} &= \frac{\partial y^{(1)a}}{\partial u^\alpha} du^\alpha + \frac{\partial y^{(1)a}}{\partial v^{(1)\alpha}} dv^{(1)\alpha} \\ dy^{(2)a} &= \frac{\partial y^{(2)a}}{\partial u^\alpha} du^\alpha + \frac{\partial y^{(2)a}}{\partial v^{(1)\alpha}} dv^{(1)\alpha} + \frac{\partial y^{(2)a}}{\partial v^{(2)\alpha}} dv^{(2)\alpha}. \end{aligned} \tag{3.4}$$

We use the previous theory to study the induced geometrical objects from $Osc^2 M$ to $Osc^2 \check{M}$.

Let us consider a Finsler space ([11]) $F^n = \left(M, F\left(x, y^{(1)}\right)\right)$ having $g_{ab}\left(x, y^{(1)}\right) = \dfrac{1}{2}\dfrac{\partial^2 F^2}{\partial y^{(1)a}\partial y^{(1)b}}$ as the fundamental tensor field. The restriction $\check{F}$ of the fundamental function F to the submanifold $Osc\check{M}$ is given by

$$\check{F}\left(u, v^{(1)}\right) = F\left(x\left(u\right), y^{(1)}\left(u, v^{(1)}\right)\right)$$

and the pair $\check{F}^m = \left(\check{M}, \check{F}\right)$ is a Finsler space and it is called the **induced Finsler subspaces** of the Finsler space F^n.

There exists a nonlinear connection on the manifold $Osc^2 M$ determined only by $g_{ab}\left(x, y^{(1)}\right)$. The dual coefficients of this nonlinear connection are [9]

$$\underset{(1)}{M}{}^a_b = \frac{\partial G^a}{\partial y^{(1)b}}$$

$$\underset{(2)}{M}{}^a_b = \frac{1}{2}\left(\Gamma \underset{(1)}{M}{}^a_b - \underset{(1)}{M}{}^a_d \underset{(1)}{M}{}^d_b\right),$$

where

$$G^a = \frac{1}{2}\gamma^a_{bc}\left(x, y^{(1)}\right) y^{(1)b} y^{(1)c},$$

$$\Gamma = y^{(1)a}\tfrac{\partial}{\partial x^a} + 2y^{(2)a}\tfrac{\partial}{\partial y^{(1)a}}$$

and $\gamma^a_{bc}\left(x, y^{(1)}\right)$ are the Christoffel symbols of the fundamental tensor g_{ab}.

Next, we consider

$$B_{\alpha}^{a}(u)=\frac{\partial x^{a}}{\partial u^{\alpha}}$$

and $\mathbb{G} = g_{ab}dx^a \otimes dx^b + g_{ab}\delta y^{(1)a} \otimes \delta y^{(1)a} + g_{ab}\delta y^{(2)a} \otimes \delta y^{(2)a}$, the Sasaki prolongation of the metric g along $Osc^2 M$.

Thus, $\left\{B_1^a, B_2^a, ..., B_m^a\right\}$ are m-linear independent d-vector fields on $Osc^2\check{M}$. Also, $\left\{B_\alpha^1, B_\alpha^2, ..., B_\alpha^n\right\}$ are d-covector fields, with respect to the next transformations of coordinates

$$\begin{cases} \bar{u}^{\alpha}=\bar{u}^{\alpha}\left(u^{1},...,u^{m}\right), rank\left\|\dfrac{\partial\bar{u}^{\alpha}}{\partial u^{\beta}}\right\|=m \\ \\ \bar{v}^{(1)\alpha}=\dfrac{\partial\bar{u}^{\alpha}}{\partial u^{\beta}}v^{(1)\beta} \\ \\ 2\bar{v}^{(2)\alpha}=\dfrac{\partial\bar{v}^{(1)\alpha}}{\partial u^{\beta}}v^{(1)\beta}+2\dfrac{\partial\bar{v}^{(1)\alpha}}{\partial v^{(1)\beta}}v^{(2)\beta}. \end{cases} \tag{3.5}$$

Of course, d-vector fields $\left\{B_1^a, ..., B_m^a\right\}$ are tangent to the submanifold $\check{M}$.

We say that a d-vector field $\xi^a\left(x, y^{(1)}, y^{(2)}\right)$ is **normal** to $Osc^2\check{M}$ if, on $\check{\pi}^{-1}\left(\check{U}\right) \subset Osc^2\check{M}$, we have

$$\begin{array}{l} g_{ab}\left(x\left(u\right),y^{(1)}\left(u,v^{(1)},v^{(2)}\right),y^{(2)}\left(u,v^{(1)},v^{(2)}\right)\right)B_{\alpha}^{a}\left(u\right)\cdot \\ \cdot\xi^{b}\left(x\left(u\right),y^{(1)}\left(u,v^{(1)},v^{(2)}\right),y^{(2)}\left(u,v^{(1)},v^{(2)}\right)\right)=0. \end{array}$$

Consequently, on $\check{\pi}^{-1}\left(\check{U}\right) \subset Osc^2\check{M}$ there exist $n-m$ unit vector fields $B_{\bar{\alpha}}^a$, $(\bar{\alpha}=1,...,n-m)$ normal along $Osc^2\check{M}$, and to each other:

$$g_{ab}B_{\alpha}^{a}B_{\bar{\beta}}^{b}=0,\quad g_{ab}B_{\bar{\alpha}}^{a}B_{\bar{\beta}}^{b}=\delta_{\bar{\alpha}\bar{\beta}},\left(\bar{\alpha},\bar{\beta}=1,...,n-m\right). \tag{3.6}$$

The system of d-vectors $B_{\bar{\alpha}}^a$ $(\bar{\alpha}=1,...,n-m)$ is determined up to orthogonal transformations of the form

$$B_{\bar{\alpha}'}^{a}=A_{\bar{\alpha}'}^{\bar{\beta}}B_{\bar{\beta}}^{a},\ \left\|A_{\bar{\alpha}'}^{\bar{\alpha}}\right\|\in\mathcal{O}\left(n-m\right), \tag{3.7}$$

where $\bar{\alpha}, \bar{\beta}, ...$ run over the set $(1, 2, .., n-m)$.

We say that the system of d-vectors $\{B_\alpha^a, B_{\bar{\alpha}}^a\}$ determines a frame in $Osc^2 M$ along to $Osc^2\check{M}$.

Its dual frame will be denoted by $\left\{B_a^\alpha\left(u, v^{(1)}, ..., v^{(2)}\right), B_a^{\bar{\alpha}}\left(u, v^{(1)}, ..., v^{(2)}\right)\right\}$. This is also defined on an open set $\check{\pi}^{-1}\left(\check{U}\right) \subset Osc^2\check{M}$, $\check{U}$ being a domain of a local chart on the submanifold $\check{M}$.

The conditions of duality are given by:

$$B_{\beta}^{a}B_{a}^{\alpha}=\delta_{\beta}^{\alpha},\quad B_{\beta}^{a}B_{a}^{\bar{\alpha}}=0,\quad B_{a}^{\alpha}B_{\bar{\beta}}^{a}=0,\quad B_{a}^{\bar{\alpha}}B_{\bar{\beta}}^{a}=\delta_{\bar{\beta}}^{\bar{\alpha}} \tag{3.8}$$

$$B_{\alpha}^{a}B_{b}^{\alpha}+B_{\bar{\alpha}}^{a}B_{b}^{\bar{\alpha}}=\delta_{b}^{a}. \tag{3.9}$$

Using (3.6), we deduce:

$$g_{\alpha\beta}B_{a}^{\alpha}=g_{ab}B_{\beta}^{a},\ \delta_{\bar{\alpha}\bar{\beta}}B_{b}^{\bar{\beta}}=g_{ab}B_{\bar{\alpha}}^{a}. \tag{3.10}$$

So, we can look at the set

$$\mathcal{R}=\left\{\left(u,v^{(1)},v^{(2)}\right);B_{\alpha}^{a}\left(u\right),B_{\bar{\alpha}}^{a}\left(u,v^{(1)},v^{(2)}\right)\right\}$$

$\left(u,v^{(1)},v^{(2)}\right)\in\check{\pi}^{-1}\left(\check{U}\right)$ as a moving frame. Now, we shall represent in $\mathcal{R}$ the d-tensor fields from the space Osc^2M , restricted to the open set $\check{\pi}^{-1}\left(\check{U}\right)$.

4. Induced nonlinear connections

Now, let us consider the canonical nonlinear connection N on the Osc^2M. Then its dual coefficients $\underset{(1)}{M}{}_{b}^{a},\underset{(2)}{M}{}_{b}^{a}$ depends only by the metric g. We will prove that the restriction of the nonlinear connection N to $Osc^2\check{M}$ uniquely determines an induced nonlinear connection Ň on $Osc^2\check{M}$. Of course, Ň is well determined by means of its dual coefficients $\left(\underset{(1)}{\check{M}}{}_{\beta}^{\alpha},\underset{(2)}{\check{M}}{}_{\beta}^{\alpha}\right)$ or by means of its adapted cobasis $\left(du^{\alpha},\delta v^{(1)\alpha},\delta v^{(2)\alpha}\right)$.

Definition 4.1. A non-linear connection $\check{N}$ on $Osc^2\check{M}$ is called **induced** by the nonlinear connection N if we have

$$\delta v^{(1)\alpha}=B_{a}^{\alpha}\delta y^{(1)a},\quad \delta v^{(2)\alpha}=B_{a}^{\alpha}\delta y^{(2)a}. \tag{4.1}$$

Proposition 4.2. [16] *The dual coefficients of the non-linear connection $\check{N}$ are*

$$\begin{aligned}
&\underset{(1)}{\check{M}}{}_{\beta}^{\alpha}=B_{a}^{\alpha}\left(B_{0\beta}^{a}+\underset{(1)}{M}{}_{b}^{a}B_{\beta}^{b}\right)\\
&\underset{(2)}{\check{M}}{}_{\beta}^{\alpha}=B_{a}^{\alpha}\left(\frac{1}{2}\frac{\partial B_{\delta\gamma}^{a}}{\partial u^{\beta}}v^{(1)\delta}v^{(1)\gamma}+B_{\delta\beta}^{a}v^{(2)\delta}+\underset{(1)}{M}{}_{b}^{a}B_{0\beta}^{b}+\underset{(2)}{M}{}_{b}^{a}B_{\beta}^{b}\right),
\end{aligned} \tag{4.2}$$

where $\underset{(1)}{M}{}_{b}^{a}$, $\underset{(2)}{M}{}_{b}^{a}$ are the dual coefficients of the non-linear connection N.

Theorem 4.3. [16] *The cobasis $\left(dx^{a},\delta y^{(1)a},\delta y^{(2)a}\right)$ restricted to $Osc^2\check{M}$ is uniquely represented in the moving frame $\mathcal{R}$ in the following form:*

$$\begin{cases}
dx^{a}=B_{\beta}^{a}du^{\beta}\\
\delta y^{(1)a}=B_{\alpha}^{a}\delta v^{(1)\alpha}+B_{\bar{\alpha}}^{a}\underset{(1)}{K}{}_{\beta}^{\bar{\alpha}}du^{\beta}\\
\delta y^{(2)a}=B_{\alpha}^{a}\delta v^{(2)\alpha}+B_{\bar{\beta}}^{a}\underset{(1)}{K}{}_{\alpha}^{\bar{\beta}}\delta v^{(1)\alpha}+B_{\bar{\beta}}^{a}\underset{(2)}{K}{}_{\alpha}^{\bar{\beta}}du^{\alpha},
\end{cases} \tag{4.3}$$

where

$$\underset{(1)}{K}{}^{\bar{\alpha}}_{\beta} = B^{\bar{\alpha}}_{a}\left(B^{a}_{0\beta} + \underset{(1)}{M}{}^{a}_{b} B^{b}_{\beta}\right)$$

$$\underset{(2)}{K}{}^{\bar{\alpha}}_{\beta} = B^{\bar{\alpha}}_{a}\left(\frac{1}{2}\frac{\partial B^{a}_{\delta\gamma}}{\partial u^{\beta}} v^{(1)\delta} v^{(1)\gamma} + B^{b}_{\delta\beta} v^{(2)\delta} + \underset{(1)}{M}{}^{a}_{b} B^{b}_{0\beta} + \underset{(2)}{M}{}^{a}_{b} B^{b}_{\beta} - \right. \tag{4.4}$$

$$\left. - B^{\bar{\alpha}}_{f} B^{\gamma}_{d}\left(B^{f}_{\gamma} + \underset{(1)}{M}{}^{f}_{b} B^{b}_{\gamma}\right)\left(B^{d}_{0\beta} + \underset{(1)}{M}{}^{d}_{g} B^{g}_{\beta}\right)\right)$$

are mixed d-tensor fields.

Proof. The first relation is obvious. From (3.2) and (4.2) we obtain (4.3). □

Generally, a set of functions $T^{i\ldots\alpha\ldots\bar{\alpha}}_{j\ldots\beta\ldots\bar{\beta}}\left(u, v^{(1)}, v^{(2)}\right)$ which are d-tensors in the index i, j,..., and d-tensors in the index $\alpha, \beta, \ldots$and tensors with respect to the transformations (3.7) in the index $\bar{\alpha}, \bar{\beta}, \ldots$ is calld a *mixed* d-tensor field on $Osc^2\check{M}$.

5. The relative covariant derivatives

We shall construct the operators $\underset{(i)}{\nabla}$ of relative (or mixed) covariant derivation in the algebra of mixed d-tensor fields. It is clear that $\underset{(i)}{\nabla}$ will be known if its action on functions and on the vector fields of the form

$$\begin{gathered} X^{a}\left(x\left(u\right), y^{(1)}\left(u, v^{(1)}\right), y^{(2)}\left(u, v^{(1)}, v^{(2)}\right)\right) \\ X^{\alpha}\left(u, v^{(1)}, v^{(2)}\right),\ X^{\bar{\alpha}}\left(u, v^{(1)}, v^{(2)}\right) \end{gathered} \tag{5.1}$$

are known.

Let D be the canonical N-linear metric connection on the manifold Osc^2M [2]

$$\begin{aligned} \underset{(00)}{L}{}^{a}_{bc} &= \frac{1}{2} g^{ad}\left(\delta_b g_{dc} + \delta_c g_{db} - \delta_d g_{bc}\right), \\ \underset{(i0)}{L}{}^{a}_{bc} &= \underset{(jj)}{B}{}^{a}_{bc} + \frac{1}{2} g^{ad}\left(\delta_c g_{bd} - \underset{(jj)}{B}{}^{f}_{cb} g_{fd} - \underset{(jj)}{B}{}^{f}_{cd} g_{bf}\right), (j = 1, 2) \\ \underset{(k1)}{C}{}^{a}_{bc} &= \frac{1}{2} g^{ad} \delta_{1b} g_{bd}, (k = 0, 2) \\ \underset{(l2)}{C}{}^{a}_{bc} &= \frac{1}{2} g^{ad} \dot{\partial}_{2c} g_{bd}, (l = 0, 1) \\ \underset{(ii)}{C}{}^{a}_{bc} &= \frac{1}{2} g^{ad}\left(\delta_{ib} g_{dc} + \delta_{ic} g_{db} - \delta_{id} g_{bc}\right), (i = 1, 2). \end{aligned} \tag{5.2}$$

Definition 5.1. The **coupling** of the canonical N-linear metric connection D with the induced nonlinear connection $\check{N}$ along $Osc^2\check{M}$ is locally given by the set of its nine coefficients $\check{D}\Gamma\left(\check{N}\right) = \left(\underset{(i0)}{\check{L}}{}^{a}_{b\delta}, \underset{(i1)}{\check{C}}{}^{a}_{b\delta}, \underset{(i2)}{\check{C}}{}^{a}_{b\delta}\right), (i = 0, 1, 2)$ *where*

$$\begin{aligned} \underset{(i0)}{\check{L}}{}^{a}_{b\delta} &= \underset{(i0)}{L}{}^{a}_{bd}B^{d}_{\delta} + \underset{(i1)}{C}{}^{a}_{bd}B^{d}_{\bar{\delta}}\underset{(1)}{K}{}^{\bar{\delta}}_{\delta} + \underset{(i2)}{C}{}^{a}_{b\delta}B^{d}_{\bar{\delta}}\underset{(2)}{K}{}^{\bar{\delta}}_{\delta} \\ \underset{(i1)}{\check{C}}{}^{a}_{b\delta} &= \underset{(i1)}{C}{}^{a}_{bd}B^{d}_{\delta} + \underset{(i2)}{C}{}^{a}_{bd}B^{d}_{\bar{\delta}}\underset{(1)}{K}{}^{\bar{\delta}}_{\delta} \qquad (i = 0, 1, 2) \\ \underset{(i2)}{\check{C}}{}^{a}_{b\delta} &= \underset{(i2)}{C}{}^{a}_{bd}B^{d}_{\delta}. \end{aligned} \tag{5.3}$$

We have the operators $\underset{(i)}{\check{D}}$ and $\underset{(i)}{D}$ $(i = 0, 1, 2)$ with the property

$$\underset{(i)}{\check{D}}X^{a} = \underset{(i)}{D}X^{a} \text{ (modulo (4.3))} \tag{5.4}$$

where

$$\underset{(i)}{D}X^{a} = dX^{a} + X^{b}\underset{(i)}{\omega}{}^{a}_{b}, \tag{5.5}$$

and

$$\underset{(i)}{\check{D}}X^{a} = dX^{a} + X^{b}\underset{(i)}{\check{\omega}}{}^{a}_{b}. \tag{5.6}$$

Here $\underset{(i)}{\omega}{}^{a}_{b}$ and $\underset{(i)}{\check{\omega}}{}^{a}_{b}$ are the 1-forms of the canonical N-linear metric connection D and of the coupling $\check{D}$ respectively.

Of course, we can write $\underset{(i)}{\check{D}}X^{a}$ in the form

$$\underset{(i)}{\check{D}}X^{a} = X^{a}{}_{\shortmid i\delta}\, du^{\delta} + X^{a}\overset{(1)}{\mid}_{i\delta}\, \delta v^{(1)\delta} + X^{a}\overset{(2)}{\mid}_{i\delta}\, \delta v^{(2)\delta}.$$

Definition 5.2. We call the **induced tangent connection** on $Osc^2\check{M}$ by the canonical N-linear metric connection D the set of its nine coefficients $D^{\top}\Gamma\left(\check{N}\right) = \left(\underset{(i0)}{L}{}^{\alpha}_{\beta\delta}, \underset{(i1)}{C}{}^{\alpha}_{\beta\delta}, \underset{(i2)}{C}{}^{\alpha}_{\beta\delta}\right)$ $(i = 0, 1, 2)$ where

$$\begin{aligned} \underset{(i0)}{L}{}^{\alpha}_{\beta\delta} &= B^{\alpha}_{d}\left(B^{d}_{\beta\delta} + B^{f}_{\beta}\underset{(i0)}{\check{L}}{}^{d}_{f\delta}\right) \\ \underset{(i1)}{C}{}^{\alpha}_{\beta\delta} &= B^{\alpha}_{d}B^{f}_{\beta}\underset{(i1)}{\check{C}}{}^{d}_{f\delta} \qquad (i = 0, 1, 2) \\ \underset{(i2)}{C}{}^{\alpha}_{\beta\delta} &= B^{\alpha}_{d}B^{f}_{\beta}\underset{(i2)}{\check{C}}{}^{d}_{f\delta}. \end{aligned} \tag{5.7}$$

We have the operators $\underset{(i)}{D}^{\top}$ with the properties

$$\underset{(i)}{D}^{\top} X^{\alpha} = B^{\alpha}_{b} \underset{(i)}{\check{D}} X^{b}, \quad for\ X^{a} = B^{a}_{\gamma} X^{\gamma} \tag{5.8}$$

$$\underset{(i)}{D}^{\top} X^{\alpha} = dX^{\alpha} + X^{\beta} \underset{(i)}{\omega}{}^{\alpha}_{\beta}, \tag{5.9}$$

where $\underset{(i)}{\omega}{}^{\alpha}_{\beta}$ are **the connection 1-forms** of $\underset{(i)}{D}^{\top}$ $(i = 0, 1, 2)$.

As in the case of $\check{D}$ we may write

$$\underset{(i)}{D}^{\top} X^{\alpha} = X^{\alpha}_{|i\delta} du^{\delta} + X^{\alpha} \overset{(1)}{|}_{i\delta} \delta v^{(1)\delta} + X^{\alpha} \overset{(2)}{|}_{i\delta} \delta v^{(2)\delta}.$$

Definition 5.3. We call the **induced normal connection** on $Osc^{2}\check{M}$ by the canonical N-linear metric connection D the set of its nine coefficients $D^{\perp}\Gamma\left(\check{N}\right) = \left(\underset{(i0)}{L}{}^{\bar{\alpha}}_{\bar{\beta}\delta}, \underset{(i1)}{C}{}^{\bar{\alpha}}_{\bar{\beta}\delta}, \underset{(i2)}{C}{}^{\bar{\alpha}}_{\bar{\beta}\delta} \right)$ *where*

$$\begin{aligned} \underset{(i0)}{L}{}^{\bar{\alpha}}_{\bar{\beta}\delta} &= B^{\bar{\alpha}}_{d} \left(\frac{\delta B^{d}_{\bar{\beta}}}{\delta u^{\delta}} + B^{f}_{\bar{\beta}} \underset{(i0)}{\check{L}}{}^{d}_{f\delta} \right) \\ \underset{(i1)}{C}{}^{\bar{\alpha}}_{\bar{\beta}\delta} &= B^{\bar{\alpha}}_{d} \left(\frac{\delta B^{d}_{\bar{\beta}}}{\delta v^{(1)\delta}} + B^{f}_{\bar{\beta}} \underset{(i1)}{\check{C}}{}^{d}_{f\delta} \right) \quad (i = 0, 1, 2) \\ \underset{(i2)}{C}{}^{\bar{\alpha}}_{\bar{\beta}\delta} &= B^{\bar{\alpha}}_{d} \left(\frac{\partial B^{d}_{\bar{\beta}}}{\partial v^{(2)\delta}} + B^{f}_{\bar{\beta}} \underset{(i2)}{\check{C}}{}^{d}_{f\delta} \right). \end{aligned} \tag{5.10}$$

As before, we have the operators $\underset{(i)}{D}^{\perp}$ with the properties

$$\underset{(i)}{D}^{\perp} X^{\bar{\alpha}} = B^{\bar{\alpha}}_{b} \underset{(i)}{\check{D}} X^{b}, \quad for\ X^{a} = B^{a}_{\bar{\gamma}} X^{\bar{\gamma}} \tag{5.11}$$

$$\underset{(i)}{D}^{\perp} X^{\bar{\alpha}} = dX^{\bar{\alpha}} + X^{\bar{\beta}} \underset{(i)}{\omega}{}^{\bar{\alpha}}_{\bar{\beta}} \tag{5.12}$$

where $\underset{(i)}{\omega}{}^{\bar{\alpha}}_{\bar{\beta}}$ are the **connection 1-forms** of $\underset{(i)}{D}^{\perp}$ $(i = 0, 1, 2)$.

We may set

$$\underset{(i)}{D}^{\perp} X^{\bar{\alpha}} = X^{\bar{\alpha}}_{|i\delta} du^{\delta} + X^{\bar{\alpha}} \overset{(1)}{|}_{i\delta} \delta v^{(1)\delta} + X^{\bar{\alpha}} \overset{(2)}{|}_{i\delta} \delta v^{(2)\delta}.$$

Now, we can define the relative (or mixed) covariant derivatives $\underset{(i)}{\nabla}$ enounced at the begining of this section.

Theorem 5.4. *A relative (mixed) covariant derivation in the algebra of mixed d-tensor fields is an operator* $\underset{(i)}{\nabla}$ *for which the following properties hold:*

$$\underset{(i)}{\nabla} f = df, \quad \forall f \in \mathcal{F}\left(Osc^2 \check{M}\right)$$

$$\underset{(i)}{\nabla} X^a = \underset{(i)}{\check{D}} X^a, \quad \underset{(i)}{\nabla} X^\alpha = \underset{(i)}{D^\intercal} X^\alpha, \quad \underset{(i)}{\nabla} X^{\bar{\alpha}} = \underset{(i)}{D^\perp} X^{\bar{\alpha}} \ \ (i = 0, 1, 2)\,.$$

The connection 1-forms $\underset{(i)}{\check{\omega}}{}^a_b, \underset{(i)}{\omega}{}^\alpha_\beta, \underset{(i)}{\omega}{}^{\bar{\alpha}}_{\bar{\beta}}$ *will be called the* ***connection 1-forms*** *of* $\underset{(i)}{\nabla}$.

The Liouville vector fields for submanifolds, introduced by professor Miron in [9], are

$$\overset{1}{\gamma} = v^{(1)\alpha} \frac{\partial}{\partial v^{(2)\alpha}}$$

$$\overset{2}{\gamma} = v^{(1)\alpha} \frac{\partial}{\partial v^{(1)\alpha}} + 2 v^{(2)\alpha} \frac{\partial}{\partial v^{(2)\alpha}}.$$

If we represent this vector fields in the adapted basis, we get

$$\overset{1}{\gamma} = z^{(1)\alpha} \dot{\partial}_{2\alpha}, \overset{2}{\gamma} = z^{(1)\alpha} \delta_{1\alpha} + 2 z^{(2)\alpha} \dot{\partial}_{2\alpha}$$

where

$$z^{(1)\alpha} = v^{(1)\alpha}, z^{(2)\alpha} = v^{(2)\alpha} + \frac{1}{2} \underset{(1)}{M}{}^\alpha{}_\beta v^{(1)\beta}.$$

D -vector fields $z^{(1)\alpha}$ and $z^{(2)\alpha}$ are called the **Liouville d-vector fields** of submanifold $Osc^2 \check{M}$.

The $\left(z^{(1)}\right)$- and $\left(z^{(2)}\right)$-**deflection** tensor fields are:

$$\begin{aligned}
&z^{(1)\alpha}{}_{|i\beta} = \underset{i}{\overset{(1)}{D}}{}^\alpha_\beta, \qquad z^{(1)\alpha} \overset{(1)}{|}_{i\beta} = \underset{i}{\overset{(11)}{d}}{}^\alpha_\beta, \qquad z^{(1)\alpha} \overset{(2)}{|}_{i\beta} = \underset{i}{\overset{(12)}{d}}{}^\alpha_\beta, \\
&z^{(2)\alpha}{}_{|i\beta} = \underset{i}{\overset{(2)}{D}}{}^\alpha_\beta, \qquad z^{(2)\alpha} \overset{(1)}{|}_{i\beta} = \underset{i}{\overset{(21)}{d}}{}^\alpha_\beta, \qquad z^{(2)\alpha} \overset{(2)}{|}_{i\beta} = \underset{i}{\overset{(22)}{d}}{}^\alpha_\beta.
\end{aligned} \tag{5.13}$$

Proposition 5.5. The $\left(z^{(1)}\right)$*-deflection fields have the expression:*

$$\begin{aligned}
&\underset{i}{\overset{(1)}{D}}{}^\alpha_\beta = -\underset{1}{N}{}^\alpha{}_\beta + z^{(1)\gamma} \underset{(i0)}{L}{}^\alpha_{\gamma\beta}, \\
&\underset{i}{\overset{(11)}{d}}{}^\alpha_\beta = \delta^\alpha_\beta + z^{(1)\gamma} \underset{(i1)}{C}{}^\alpha_{\gamma\beta}, \\
&\underset{i}{\overset{(12)}{d}}{}^\alpha_\beta = z^{(1)\gamma} \underset{(i2)}{C}{}^\alpha_{\gamma\beta}.
\end{aligned} \tag{5.14}$$

Proof. Indeed, if we take

$$z^{(1)\alpha}{}_{|i\beta} = \delta_\beta z^{(1)\alpha} + z^{(1)\gamma} \underset{(i0)}{L}{}^{\alpha}_{\gamma\beta},$$

$$z^{(1)\alpha} \overset{(j)}{|}_{i\beta} = \delta_{j\beta} z^{(1)\alpha} + z^{(1)\gamma} \underset{(ij)}{C}{}^{\alpha}_{\gamma\beta}, \left(i = 0,1,2; j = 1,2; \delta_{2\beta} = \dot{\partial}_{2\beta}\right)$$

we find the Formulae (5.14). □

Proposition 5.6. The $\left(z^{(2)}\right)$*-deflection fields are given by*

$$\begin{aligned}
\underset{i}{\overset{(2)}{D}}{}^{\alpha}_{\beta} &= \frac{1}{2}\left(\underset{2}{N}{}^{\alpha}{}_{\beta} + \underset{2}{M}{}^{\alpha}{}_{\beta}\right) + \frac{1}{2} z^{(1)\gamma} \delta_\beta \underset{1}{N}{}^{\alpha}{}_{\gamma} + z^{(2)\gamma} \underset{(i0)}{L}{}^{\alpha}_{\gamma\beta},\\
\underset{i}{\overset{(21)}{d}}{}^{\alpha}_{\beta} &= -\frac{1}{2}\left(2\underset{2}{N}{}^{\alpha}{}_{\beta} - \underset{1}{N}{}^{\alpha}{}_{\beta}\right) + \frac{1}{2} z^{(1)\gamma} \underset{11}{B}{}^{\alpha}_{\gamma\beta} + z^{(2)\gamma} \underset{(i1)}{C}{}^{\alpha}_{\gamma\beta},\\
\underset{i}{\overset{(22)}{d}}{}^{\alpha}_{\beta} &= \delta^{\alpha}_{\beta} + \frac{1}{2} z^{(1)\gamma} \underset{12}{B}{}^{\alpha}_{\gamma\beta} + z^{(2)\gamma} \underset{(i2)}{C}{}^{\alpha}_{\gamma\beta}.
\end{aligned} \tag{5.15}$$

Proof. Indeed, if we take

$$z^{(2)\alpha}{}_{|i\beta} = \delta_\beta z^{(2)\alpha} + z^{(2)\gamma} \underset{(i0)}{L}{}^{\alpha}_{\gamma\beta},$$

$$z^{(2)\alpha} \overset{(j)}{|}_{i\beta} = \delta_{j\beta} z^{(2)\alpha} + z^{(2)\gamma} \underset{(ij)}{C}{}^{\alpha}_{\gamma\beta}, \left(i = 0,1,2; j = 1,2; \delta_{2\beta} = \dot{\partial}_{2\beta}\right)$$

we find the Formulae (5.15). □

6. Adapted components of torsion and curvature tensors

The study of the adapted components of the torsion and curvature tensors of an arbitrary N-linear connection $D\Gamma(N)$ on $Osc^2 M$ was done in [2] and [1]. In what follows, we study the adapted components of the torsion and curvature tensors for the relative (or mixed) covariant derivatives $\underset{(i)}{\nabla}$, $(i = 0,1,2)$.

Theorem 6.1. *In local coordinates, the torsion d-tensors of the relative (or mixed) covariant derivatives* $\underset{(i)}{\nabla}$ *have the next expresions:*

$$\underset{(00)}{T}{}^{\alpha}_{\beta\gamma} = \underset{(00)}{L}{}^{\alpha}_{\beta\gamma} - \underset{(00)}{L}{}^{\alpha}_{\gamma\beta}, \qquad \underset{(01)}{T}{}^{\alpha}_{\beta\gamma} = \underset{(01)}{R}{}^{\alpha}_{\beta\gamma} = \delta_\gamma \underset{1}{N}{}^{\alpha}{}_{\beta} - \delta_\beta \underset{1}{N}{}^{\alpha}{}_{\gamma},$$

$$\underset{(02)}{T}{}^{\alpha}_{\beta\gamma} = \underset{(02)}{R}{}^{\alpha}_{\beta\gamma} = \delta_\gamma \underset{2}{N}{}^{\alpha}{}_{\beta} - \delta_\beta \underset{2}{N}{}^{\alpha}{}_{\gamma} +$$

$$+ \underset{1}{N}{}^{\alpha}_{\varepsilon}\left(\delta_\gamma \underset{1}{N}{}^{\varepsilon}{}_{\beta} - \delta_\beta \underset{1}{N}{}^{\varepsilon}{}_{\gamma}\right),$$

$$\begin{aligned}
&\underset{(10)}{P}{}^{\alpha}_{\beta\gamma} = \underset{(10)}{C}{}^{\alpha}_{\beta\gamma}, && \underset{(11)}{P}{}^{\alpha}_{\beta\gamma} = \delta_{1\gamma}\underset{1}{N}{}^{\alpha}{}_{\beta} - \underset{(10)}{L}{}^{\alpha}_{\gamma\beta}, \\
&\underset{(20)}{P}{}^{\alpha}_{\beta\gamma} = \underset{(20)}{C}{}^{\alpha}_{\beta\gamma}, && \underset{(12)}{P}{}^{\alpha}_{\beta\gamma} = \delta_{1\gamma}\underset{2}{N}{}^{\alpha}{}_{\beta} - \delta_{\beta}\underset{1}{N}{}^{\alpha}{}_{\gamma} + \underset{1}{N}{}^{\alpha}{}_{\varepsilon}\left(\delta_{1\gamma}\underset{1}{N}{}^{\varepsilon}{}_{\beta}\right), \\
&\underset{(21)}{P}{}^{\alpha}_{\beta\gamma} = \dot{\partial}_{2\gamma}\underset{1}{N}{}^{\alpha}{}_{\beta}, && \underset{(22)}{P}{}^{\alpha}_{\beta\gamma} = \dot{\partial}_{2\gamma}\underset{2}{N}{}^{\alpha}{}_{\beta} + \underset{1}{N}{}^{\alpha}{}_{\varepsilon}\left(\dot{\partial}_{2\gamma}\underset{1}{N}{}^{\varepsilon}{}_{\beta}\right) - \underset{(20)}{L}{}^{\alpha}_{\gamma\beta},
\end{aligned} \tag{6.1}$$

$$\begin{aligned}
&\underset{(11)}{S}{}^{\alpha}_{\beta\gamma} = \underset{(11)}{C}{}^{\alpha}_{\beta\gamma} - \underset{(11)}{C}{}^{\alpha}_{\gamma\beta}, && \underset{(12)}{S}{}^{\alpha}_{\beta\gamma} = \underset{(12)}{R}{}^{\alpha}_{\beta\gamma} = \delta_{1\gamma}\underset{1}{N}{}^{\alpha}{}_{\beta} - \delta_{1\beta}\underset{1}{N}{}^{\alpha}{}_{\gamma}, \\
&\underset{(21)}{Q}{}^{\alpha}_{\beta\gamma} = \underset{(12)}{C}{}^{\alpha}_{\beta\gamma}, && \underset{(22)}{Q}{}^{\alpha}_{\beta\gamma} = \dot{\partial}_{2\gamma}\underset{1}{N}{}^{\alpha}{}_{\beta} - \underset{(21)}{C}{}^{\alpha}_{\gamma\beta}, \\
&\underset{(21)}{S}{}^{\alpha}_{\beta\gamma} = 0, && \underset{(22)}{S}{}^{\alpha}_{\beta\gamma} = \underset{(22)}{C}{}^{\alpha}_{\beta\gamma} - \underset{(22)}{C}{}^{\alpha}_{\gamma\beta}.
\end{aligned}$$

Proof. Using the general local expressions from [2] and [1], which give the d-components of the torsion tensor of an N-linear connection, $D\Gamma(N)$, we deduce that the adapted components of the mixed covariant derivatives $\underset{(i)}{\nabla}$, $(i = 0, 1, 2)$ are given by the formulas from theorem. □

The following d-tensor fields will be needed in our calculations.

$$\begin{aligned}
&\underset{(0)}{\overset{i}{T}}{}^{\alpha}_{\beta\gamma} = \underset{(i0)}{L}{}^{\alpha}_{\beta\gamma} - \underset{(i0)}{L}{}^{\alpha}_{\gamma\beta} && \underset{(jj)}{\overset{i}{P}}{}^{\alpha}_{\beta\gamma} = \underset{(jj)}{B}{}^{\alpha}_{\beta\gamma} - \underset{(i0)}{L}{}^{\alpha}_{\gamma\beta} \\
&\underset{(22)}{\overset{i}{Q}}{}^{\alpha}_{\beta\gamma} = \underset{(21)}{B}{}^{\alpha}_{\beta\gamma} - \underset{(i1)}{C}{}^{\alpha}_{\gamma\beta}, && \underset{(j)}{\overset{i}{S}}{}^{\alpha}_{\beta\gamma} = \underset{(ij)}{C}{}^{\alpha}_{\beta\gamma} - \underset{(ij)}{C}{}^{\alpha}_{\gamma\beta},
\end{aligned} \tag{6.2}$$

$$(i = 0, 1, 2; j = 1, 2).$$

We remark that we have

$$\begin{aligned}
&\underset{(0)}{\overset{0}{T}}{}^{\alpha}_{\beta\gamma} = \underset{(00)}{T}{}^{\alpha}_{\beta\gamma}, && \underset{(jj)}{\overset{j}{P}}{}^{\alpha}_{\beta\gamma} = \underset{(jj)}{P}{}^{\alpha}_{\beta\gamma}, \\
&\underset{(22)}{\overset{2}{Q}}{}^{\alpha}_{\beta\gamma} = \underset{(22)}{\overset{2}{Q}}{}^{\alpha}_{\beta\gamma}, && \underset{(j)}{\overset{j}{S}}{}^{\alpha}_{\beta\gamma} = \underset{(j)}{S}{}^{\alpha}_{\beta\gamma}, (j = 1, 2).
\end{aligned}$$

Theorem 6.2. *In local coordinates, the curvature d-tensors of the relative (or mixed) covariant derivatives* $\underset{(i)}{\nabla}$ *have the next expresions:*

$$\begin{aligned}
\underset{(0i)}{R}{}_{b}{}^{a}{}_{\gamma\delta} &= \delta_\delta \underset{(i0)}{L}{}^{a}_{b\gamma} - \delta_\beta \underset{(i0)}{L}{}^{a}_{b\delta} + \underset{(i0)}{L}{}^{e}_{b\gamma}\underset{(i0)}{L}{}^{a}_{e\delta} - \underset{(i0)}{L}{}^{e}_{b\delta}\underset{(i0)}{L}{}^{a}_{e\gamma} + \\
&+ \underset{(i1)}{C}{}^{a}_{b\sigma}\underset{(01)}{R}{}^{\sigma}_{\gamma\delta} + \underset{(i2)}{C}{}^{a}_{b\sigma}\underset{(02)}{R}{}^{\sigma}_{\gamma\delta}, \\
\underset{(1i)}{P}{}_{b}{}^{a}{}_{\gamma\delta} &= \delta_{1\delta}\underset{(i0)}{L}{}^{a}_{b\gamma} - \underset{(i1)}{C}{}^{a}_{b\delta|_{i\gamma}} + \underset{(i1)}{C}{}^{a}_{b\sigma}\overset{i}{\underset{(11)}{P}}{}^{\sigma}_{\gamma\delta} + \underset{(i2)}{C}{}^{a}_{b\sigma}\overset{i}{\underset{(12)}{P}}{}^{\sigma}_{\gamma\delta}, \\
\underset{(2i)}{P}{}_{b}{}^{a}{}_{\gamma\delta} &= \dot{\partial}_{2\delta}\underset{(i0)}{L}{}^{a}_{b\gamma} - \underset{(i2)}{C}{}^{a}_{b\delta|_{i\gamma}} + \underset{(i1)}{C}{}^{a}_{b\sigma}\overset{i}{\underset{(21)}{P}}{}^{\sigma}_{\gamma\delta} + \underset{(i2)}{C}{}^{a}_{b\sigma}\overset{i}{\underset{(22)}{P}}{}^{\sigma}_{\gamma\delta},
\end{aligned}$$

$$\begin{aligned}
\underset{(2i)}{Q}{}_{b}{}^{a}{}_{\gamma\delta} &= \dot{\partial}_{2\delta}\underset{(i1)}{C}{}^{a}_{b\gamma} - \delta_{1\gamma}\underset{(i2)}{C}{}^{a}_{b\delta} + \underset{(i1)}{C}{}^{e}_{b\gamma}\underset{(i2)}{C}{}^{a}_{e\delta} - \underset{(i1)}{C}{}^{e}_{b\delta}\underset{(i1)}{C}{}^{a}_{e\gamma} + \\
&+ \underset{(i2)}{C}{}^{a}_{b\sigma}\underset{(21)}{P}{}^{\sigma}_{\gamma\delta} \\
\underset{(1i)}{S}{}_{b}{}^{a}{}_{\gamma\delta} &= \delta_{1\delta}\underset{(i1)}{C}{}^{a}_{b\gamma} - \delta_{1\gamma}\underset{(i1)}{C}{}^{a}_{b\delta} + \underset{(i1)}{C}{}^{e}_{b\gamma}\underset{(i1)}{C}{}^{a}_{e\delta} - \underset{(i1)}{C}{}^{e}_{b\delta}\underset{(i1)}{C}{}^{a}_{e\gamma} + \\
&\underset{(i2)}{C}{}^{a}_{b\sigma}\underset{(12)}{R}{}^{\sigma}_{\gamma\delta} \\
\underset{(2i)}{S}{}_{b}{}^{a}{}_{\gamma\delta} &= \dot{\partial}_{2\delta}\underset{(i2)}{C}{}^{a}_{b\gamma} - \dot{\partial}_{2\gamma}\underset{(i1)}{C}{}^{a}_{b\delta} + \underset{(i2)}{C}{}^{e}_{b\gamma}\underset{(i2)}{C}{}^{a}_{e\delta} - \underset{(i2)}{C}{}^{e}_{b\delta}\underset{(i2)}{C}{}^{a}_{e\gamma}
\end{aligned} \tag{6.3}$$

and

$$\begin{aligned}
\underset{(0i)}{R}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} &= \delta_\delta \underset{(i0)}{L}{}^{\alpha}_{\beta\gamma} - \delta_\beta \underset{(i0)}{L}{}^{\alpha}_{\beta\delta} + \underset{(i0)}{L}{}^{\varepsilon}_{\beta\gamma}\underset{(i0)}{L}{}^{\alpha}_{\varepsilon\delta} - \underset{(i0)}{L}{}^{\varepsilon}_{\beta\delta}\underset{(i0)}{L}{}^{\alpha}_{\varepsilon\gamma} + \\
&+ \underset{(i1)}{C}{}^{\alpha}_{\beta\sigma}\underset{(01)}{R}{}^{\sigma}_{\gamma\delta} + \underset{(i2)}{C}{}^{\alpha}_{\beta\sigma}\underset{(02)}{R}{}^{\sigma}_{\gamma\delta}, \\
\underset{(1i)}{P}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} &= \delta_{1\gamma}\underset{(i0)}{L}{}^{\alpha}_{\beta\gamma} - \underset{(i1)}{C}{}^{\alpha}_{\beta\delta|_{i\gamma}} + \underset{(i1)}{C}{}^{\alpha}_{\beta\sigma}\overset{i}{\underset{(11)}{P}}{}^{\sigma}_{\gamma\delta} + \underset{(i2)}{C}{}^{\alpha}_{\beta\sigma}\overset{i}{\underset{(12)}{P}}{}^{\sigma}_{\gamma\delta}, \\
\underset{(2i)}{P}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} &= \dot{\partial}_{2\delta}\underset{(i0)}{L}{}^{\alpha}_{\beta\gamma} - \underset{(i2)}{C}{}^{\alpha}_{\beta\delta|_{i\gamma}} + \underset{(i1)}{C}{}^{\alpha}_{\beta\sigma}\overset{i}{\underset{(21)}{P}}{}^{\sigma}_{\gamma\delta} + \underset{(i2)}{C}{}^{\alpha}_{\beta\sigma}\overset{i}{\underset{(22)}{P}}{}^{\sigma}_{\gamma\delta},
\end{aligned}$$

$$
\begin{aligned}
\underset{(2i)}{Q}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} &= \dot{\partial}_{2\delta}\underset{(i1)}{C}{}^{\alpha}_{\beta\gamma} - \delta_{1\gamma}\underset{(i2)}{C}{}^{\alpha}_{\beta\delta} + \underset{(i1)}{C}{}^{\varepsilon}_{\beta\gamma}\underset{(i2)}{C}{}^{\alpha}_{\varepsilon\delta} - \underset{(i1)}{C}{}^{\varepsilon}_{\beta\delta}\underset{(i1)}{C}{}^{\alpha}_{\varepsilon\gamma} + \\
&+ \underset{(i2)}{C}{}^{\alpha}_{\beta\sigma}\underset{(21)}{P}{}^{\sigma}_{\gamma\delta} \\
\underset{(1i)}{S}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} &= \delta_{1\delta}\underset{(i1)}{C}{}^{\alpha}_{\beta\gamma} - \delta_{1\gamma}\underset{(i1)}{C}{}^{\alpha}_{\beta\delta} + \underset{(i1)}{C}{}^{\varepsilon}_{\beta\gamma}\underset{(i1)}{C}{}^{\alpha}_{\varepsilon\delta} - \underset{(i1)}{C}{}^{\varepsilon}_{\beta\delta}\underset{(i1)}{C}{}^{\alpha}_{\varepsilon\gamma} + \\
&\underset{(i2)}{C}{}^{\alpha}_{\beta\sigma}\underset{(12)}{R}{}^{\sigma}_{\gamma\delta} \\
\underset{(2i)}{S}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} &= \dot{\partial}_{2\delta}\underset{(i2)}{C}{}^{\alpha}_{\beta\gamma} - \dot{\partial}_{2\gamma}\underset{(i1)}{C}{}^{\alpha}_{\beta\delta} + \underset{(i2)}{C}{}^{\varepsilon}_{\beta\gamma}\underset{(i2)}{C}{}^{\alpha}_{\varepsilon\delta} - \underset{(i2)}{C}{}^{\varepsilon}_{\beta\delta}\underset{(i2)}{C}{}^{\alpha}_{\varepsilon\gamma}
\end{aligned} \tag{6.4}
$$

and

$$
\begin{aligned}
\underset{(0i)}{R}{}_{\overline{\beta}}{}^{\overline{\alpha}}{}_{\gamma\delta} &= \delta_{\delta}\underset{(i0)}{L}{}^{\overline{\alpha}}_{\overline{\beta}\gamma} - \delta_{\overline{\beta}}\underset{(i0)}{L}{}^{\overline{\alpha}}_{\overline{\beta}\delta} + \underset{(i0)}{L}{}^{\overline{\varepsilon}}_{\overline{\beta}\gamma}\underset{(i0)}{L}{}^{\overline{\alpha}}_{\overline{\varepsilon}\delta} - \underset{(i0)}{L}{}^{\overline{\varepsilon}}_{\overline{\beta}\delta}\underset{(i0)}{L}{}^{\overline{\alpha}}_{\overline{\varepsilon}\gamma} + \\
&+ \underset{(i1)}{C}{}^{\overline{\alpha}}_{\overline{\beta}\sigma}\underset{(01)}{R}{}^{\sigma}_{\gamma\delta} + \underset{(i2)}{C}{}^{\overline{\alpha}}_{\overline{\beta}\sigma}\underset{(02)}{R}{}^{\sigma}_{\gamma\delta}, \\
\underset{(1i)}{P}{}_{\overline{\beta}}{}^{\overline{\alpha}}{}_{\gamma\delta} &= \delta_{1\gamma}\underset{(i0)}{L}{}^{\overline{\alpha}}_{\overline{\beta}\gamma} - \underset{(i1)}{C}{}^{\overline{\alpha}}_{\overline{\beta}\delta}|_{i\gamma} + \underset{(i1)}{C}{}^{\overline{\alpha}}_{\overline{\beta}\sigma}\underset{(11)}{\overset{i}{P}}{}^{\sigma}_{\gamma\delta} + \underset{(i2)}{C}{}^{\overline{\alpha}}_{\overline{\beta}\sigma}\underset{(12)}{\overset{i}{P}}{}^{\sigma}_{\gamma\delta}, \\
\underset{(2i)}{P}{}_{\overline{\beta}}{}^{\overline{\alpha}}{}_{\gamma\delta} &= \dot{\partial}_{2\delta}\underset{(i0)}{L}{}^{\overline{\alpha}}_{\overline{\beta}\gamma} - \underset{(i2)}{C}{}^{\overline{\alpha}}_{\overline{\beta}\delta}|_{i\gamma} + \underset{(i1)}{C}{}^{\overline{\alpha}}_{\overline{\beta}\sigma}\underset{(21)}{\overset{i}{P}}{}^{\sigma}_{\gamma\delta} + \underset{(i2)}{C}{}^{\overline{\alpha}}_{\overline{\beta}\sigma}\underset{(22)}{\overset{i}{P}}{}^{\sigma}_{\gamma\delta}, \\
\underset{(2i)}{Q}{}_{\overline{\beta}}{}^{\overline{\alpha}}{}_{\gamma\delta} &= \dot{\partial}_{2\delta}\underset{(i1)}{C}{}^{\overline{\alpha}}_{\overline{\beta}\gamma} - \delta_{1\gamma}\underset{(i2)}{C}{}^{\overline{\alpha}}_{\overline{\beta}\delta} + \underset{(i1)}{C}{}^{\overline{\varepsilon}}_{\overline{\beta}\gamma}\underset{(i2)}{C}{}^{\overline{\alpha}}_{\overline{\varepsilon}\delta} - \underset{(i1)}{C}{}^{\overline{\varepsilon}}_{\overline{\beta}\delta}\underset{(i1)}{C}{}^{\overline{\alpha}}_{\overline{\varepsilon}\gamma} + \\
&+ \underset{(i2)}{C}{}^{\overline{\alpha}}_{\overline{\beta}\sigma}\underset{(21)}{P}{}^{\sigma}_{\gamma\delta}, \\
\underset{(1i)}{S}{}_{\overline{\beta}}{}^{\overline{\alpha}}{}_{\gamma\delta} &= \delta_{1\delta}\underset{(i1)}{C}{}^{\overline{\alpha}}_{\overline{\beta}\gamma} - \delta_{1\gamma}\underset{(i1)}{C}{}^{\overline{\alpha}}_{\overline{\beta}\delta} + \underset{(i1)}{C}{}^{\overline{\varepsilon}}_{\overline{\beta}\gamma}\underset{(i1)}{C}{}^{\overline{\alpha}}_{\overline{\varepsilon}\delta} - \underset{(i1)}{C}{}^{\overline{\varepsilon}}_{\overline{\beta}\delta}\underset{(i1)}{C}{}^{\overline{\alpha}}_{\overline{\varepsilon}\gamma} + \ . \\
&\underset{(i2)}{C}{}^{\overline{\alpha}}_{\overline{\beta}\sigma}\underset{(12)}{R}{}^{\sigma}_{\gamma\delta}, \\
\underset{(2i)}{S}{}_{\overline{\beta}}{}^{\overline{\alpha}}{}_{\gamma\delta} &= \dot{\partial}_{2\delta}\underset{(i2)}{C}{}^{\overline{\alpha}}_{\overline{\beta}\gamma} - \dot{\partial}_{2\gamma}\underset{(i1)}{C}{}^{\overline{\alpha}}_{\overline{\beta}\delta} + \underset{(i2)}{C}{}^{\overline{\varepsilon}}_{\overline{\beta}\gamma}\underset{(i2)}{C}{}^{\overline{\alpha}}_{\overline{\varepsilon}\delta} - \underset{(i2)}{C}{}^{\overline{\varepsilon}}_{\overline{\beta}\delta}\underset{(i2)}{C}{}^{\overline{\alpha}}_{\overline{\varepsilon}\gamma}
\end{aligned} \tag{6.5}
$$

$$
\left(i = 0, 1, 2;\ \underset{(12)}{\overset{i}{P}}{}^{\alpha}_{\beta\gamma} = \underset{(12)}{P}{}^{\alpha}_{\beta\gamma};\ \underset{(21)}{\overset{i}{P}}{}^{\alpha}_{\beta\gamma} = \underset{(21)}{P}{}^{\alpha}_{\beta\gamma};\ \underset{(22)}{R}{}^{\alpha}_{\beta\gamma} = 0; \delta_{0a} = \delta_a, \delta_{2a} = \dot{\partial}_{2a}\right).
$$

Proof. The general formulas that express the local curvature d-tensors of an arbitrary N-linear connection (for more details, see [2] and [1]), applied to the relative covariant derivatives $\underset{(i)}{\nabla}$, $(i = 0, 1, 2)$, imply the above formulas. □

7. The Ricci identities

Let $\check{D}\Gamma\left(\check{N}\right) = \left(\underset{(i0)}{\check{L}}{}^{a}_{b\delta}, \underset{(i1)}{\check{C}}{}^{a}_{b\delta}, \underset{(i2)}{\check{C}}{}^{a}_{b\delta}\right)$ be the coupling of the canonical N-linear metric connection D (5.2) with the induced nonlinear connection N along the manifold Osc^2M, $D^{\top}\Gamma\left(\check{N}\right) = \left(\underset{(i0)}{L}{}^{\alpha}_{\beta\delta}, \underset{(i1)}{C}{}^{\alpha}_{\beta\delta}, \underset{(i2)}{C}{}^{\alpha}_{\beta\delta}\right)$ and $D^{\perp}\Gamma\left(\check{N}\right) = \left(\underset{(i0)}{L}{}^{\bar{\alpha}}_{\bar{\beta}\delta}, \underset{(i1)}{C}{}^{\bar{\alpha}}_{\bar{\beta}\delta}, \underset{(i2)}{C}{}^{\bar{\alpha}}_{\bar{\beta}\delta}\right)$ $(i = 0, 1, 2)$ the induced tangent connection on $Osc^2\check{M}$ and the induced normal connection on $Osc^2\check{M}$, respectively.

Theorem 7.1. [3] *For any d-vector fields X^{α}, the following Ricci identities hold:*

$$
\begin{aligned}
X^{\alpha}{}_{|i\beta|i\gamma} - X^{\alpha}{}_{|i\beta|i\gamma} &= X^{\delta} \underset{(0i)}{R}{}_{\delta}{}^{\alpha}{}_{\beta\gamma} - \underset{(0)}{\overset{(i)}{T}}{}^{\sigma}_{\beta\gamma} X^{\alpha}{}_{|i\sigma} - \underset{(01)}{R}{}^{\sigma}_{\beta\gamma} X^{\alpha} \overset{(1)}{|}{}_{i\sigma} - \\
&\quad - \underset{(02)}{R}{}^{\sigma}_{\beta\gamma} X^{\alpha} \overset{(2)}{|}{}_{i\sigma}, \\
X^{\alpha}{}_{|i\beta} \overset{(1)}{|}{}_{i\gamma} - X^{\alpha} \overset{(1)}{|}{}_{i\gamma}{}_{|i\beta} &= X^{\delta} \underset{(1i)}{P}{}_{\delta}{}^{\alpha}{}_{\beta\gamma} - \underset{(i1)}{C}{}^{\sigma}_{\beta\gamma} X^{\alpha}{}_{|i\sigma} - \underset{(11)}{\overset{(i)}{P}}{}^{\sigma}_{\beta\gamma} X^{\alpha} \overset{(1)}{|}{}_{i\sigma} \\
&\quad - \underset{(12)}{P}{}^{\sigma}_{\beta\gamma} X^{\alpha} \overset{(2)}{|}{}_{i\sigma}, \\
X^{\alpha}{}_{|i\beta} \overset{(2)}{|}{}_{i\gamma} - X^{\alpha} \overset{(2)}{|}{}_{i\gamma}{}_{|i\beta} &= X^{\delta} \underset{(2i)}{P}{}_{\delta}{}^{\alpha}{}_{\beta\gamma} - \underset{(i2)}{C}{}^{\sigma}_{\beta\gamma} X^{\alpha}{}_{|i\sigma} - \underset{(21)}{P}{}^{\sigma}_{\beta\gamma} X^{\alpha} \overset{(1)}{|}{}_{i\sigma} \\
&\quad - \underset{(22)}{\overset{(i)}{P}}{}^{\sigma}_{\beta\gamma} X^{\alpha} \overset{(2)}{|}{}_{i\sigma},
\end{aligned}
\tag{7.1}
$$

$$X^{\alpha}\overset{(1)}{|}_{i\beta}\overset{(2)}{|}_{i\gamma}-X^{\alpha}\overset{(2)}{|}_{i\gamma}\overset{(1)}{|}_{i\beta} = X^{\delta}\underset{(21)}{Q}{}_{\delta}{}^{\alpha}{}_{\beta\gamma}-\underset{(i2)}{C}{}^{\sigma}_{\beta\gamma}X^{\alpha}\overset{(1)}{|}_{i\sigma}-$$

$$-\underset{(22)}{\overset{(i)}{Q}}{}^{\sigma}_{\beta\gamma}X^{\alpha}\overset{(2)}{|}_{i\sigma},$$

$$X^{\alpha}\overset{(j)}{|}_{i\beta}\overset{(j)}{|}_{i\gamma}-X^{\alpha}\overset{(j)}{|}_{i\gamma}\overset{(j)}{|}_{i\beta} = X^{\delta}\underset{(ji)}{S}{}_{\delta}{}^{\alpha}{}_{\beta\gamma}-\underset{(j)}{\overset{(i)}{S}}{}^{\sigma}_{\beta\gamma}X^{\alpha}\overset{(1)}{|}_{i\sigma}-$$

$$-\underset{(j2)}{R}{}^{\sigma}_{\beta\gamma}X^{\alpha}\overset{(2)}{|}_{i\sigma}$$

where $\underset{(22)}{R}{}^{\alpha}_{\beta\gamma}=0,(i=0,1,2,j=1,2)$ *and*

$$X=X^{(0)\alpha}\delta_{\alpha}+X^{(1)\alpha}\delta_{1\alpha}+X^{(2)\alpha}\dot{\partial}_{2\alpha}$$

is an arbitrary d-vector field on the submanifold $\check{E}=Osc^2\check{M}$.

Proof. Let (Y_A) and (ω^A) , where $A\in\{(i)\,a,i=0,1,2\}$, be on $\check{E}=Osc^2\check{M}$ the bases and the dual bases adapted to the nonlinear connection N, and let $X=X^FY_F$ be a d-vector field on E^*. In this context, using the following true equalities (applied for the induced tangent connection $D^{\top}\Gamma\left(\check{N}\right)$):

(1) $D_{Y_C}Y_B=\Gamma^F_{BC}Y_F$,

(2) $[Y_B,Y_C]=R^F_{BC}Y_F$,

(3) $\mathbb{T}(Y_C,Y_B)=\mathbb{T}^F_{BC}Y_F=\{\Gamma^F_{BC}-\Gamma^F_{CB}-R^F_{CB}\}Y_F$,

(4) $\mathbb{R}(Y_C,Y_B)Y_A=\mathbb{R}^F_{ABC}Y_F$,

(5) $D_{Y_C}\omega^B=-\Gamma^B_{FC}\omega^F$,

(6) $[\mathbb{R}(Y_C,Y_B)X]\otimes\omega^B\otimes\omega^C=\{D_{Y_C}D_{Y_B}X-$
$-D_{Y_B}D_{Y_C}X-D_{[Y_C,Y_B]}X\}\otimes\omega^B\otimes\omega^C$,

by a direct calculation, we find that

$$X^A_{:B:C}-X^A_{:C:B}=X^F\mathbb{R}^A_{FBC}-X^A_{:F}\mathbb{T}^F_{BC}, \tag{7.2}$$

where "$_{:G}$" represents one from the local covariant derivatives "$_{|i\delta}$", "$\overset{(1)}{|}_{i\delta}$" or "$\overset{(2)}{|}_{i\delta}$" produced by the induced tangent connection $D^{\top}\Gamma\left(\check{N}\right)$.

Taking into account in (7.2) that the indices $A,B,C,\ldots$ belong to the set $\{(i)\,a,i=0,1,2\}$ by complicated computations, we find what we were looking for. □

The Ricci identities (7.1) applied to the Liouville d-vector fields $z^{(1)\alpha}$ and $z^{(2)\alpha}$ lead to the next theorem.

Theorem 7.2. *The deflection tensor fields satisfy the following identities:*

$$
\begin{aligned}
\underset{i}{\overset{(j)}{D}}{}^{\alpha}_{\beta|_{i\gamma}} - \underset{i}{\overset{(j)}{D}}{}^{\alpha}_{\gamma|_{i\beta}} &= z^{(j)\delta} \underset{(0i)}{R}{}_{\delta}{}^{\alpha}{}_{\beta\gamma} - \underset{i}{\overset{(j)}{D}}{}^{\alpha}_{\delta} \underset{(0)}{\overset{(i)}{T}}{}^{\delta}_{\beta\gamma} - \\
& - \underset{i}{\overset{(j1)}{d}}{}^{\alpha}_{\delta} \underset{(01)}{R}{}^{\delta}_{\beta\gamma} - \underset{i}{\overset{(j2)}{d}}{}^{\alpha}_{\delta} \underset{(02)}{R}{}^{\delta}_{\beta\gamma}, \\
\underset{i}{\overset{(j)}{D}}{}^{\alpha}_{\beta} \overset{(1)}{|}_{i\gamma} - \underset{i}{\overset{(j1)}{d}}{}^{\alpha}_{\gamma|_{i\beta}} &= z^{(j)\delta} \underset{(1i)}{P}{}_{\delta}{}^{\alpha}{}_{\beta\gamma} - \underset{i}{\overset{(j)}{D}}{}^{\alpha}_{\delta} \underset{(i1)}{C}{}^{\delta}_{\beta\gamma} - \\
& - \underset{i}{\overset{(j1)}{d}}{}^{\alpha}_{\delta} \underset{(11)}{\overset{(i)}{P}}{}^{\sigma}_{\beta\gamma} - \underset{i}{\overset{(j2)}{d}}{}^{\alpha}_{\delta} \underset{(12)}{P}{}^{\delta}_{\beta\gamma}, \\
\underset{i}{\overset{(j)}{D}}{}^{\alpha}_{\beta} \overset{(2)}{|}_{i\gamma} - \underset{i}{\overset{(j2)}{d}}{}^{\alpha}_{\gamma|_{i\beta}} &= z^{(j)\delta} \underset{(2i)}{P}{}_{\delta}{}^{\alpha}{}_{\beta\gamma} - \underset{i}{\overset{(j)}{D}}{}^{\alpha}_{\delta} \underset{(i2)}{C}{}^{\sigma}_{\beta\gamma} - \\
& - \underset{i}{\overset{(j1)}{d}}{}^{\alpha}_{\delta} \underset{(21)}{P}{}^{\delta}_{\beta\gamma} - \underset{i}{\overset{(j2)}{d}}{}^{\alpha}_{\delta} \underset{(22)}{\overset{(i)}{P}}{}^{\sigma}_{\beta\gamma}, \\
\underset{i}{\overset{(j1)}{d}}{}^{\alpha}_{\beta} \overset{(2)}{|}_{i\gamma} - \underset{i}{\overset{(j2)}{d}}{}^{\alpha}_{\gamma} \overset{(1)}{|}_{i\beta} &= z^{(j)\delta} \underset{(2i)}{Q}{}_{\delta}{}^{\alpha}{}_{\beta\gamma} - X^{\alpha} \overset{(1)}{|}_{i\sigma} - \\
& - \underset{i}{\overset{(j1)}{d}}{}^{\alpha}_{\delta} \underset{(i2)}{C}{}^{\delta}_{\beta\gamma} - \underset{i}{\overset{(j2)}{d}}{}^{\alpha}_{\delta} \underset{(22)}{\overset{(i)}{Q}}{}^{\delta}_{\beta\gamma}, \\
\underset{i}{\overset{(ji)}{d}}{}^{\alpha}_{\beta} \overset{(l)}{|}_{i\gamma} - \underset{i}{\overset{(ji)}{d}}{}^{\alpha}_{\gamma} \overset{(l)}{|}_{i\beta} &= z^{(j)\delta} \underset{(li)}{S}{}_{\delta}{}^{\alpha}{}_{\beta\gamma} - \\
& - \underset{i}{\overset{(jl)}{d}}{}^{\alpha}_{\delta} \underset{(i)}{\overset{(l)}{S}}{}^{\delta}_{\beta\gamma} - \underset{i}{\overset{(j2)}{d}}{}^{\alpha}_{\delta} \underset{(l2)}{R}{}^{\delta}_{\beta\gamma},
\end{aligned}
\tag{7.3}
$$

$\left(i = 0, 1, 2; j, l = 1, 2;\ \underset{(22)}{R}{}^{\alpha}_{\beta\gamma} = 0.\right)$

Also, if the $\left(z^{(1)}\right)$-and $\left(z^{(2)}\right)$-deflection tensors have the following particular form

$$
\begin{aligned}
&\underset{i}{\overset{(1)}{D}}{}^{\alpha}_{\beta} = 0, \qquad \underset{i}{\overset{(11)}{d}}{}^{\alpha}_{\beta} = \delta^{\alpha}_{\beta}, \qquad \underset{i}{\overset{(12)}{d}}{}^{\alpha}_{\beta} = 0 \\
&\underset{i}{\overset{(2)}{D}}{}^{\alpha}_{\beta} = 0, \qquad \underset{i}{\overset{(21)}{d}}{}^{\alpha}_{\beta} = 0, \qquad \underset{i}{\overset{(22)}{d}}{}^{\alpha}_{\beta} = \delta^{\alpha}_{\beta}
\end{aligned}
\tag{7.4}
$$

then, the fundamental identities from (7.3) are very important, especially for applications.

Proposition 7.3. With the deflection tensor which are given by (7.4), the following identities hold:

$$
\begin{aligned}
&z^{(j)\delta}\underset{(0i)}{R}{}_{\delta}{}^{\alpha}{}_{\beta\gamma}=\underset{(0j)}{R}{}^{\alpha}_{\beta\gamma}, && z^{(1)\delta}\underset{(2i)}{P}{}_{\delta}{}^{\alpha}{}_{\beta\gamma}=\underset{(21)}{P}{}^{\alpha}_{\beta\gamma}, && z^{(2)\delta}\underset{(1i)}{P}{}_{\delta}{}^{\alpha}{}_{\beta\gamma}=\underset{(12)}{P}{}^{\alpha}_{\beta\gamma},\\
&z^{(j)\delta}\underset{(ji)}{P}{}_{\delta}{}^{\alpha}{}_{\beta\gamma}=\underset{(jj)}{P}{}^{\alpha}_{\beta\gamma}, && z^{(1)\delta}\underset{(2i)}{Q}{}_{\delta}{}^{\alpha}{}_{\beta\gamma}=\underset{(i2)}{C}{}^{\alpha}_{\beta\gamma}, && z^{(2)\delta}\underset{(2i)}{Q}{}_{\delta}{}^{\alpha}{}_{\beta\gamma}=\underset{(22)}{\overset{i}{Q}}{}^{\alpha}_{\beta\gamma},\\
&z^{(j)\delta}\underset{(ji)}{S}{}_{\delta}{}^{\alpha}{}_{\beta\gamma}=\underset{(i)}{\overset{j}{S}}{}^{\alpha}_{\beta\gamma}, && z^{(1)\delta}\underset{(2i)}{S}{}_{\delta}{}^{\alpha}{}_{\beta\gamma}=0, && z^{(2)\delta}\underset{(1i)}{S}{}_{\delta}{}^{\alpha}{}_{\beta\gamma}=\underset{(12)}{R}{}^{\alpha}_{\beta\gamma}.
\end{aligned}
\tag{7.5}
$$

$(i=0,1,2; j=1,2)$.

Proof. Using the Ricci identities of the Liouville d-vector fields $z^{(1)\alpha}$ and $z^{(2)\alpha}$ from the last theorem and the particular form of the $\left(z^{(1)}\right)$-and $\left(z^{(2)}\right)$-deflection tensors from (7.4) we get the Formulae (7.5). □

Remark 7.4. The deflection d-tensor identities (7.3) will be used in the near future for the construction of the geometrical Maxwell equations that will govern the abstract "electromagnetism" in the Lagrange subspaces of second order (this is our work in progress).

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