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Modeling Dynamic Production Systems with Network Structure

F. Koushki

Department of Electrical, Biomedical and Mechatronics Engineering, Qazvin Branch, Islamic Azad University, Qazvin, Iran.
E-mail: fkoushki@gmail.com

Abstract. This paper deals with the problem of optimizing two-stage structure decision making units (DMUs) where the activity and the performance of two-stage DMU in one period effect on its efficiency in the next period. To evaluate such systems the effect of activities in one period on ones in the next term must be considered. To do so, here a dynamic DEA approach presented to measure the performance of such network units. According to the results of proposed dynamic model the inefficiencies of DMUs improve considerably. Additionally, in models which measure efficiency score, undesirable outputs are mostly treated as inputs, which do not reflect the true production process. This paper proposes an alternative method in dealing with bad outputs. Statistical analysis of sub-efficiencies, i.e. efficiency score of each stage, during all periods represents useful information about the total performance of the stage over all periods.

Keywords: Data envelopment analysis (DEA), Network DEA, Bad outputs, Dynamic DEA, Sub-efficiency.

2010 Mathematics subject classification: 90C05, 90B10, 90C35.

1. Introduction

Data envelopment analysis (DEA) is a nonparametric technique based on mathematical programming to evaluate performance of homogenous multi input/output decision making units. In decision-making units, which use multiple

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inputs to produce multiple outputs, managers make decisions about how to use, integrate and process the inputs and resources.

In traditional DEA models, the efficiency of DMUs measured by assuming that the inputs are used to produce only final outputs. In contrast, some production systems have a network structure, such as when production by one division or sub-process results in an intermediate output that is an input to another sub-process. In these cases, managers are likely to glean more information from sub-process or divisional efficiency measures than from firm level efficiency measures. Botti et al. [3] use DEA to estimate the efficiency of French hotel chains. They found that the plural form of organization consisting of a mixture of company-owned subsidiaries and franchised units was more efficient than hotel chains dominated by franchises and hotel chains dominated by company owned subsidiaries.

There are many decision making units with network structure in which the outputs of one division or sub-process are the inputs to another sub-process. Färe and Primont [8], first time, applied a DEA approach to evaluate performance of multi-plant firms as multi-stage structure DMUs. Färe and Grosskopf [6, 7] suggested models to measure efficiency score of multi-stage DMUs in static and dynamic cases. In dynamic case, activities of DMU in one period effect on ones in the next period.

There have been many studies dealing with systems with two-stage structure. Wang et al. [16] proposed a DEA model to measure efficiency score of two-stage structure DMUs without consideration of intermediate products. Seiford and Zhu [15] extended their approach and applied modified model to assess the efficiency of the top 55 US commercial banks.

Chen and Zhu [4] improved models presented by Seiford and Zhu with consideration the intermediate products to project two-stage structure DMUs on efficient frontier. Kao and Hwang [11] evaluated efficiency score of two-stage DMU as product of efficiencies of its stages. Chen et al. [5] measured the efficiency score of two-stage structure DMU as a weighted mean of efficiency scores of stages. Aslani et al. [2] proposed radial basis function neural networks with the K-means clustering method for evaluation of bank branches. Yang et al. [18] presented a non-linear programming to measure efficiency of two-member supply chains, as two-stage DMUs.

Paradi et al. [13] developed a two-stage DEA approach for simultaneously benchmarking the performance of operating units and a modified slacks-based measure model to aggregate the obtained efficiency scores from stage one and generate a composite performance index for each unit. Fukuyama and Mirdehghan [9] proposed slack-based network approach for identifying the efficiency status of each DMU and its divisions. Amirteimoori [1] and Liu [12] proposed DEA approaches for performance assessment of two-stage decision process in existence of imperfect outputs and Fuzzy data respectively. Wang et al. [17]

utilized the network DEA approach to evaluate the efficiencies of major Chinese commercial banks.

Banks have two-stage structure in which labor, physical capital, and financial equity capital are inputs of the first stage to raise deposits, which are as intermediate output. In the second stage, deposits are used to produce loans and security investments as the outputs of this stage.

Bad loans produced in one period might limit the way labor, physical capital, and equity capital are able to produce deposits in the next period. It means that performance of DMU in one period effect on its performance in the next period. Therefore, a dynamic model is required to evaluate such systems.

In this paper, a dynamic DEA approach is presented to measure the performance of such network units. In models which evaluate DMUs, the bad outputs mostly are treated as inputs so that the true production process is not reflected. Here an alternative method is considered in dealing with bad outputs. In section 2, some preliminaries of two-stage structure DMUs are given; additionally, non-redial dynamic model to measure inefficiency, for several periods, of whole DMU process at each period and during all periods is proposed. In proposed dynamic model, bad outputs are treated properly. In section 3, the superiority of the dynamic model over the traditional separation model ,which measures inefficiency of DMU in each period separately without considering the relation between the periods, is demonstrated during an illustrative example. Statistical analysis of efficiency scores of each stage, named sub-efficiencies, during all periods is presented in section 4.

2. Dynamic Model

Let x_j, y_j, b_j and be the N-dimensional input, M-dimensional final output, L-dimensional bad output vectors and vector of variables associated with intermediate output, respectively, of DMU_j, $j = 1, \ldots, J$. Deposits are considered as an intermediate output from stage 1 and an input to stage 2.The network slacks-based inefficiency [10] is defined as follows

$$NSBI(\mathrm{x}_0, \mathrm{y}_0, \mathrm{b}_0) = \max_{\mathrm{s}^-, \mathrm{s}^+, s^{\#}, \lambda^1, \lambda^2} \left\{ \frac{\frac{1}{N}\sum_{n=1}^{N} \frac{s_n^-}{x_n} + \frac{1}{M}\sum_{m=1}^{M} \frac{s_m^+}{y_m} + \frac{1}{L}\sum_{l=1}^{L} \frac{s_l^{\#}}{b_l}}{3} \right|$$

$$\mathrm{x}_0 = \sum_{j=1}^{J} \mathrm{x}_j \lambda_j^1 + \mathrm{s}^-, \quad \mathrm{y}_0 = \sum_{j=1}^{J} \mathrm{y}_j \lambda_j^2 - \mathrm{s}^+, \quad \mathrm{b}_0 = \sum_{j=1}^{J} \mathrm{b}_j \lambda_j^2 + \mathrm{s}^{\#}$$

$$\left.\begin{aligned}&\sum_{j=1}^{J} \mathrm{z}_j\lambda_j^1 \geq \mathrm{z}_0, \quad \sum_{j=1}^{J} \mathrm{z}_j\lambda_j^2 \leq \mathrm{z}_0\\&\sum_{j=1}^{J} \lambda_j^1 = 1, \quad \sum_{j=1}^{J} \lambda_j^2 = 1\\&\lambda_j^1 \geq 0,\ \lambda_j^2 \geq 0,\ \mathrm{s}^- \geq 0,\ \mathrm{s}^+ \geq 0,\ \mathrm{s}^\# \geq 0\end{aligned}\right\} \tag{2.1}$$

λ^1, λ^2 are vectors of intensity variables for the two stages.

The production of bad loans in one period might limit the way labor, physical capital, and equity capital are able to produce deposits in the next period.

In addition, the profit earned by bank activities in one period, makes the financial capital increase in the next period.

This paper improves static model (2.1) and uses a dynamic model to measure the inefficiency of DMU_0 at each term and the overall inefficiency, in which the effects of activities at each term on the next term are considered.

At each term t, the whole profit earned by bank activities will increase the financial capital at the next term. Let p^t be the value of whole profit at term t. Suppose x_{01}^t indicates the financial capital in the input vector of DMU_0 at term t.Hence, at term t, $t = 2, \ldots, T$, this value will be $x_{01}^t + p^{t-1}$.The whole profit earned in each term, is a proportion of the financial capital in that term. Thus, we have $p^t = \alpha x_{01}^t \quad 0 < \alpha < 1$.

The bad outputs at each term t, are nonperforming loans with borrowers unable to make full or even partial repayment. Therefore, bad outputs at term t, b_0^t, make financial capital reduce as much as $\sum_{l=1}^{L} b_{01}^t$ at the next term. At term t, $t = 2, \ldots, T$, this value will be $x_{01}^t - \sum_{l=1}^{L} b_{01}^{t-1}$.

Previous models do not reflect the true production process, as they consider bad outputs as inputs. Here, another method is considered to treat bad outputs as follow calculations:

Let $b_{sr} = max_j\{b_{jr}\} \quad r = 1, \ldots, L$; Then, find $u_r \quad r = 1, \ldots, L$ such that $b_{sr} = -b_{sr} + u_r \quad r = 1, \ldots, L$ Then let $\bar{b}_{jr} = -b_{jr} + u_r \quad j = 1, \ldots, J \quad \ldots r = 1, \ldots, L$.

According to the above method, the values of $\bar{b}$ are desirable outputs and the increment of them is favorite and results in decrease of the values of b as bad outputs.

In addition, deposits as the outputs of the first stage are the inputs to the second stage. This continuity and series relationship between two stages implies that $\sum_{j=1}^{J} \lambda_j^1 Z_j = \sum_{j=1}^{J} \lambda_j^2 Z_j$ instead of inequalities $\sum_{j=1}^{J} z_j \lambda_j^1 \geq z_0$, $\sum_{j=1}^{J} z_j \lambda_j^2 \leq z_0$ in model (2.1).

Therefore, the network slacks-based overall inefficiency is defined as follows:

$$\max \frac{1}{T}\sum_{t=1}^{T} w^t \left[\frac{\frac{1}{N}\sum_{n=1}^{N}\frac{s_n^{-t}}{x_{0n}^t}+\frac{1}{M}\sum_{m=1}^{M}\frac{s_m^{+t}}{y_{0m}^t}+\frac{1}{L}\sum_{l=1}^{L}\frac{s_l^{\#^t}}{\bar{b}_{0l}^t}}{3}\right]$$

$$\begin{aligned}
s.t.\ &\sum_{j=1}^{J}\mathrm{x}_j^1\lambda_j^{1^1}+\mathrm{s}^{-^1}=\mathrm{x}_0^1\\
&\sum_{j=1}^{J}x_{j^1}^t\lambda_j^{1^t}+s_1^{-t}=x_{01}^t+p^{t-1}-\sum_{l=1}^{L}b_{0l}^{t-1}\quad t=2,\ldots,T\\
&\sum_{j=1}^{J}x_{jn}^t\lambda_j^{1^t}+s_n^{-t}=x_{0n}^t\quad n=2,\ldots,N\quad t=2,\ldots,T\\
&\sum_{j=1}^{J}\mathrm{y}_j^t\lambda_j^{2^t}+\mathrm{s}^{+t}=\mathrm{y}_0^t\quad t=1,\ldots,T\\
&\sum_{j=1}^{J}\bar{\mathrm{b}}_j^t\lambda_j^{2^t}+\mathrm{s}^{\#^t}=\bar{\mathrm{b}}_0^t\quad t=1,\ldots,T\\
&\sum_{j=1}^{J}z_j^t\lambda_j^{1^t}=\sum_{j=1}^{J}z^t\lambda_j^{2^t}\quad t=1,\ldots,T\\
&\sum_{j=1}^{J}\lambda_j^{1^t}=1\quad t=1,\ldots,T\\
&\sum_{j=1}^{J}\lambda_j^{2^t}=1\quad t=1,\ldots,T\\
&\lambda^{1^t}\geq 0,\lambda^{2^t}\geq 0\quad t=1,\ldots,T\\
&\mathrm{S}^{-^t}\geq 0,\mathrm{S}^{+^t}\geq 0,\mathrm{S}^{\#^t}\geq 0\quad t=1,\ldots,T
\end{aligned}\tag{2.2}$$

w^t is the weight of term t ,which is determined by manager with attention to the importance of term t , satisfying $\sum_{t=1}^{T} w^t = T$

In this model we assume that data is independent of passing the time. Otherwise a rate of interest per term should be considered in modeling.

Let $\left(\{\lambda^{t^*}\},\{\mathrm{s}^{-^{t^*}}\},\{\mathrm{s}^{+^{t^*}}\},\{\mathrm{s}^{\#^{t^*}}\}\right)$ be an optimal solution. We define the inefficiency score for DMU_0 at term by

$$\beta_0^{t^*}=\frac{\frac{1}{N}\sum_{n=1}^{N}\frac{s_n^{-t^*}}{x_{0n}^t}+\frac{1}{M}\sum_{m=1}^{M}\frac{s_m^{+t^*}}{y_{0m}^t}+\frac{1}{L}\sum_{l=1}^{L}\frac{s_l^{\#t^*}}{\bar{b}_{01}^t}}{3}$$

The overall inefficiency is the weighted average of the term efficiencies as described below:

$$\beta_0^* = \frac{1}{T}\sum_{t=1}^{T} w^t \beta_0^{t^*}$$

If all optimal solutions satisfy $\beta_0^{t^*} = 0$, then DMU_0 is efficient for the term t. This implies that the optimal slacks for the term are all zero.

If $\beta_0^* = 0$, then DMU_0 is overall efficient and all optimal slacks are zero.

Thus we have:

Theorem 2.1. DMU_0 *is overall efficient, if and only if it is efficient for all terms.*

The term inefficiencies and the overall inefficiency satisfy the following optimality property.

Theorem 2.2. *If for two* DMUs *a and b, their terms of inefficiencies satisfy the inequality $\beta_a^{t^*} \geq \beta_b^{t^*}$ $(\forall t)$, then it holds that $\beta_a^* \geq \beta_b^*$. Addition to assumptions, if there exists a term t such that $\beta_a^{t^*} > \beta_b^{t^*}$, then we have the strict inequality $\beta_a^* > \beta_b^*$.*

Proof. From the definition of overall inefficiency, this theorem holds. □

We define the projected of DMU_0 as follows:

$$\begin{aligned}
&\bar{\mathrm{x}}_0^1 = \mathrm{x}_0^1 - \mathrm{s}^{-1^*}\\
&\bar{\mathrm{x}}_{01}^t = \mathrm{x}_{01}^t + p^{t-1} - \sum_{l-1}^{L} \mathrm{b}_{01}^{t-1} - \mathrm{s}_1^{-t^*} \quad t = 2, \ldots, T\\
&\bar{\mathrm{x}}_{0n}^t = \mathrm{x}_{0n}^t - \mathrm{s}_n^{-t^*} \quad n = 2, \ldots, N \quad t = 2, \ldots, T\\
&\bar{\mathrm{y}}_0^t = \mathrm{y}_0^t + \mathrm{s}^{+t^*} \quad t = 1, \ldots, T\\
&\bar{\bar{\mathrm{b}}}_0^t = \bar{\mathrm{b}}_0^t + \mathrm{s}^{\#t^*} \quad t = 1, \ldots, T
\end{aligned}$$

Theorem 2.3. *The projected* DMU_0 *is overall efficient.*

Proof. We evaluate the overall efficiency of the projected DMU.

Let $\left(\{\bar{\lambda}^{t^*}\},\left\{\bar{s}^{-t^*}\right\},\left\{\bar{s}^{+t^*}\right\},\left\{\bar{s}^{\#t^*}\right\}\right)$ be an optimal solution. Then we have:

$$\bar{\mathrm{x}}_0^1=\sum_{j=1}^{J}\mathrm{x}_0^1\bar{\lambda}_j^{1^{1^*}}+\bar{\mathrm{s}}^{-1^*}$$

$$\bar{x}_{01}^t+p^{t-1}-\sum_{l=1}^{L}b_{01}^{t-1}=\sum_{j=1}^{J}x_{jl}^t\bar{\lambda}_j^{1^{t^*}}+\bar{s}_1^{-t^*}\qquad t=2,\ldots,T$$

$$\bar{x}_{0n}^t=\sum_{j=1}^{J}x_{jn}^t\bar{\lambda}_j^{1^{t^*}}+\bar{s}^{-t^*}\quad n=2,\ldots,N\qquad t=2,\ldots,T$$

$$\bar{\mathrm{y}}_0^t=\sum_{j=1}^{J}\mathrm{y}_j^t\bar{\lambda}_j^{2^{t^*}}-\bar{\mathrm{s}}^{+^{t^*}}\qquad t=1,\ldots,T$$

$$\bar{\bar{\mathrm{b}}}_0^t=\sum_{j=1}^{J}\bar{\mathrm{b}}_j^t\bar{\lambda}_j^{2t^*}-\bar{\mathrm{s}}^{\#^{t^*}}\qquad t=1,\ldots,T$$

Replacing $\bar{\mathrm{x}}_0^1$, $\bar{\mathrm{x}}_{01}^t+\mathrm{p}^{t-1}-\sum_{l=1}^{L}\mathrm{b}_{01}^{t-1}$, $\bar{\mathrm{x}}_{0n}^t$, $\bar{\mathrm{y}}_0^t$ and $\bar{\bar{\mathrm{b}}}_0^t$, we have:

$$\mathrm{x}_0^1=\sum_{j=1}^{J}\mathrm{x}^1\bar{\lambda}_j^{1^{1^*}}+\mathrm{s}^{-1^*}+\bar{\mathrm{s}}^{-1^*}$$

$$x_{01}^t+p^{t-1}-\sum_{l=1}^{L}b_{01}^{t-1}=\sum_{j=1}^{J}x_{jl}^t\bar{\lambda}_j^{1^{t^*}}+s_1^{-t^*}+\bar{s}_1^{-t^*}\quad t=2,\ldots,T$$

$$x_{0n}^t=\sum_{j=1}^{J}x_{jn}^t\bar{\lambda}_j^{1^{t^*}}+s_n{}^{-t^*}+\bar{s}_n^{-t^*}\quad n=2,\ldots,N\quad t=2,\ldots,T$$

$$\mathrm{y}_0^t=\sum_{j=1}^{J}\mathrm{y}_j^t\bar{\lambda}_j^{2^{t^*}}-\mathrm{s}^{+^{t^*}}-\bar{\mathrm{s}}^{+^{t^*}}\quad t=1,\ldots,T$$

$$\bar{\mathrm{b}}_0^t=\sum_{j=1}^{J}\bar{\mathrm{b}}_j^t\bar{\lambda}_j^{2^{t^*}}-\mathrm{s}^{\#^{t^*}}-\bar{\mathrm{s}}^{\#^{t^*}}\quad t=1,\ldots,T$$

Considering this expression we have the overall inefficiency,

$$\beta_0^{**}=\frac{1}{T}\sum_{t=1}^{T}w^t\frac{1}{3}\Big[\frac{1}{N}\sum_{n=1}^{N}\frac{s_n^{-t^*}+\bar{s}_n^{-t^*}}{x_{0n}^t}+\frac{1}{M}\sum_{m=1}^{M}\frac{s_m^{+t^*}+\bar{s}_m^{+t^*}}{y_{0m}^t}+\frac{1}{L}\sum_{l=1}^{L}\frac{s_l^{\#t^*}+\bar{s}_l^{\#t^*}}{\bar{b}_{01}^t}\Big].$$

If any member of $(\{\bar{s}^{-t^*}\},\{\bar{s}^{+t^*}\},\{\bar{s}^{\#t^*}\}$ is positive, then it holds that $\beta_0^{**}>\beta_0^*$.

This contradicts the optimality of β_0^*. Thus, we have $\bar{\mathrm{s}}^{-t^*}=0$, $\bar{\mathrm{s}}^{+t^*}=0$ and $\bar{\mathrm{s}}^{\#t^*}=0$, $\forall t$.

Hence the projected DMU is overall efficient. □

3. An Illustrative Example

The data, which is drawn from 20 branches of some bank, is shown in tables 1 and 2. Inputs to the first stage are financial capital (x_1), personnel cost (x_2), official cost (x_3), and rent cost (x_4). The intermediate output of the first stage is raised funds (z). In the second stage, the desirable output is all assets associated with business activities (y) and the undesirable output is sum of all loans to bankrupt borrowers and the balance of non-accrual delinquent loans (b).

All data is in billion.

	x^1_{j1}	x^1_{j2}	x^1_{j3}	x^1_{j4}	y^1_j	p_j	b^1_j	z^1_j
1	31169	2326	13	300	37777	9988	592	46818
2	19072	2151	50	2304	20193	1662	201	23260
3	19538	1881	29	204	27228	13591	162	31416
4	75773	5395	36	1232	68971	33806	137	77388
5	52543	4770	26	8064	52631	31154	241	61702
6	61518	4235	11	144	56788	176	292	64503
7	283725	25501	15	4480	283898	66081	429	284957
8	28578	2679	5	60	26229	12793	149	33655
9	37128	3592	13	2520	39582	10199	159	42132
10	280455	13509	17	5760	453842	693940	472	494410
11	25678	2700	23	4560	39670	975	403	43567
12	103567	19153	37	411	176000	11392	1100	190234
13	40306	3200	20	150	55460	3627	506	59445
14	252300	24000	30	2114	293523	12615	4200	300700
15	80445	6532	17	190	106000	5631	980	111760
16	93523	8800	19	645	110541	9352	1013	121956
17	300123	44000	36	270	352400	12004	4921	379000
18	35134	3500	19	460	49763	4918	512	63000
19	53523	3960	21	89	76534	4817	603	87690
20	20200	2240	33	211	40370	1414	450	44300

Table 1. Data

The values of $\bar{b}$ associated with bad outputs calculated in table 3.

We assign weights of one for each term and use the dynamic model for 2 terms.

α is considered in interval [0.02,0.14].

Here GAMS is used to estimate each inefficiency measure.

To clarify features of dynamic model, the results are compared with those of separate model, in table 5, which measures inefficiency score separately for

	x_{j1}^2	x_{j2}^2	x_{j3}^2	x_{j4}^2	y_j^2	b_j^2	z_j^2
1	56214	4986	11	400	56646	102	58806
2	24721	3587	49	2880	27870	62	34921
3	44845	3526	15	228	46632	77	55007
4	111057	6537	46	1540	103790	196	111200
5	92615	7905	29	10080	99948	294	100856
6	64387	5217	13	168	78581	278	89109
7	350609	30850	12	5600	445937	243	451038
8	46299	4144	7	96	39411	114	46557
9	52172	5131	16	3150	44358	359	54430
10	1187878	29311	12	7200	1187838	1114	1198351
11	48967	2700	23	9609	50320	211	57640
12	198600	3350	21	712	207489	2300	215432
13	73522	4800	40	270	80960	398	85921
14	403755	19202	87	4600	443213	5670	459700
15	165988	3500	30	286	186400	1530	190917
16	201546	5690	59	1580	217564	2768	223600
17	453214	9489	98	570	483790	6403	490000
18	73966	5780	29	540	88917	412	95412
19	110731	7917	37	150	146712	1130	153600
20	52700	4112	21	413	76215	117	83700

Table 2. Data

each term. Over all inefficiency of separate model is calculated as an average of term efficiencies during the two terms. Bank activities in each term affect the inputs of the next term; thus they affect the performance of the next term. In dynamic model, the effects of each term activities are considered. The results of dynamic model in table 4 show that by considering the effects of each term activities on the next term, the inefficiencies of DMUs improve considerably.

4. Sub-efficiencies Statistical Analysis

Financial capital, personnel cost, official cost, and rent cost are inputs to stage 1. Deposits are produced as an output from stage 1 and an input to stage 2. The desirable output is all assets associated with business activities and the undesirable output is sum of all loans to bankrupt borrowers and the balance of non-accrual delinquent loans.

Thus we can associate an efficiency score with each stage, considering their input and output respectively, named sub-efficiency and is given by following SBM models for stage 1 and stage 2, respectively:

$\bar{b}_j^1$	$\bar{b}_j^2$
9250	12704
9641	12744
9680	12729
9705	12610
9601	12512
9550	12528
9413	12563
9693	12692
9683	12447
9370	11692
9439	12595
8742	10506
9336	12408
5642	11276
8862	10038
8829	6403
4921	12394
9330	11676
9239	3960
9392	12689

TABLE 3. Translated outputs

$$\theta_1^* = \min \frac{1 - \frac{1}{N}\sum_{n=1}^{N}\frac{S_n^{\#^-}}{x_{n0}}}{1 + \frac{1}{Q}\sum_{q=1}^{Q}\frac{S_q^{\#^-}}{z_{q0}}}$$
$$s.t. \sum_{j=1}^{J}\lambda_j^1 X_j + \mathrm{S}^{\#^-} = \mathrm{X}_0$$
$$\sum_{j=1}^{J}\lambda_j^1 Z_j + \mathrm{S}^{\#^+} = Z_0$$
$$\mathrm{S}^{\#^-},\ \mathrm{S}^{\#^+}, \lambda^1 \geq 0 \tag{4.1}$$

$$\theta_2^* = \min \frac{1 - \frac{1}{Q}\sum_{q=1}^{Q}\frac{S_q^{'-}}{Z_{q0}}}{1 + \frac{1}{M}\sum_{m=1}^{M}\frac{S_m^{'+}}{y_{m0}} + \frac{1}{L}\sum_{l=1}^{L}\frac{S_l^{''+}}{\bar{b}_{l0}}}$$
$$s.t. \sum_{j=1}^{J}\lambda_j^2 Z_j + \mathrm{S}^{'-} = Z_0$$
$$\sum_{j=1}^{J}\lambda_j^2 Y_j - \mathrm{S}^{'+} = Y_0$$
$$\sum_{j=1}^{J}\lambda_j^2 \bar{b}_j - \mathrm{S}^{''+} = \bar{b}_0$$
$$\mathrm{S}^{'-}, \mathrm{S}^{'+}, \mathrm{S}^{''+}, \lambda^2 \geq 0 \tag{4.2}$$

	Overall inefficiency	Term inefficiency	
		t_1	t_2
1	0.09645	0.05471	0.13819
2	0.00000	0.00000	0.00000
3	0.00651	0.00000	0.01302
4	0.26405	0.26266	0.26543
5	0.28185	0. 30731	0.25639
6	0.06504	0.09983	0.03024
7	0.10009	0.20018	0.00000
8	0.00818	0.00000	0.01636
9	0.24103	0.22964	0.25241
10	0.00000	0.00000	0.00000
11	0.09401	0.16137	0.02665
12	0.04830	0.04009	0.05652
13	0.13035	0.09626	0.16444
14	0.21263	0.27047	0.15479
15	0.04830	0.02956	0.06705
16	0.29455	0.15699	0.43211
17	0.11784	0.23276	0.00292
18	0.13287	0.06317	0.20257
19	0.29819	0.03967	0.55671
20	0.01667	0.01631	0.01703

TABLE 4. Results of dynamic model

Let θ_k^{t1} be the sub-efficiency score of stage 1 of DMU$_k$ in term t and $\bar{\theta}_k^1$ be the mean value of such scores all over the T terms. Thus the DMU with larger mean value has more performance in stage 1 all over T terms.

Let ∂_k^1 be the standard deviation value of sub-efficiencies of stage 1of DMU$_k$ all over T terms.

Large ∂_k^1 is not desirable since it denotes that the sub-efficiencies are unstable during T terms so that larger standard deviation represents more dispersion in sub-efficiencies of DMU$_k$ during T terms.

The coefficient variation value associated with sub-efficiencies of stage 1 of DMU$_k$ all over T terms obtained by $cv_k^1 = \dfrac{\partial_k^1}{\bar{\theta}_k^1}$.

According to above argument, the value of $\dfrac{1}{cv_k^1} = \dfrac{\bar{\theta}_k^1}{\partial_k^1}$ can be interpreted as an efficiency indicator of stage 1 of DMU$_k$ all over T terms.

The same argument can be considered for stage 2 sub-efficiencies during T terms.

	Overall inefficiency	Term inefficiency	
		t_1	t_2
1	0.15446	0.12319	0.18574
2	0.12282	0.00000	0.24565
3	0.09791	0.03397	0.16186
4	0.32658	0.36825	0.28491
5	0.34612	0.38789	0.30436
6	0.11877	0.20731	0.03024
7	0.17641	0.23700	0.11582
8	0.08739	0.07770	0.09708
9	0.28359	0.25104	0.31614
10	0.05160	0.10321	0.00000
11	0.22516	0.21560	0.23473
12	0.09211	0.12770	0.05652
13	0.20261	0.18701	0.21822
14	0.35177	0.46890	0.23465
15	0.09122	0.11540	0.06705
16	0.34792	0.26374	0.43211
17	0.20405	0.40518	0.00292
18	0.18768	0.14617	0.22920
19	0.38693	0.12239	0.55671
20	0.10307	0.07315	0.13300

Table 5. Results of separate model

The mean, standard deviation, and coefficient variation values of sub-efficiencies of each of 2 stages of 10 branches of some bank all over 9 terms calculated in table 6. Branches 1 and 2 are sub-efficient in each stage in every term.

	$\bar{\theta}_k^1$	∂_k^1	cv_k^1	$\bar{\theta}_k^2$	∂_k^2	cv_k^2
1	1.00000	0.00000	0.00000	1.00000	0.00000	0.00000
2	1.00000	0.00000	0.00000	1.00000	0.00000	0.00000
3	1.00000	0.00000	0.00000	0.70328	0.19962	0.28384
4	0.79108	0.18978	0.23989	0.71579	0.42826	0.59830
5	0.84634	0.22973	0.27143	0.40908	0.41305	1.00970
6	1.00000	0.00000	0.00000	0.35023	0.32069	0.91565
7	0.71530	0.26468	0.37002	0.89856	0.30431	0.33866
8	1.00000	0.00000	0.00000	0.83730	0.91653	1.09462
9	0.81959	0.05522	0.06737	0.55997	0.27757	0.49568
10	1.00000	0.00000	0.00000	0.77473	0.27055	0.34921

Table 6. Sub-efficiencies statistical indicators

Branches 3 and 8 are sub-efficient in stage 1 in every term; Branch 3 has the least and branch 8 has the most sub-efficiency instability in stage 2 between branches 3 to 10 all over 9 terms.

Branch 9 has the least and branch 7 has the most sub-efficiency instability in stage 1 between branches 4,5,7,9.

In addition branch 9 has the least value of coefficient variation in stage 1 between branches 4,5,7,9 all over 9 terms that implies this branch has the best total performance in stage 1 between these branches. Similarly, branch 3has the best total performance in stage 2 between branches 3 to 10 all over 9 terms.

5. Conclusion

This paper has considered the production systems in which inputs in the first stage produce intermediate outputs transformed in the second stage of production to final outputs.

Bank production process has such network structure in which bad loans are considered as an undesirable output of the second stage. These bad loans are as a constraining factor to the first stage in the subsequent period. This paper has modeled such network production systems by using a dynamic model to measure the inefficiency at each term and the overall inefficiency while the effects of outputs in one term on the inputs of the next term are considered. In models that measure efficiency, the bad outputs are mostly treated as inputs so that the true production process is not reflected. Here an alternative method is considered in dealing with bad outputs. The superiority of the dynamic model over the traditional separation model has been demonstrated during an illustrative example. The statistical analysis of efficiency scores for every DMU stage, which named sub-efficiencies, represents useful information about the total performance of the stage over all periods. Standard deviation value of such sub-efficiencies implies the instability amount of efficiencies of each stage over all periods. The converse of coefficient variation value of such sub-efficiencies over all periods shows the total performance of each stage over all terms.

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