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p-Analog of the Semigroup Fourier-Steiltjes Algebras

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Abstract. In this paper we define the p-analog of the restricted representations and the p-analog of the Fourier–Stieltjes algebras on inverse semigroups. Also we improve some results about Herz algebras on Clifford semigroups and we give a necessary and sufficient condition for amenability of these algebras on Clifford semigroups.

Keywords: Restricted fourier–Stieltjes algebras, Restricted inverse semigroup, Restricted representations, QSL_p-spaces, p-Analog of the Fourier–Stieltjes algebras.

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1. Introduction and Preliminaries

An inverse semigroup S is a discrete semigroup such that for each $s \in S$ there exists a unique element $s^* \in S$ such that $ss^*s = s, s^*ss^* = s^*$. The set $E(S)$ of idempotents of S consists of elements of the form ss^*, $s \in S$. Actually for each abstract inverse semigroup S there is a $*$-semigroup homomorphism from S into the inverse semigroup of partial isometries on some Hilbert space[18].

Dunkl and Ramirez in [8] and T. M. Lau in [15] attempted to define a suitable substitution for Fourier and Fourier–Stieltjes algebras on semigroups. Each definition has its own difficulties. Amini and Medghalchi introduced and

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extensively studied the theory of restricted semigroups and restricted representations and restricted Fourier and Fourier–Stieltjes algebras, $A_{r,e}(S), B_{r,e}(S)$ in [2] and [3]. Also they studied the spectrum of the Fourier Stieltjes algebra for a unital foundation topological $*$-semigroup in [4]. In this section we mention some of their results.

Throughout this paper S is an inverse semigroup. Given $x, y \in S$, the restricted product of x, y is xy if $x^*x = yy^*$, and undefined, otherwise. The set S with its restricted product forms a groupoid [16, 3.1.4] which is called the associated groupoid of S. If we adjoin a zero element 0 to this groupoid, and put $0^* = 0$, we will have an inverse semigroup S_r with the multiplication rule

$$x \bullet y = \begin{cases} xy & \text{if } x^*x = yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in S \cup \{0\}$, which is called the restricted semigroup of S. A restricted representation $\{\pi, \mathcal{H}_\pi\}$ of S is a map $\pi : S \longrightarrow \mathcal{B}(\mathcal{H}_\pi)$ such that $\pi(x^*) = \pi(x)^*$ $(x \in S)$ and

$$\pi(x)\pi(y) = \begin{cases} \pi(xy) & \text{if } x^*x = yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in S$. Let $\Sigma_r = \Sigma_r(S)$ be the family of all restricted representations π of S with $\|\pi\| \leq 1$. Now it is clear that, via a canonical identification, $\Sigma_r(S) = \Sigma_0(S_r)$, consist of all $\pi \in \Sigma(S_r)$ with $\pi(0) = 0$, where the notation Σ has been used for all $*$-homomorphism from S into $\mathcal{B}(\mathcal{H})$ [2]. One of the central concepts in the analytic theory of inverse semigroups is the left regular representation $\lambda : S \longrightarrow \mathcal{B}(\ell^2(S))$ defined by

$$\lambda(x)\xi(y) = \begin{cases} \xi(x^*y) & \text{if } xx^* \geq yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for $\xi \in \ell^2(S), x, y \in S$. The restricted left regular representation $\lambda_r : S \longrightarrow \mathcal{B}(\ell^2(S))$ is defined in [2] by

$$\lambda_r(x)\xi(y) = \begin{cases} \xi(x^*y) & \text{if } xx^* = yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for $\xi \in \ell^2(S), x, y \in S$. The main objective of [2] is to change the convolution product on the semigroup algebra to restore the relation with the left regular representation.

For each $f, g \in \ell^1(S)$, define

$$(f \bullet g)(x) = \sum_{x^*x = yy^*} f(xy)g(y^*) \quad (x \in S),$$

and for all $x \in S$, $\tilde{f}(x) = \overline{f(x^*)}$. $\ell^1_r(S) := (\ell^1(S), \bullet, \tilde{\ })$ is a Banach *-algebra with an approximate identity . The left regular representation λ_r lifts to a faithful representation $\tilde{\lambda}$ of $\ell^1_r(S)$. We call the completion $C^*_{\lambda_r}(S)$ of $\ell^1_r(S)$ with the

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norm $\|\cdot\|_{\lambda_r} := \|\tilde{\lambda_r}(\cdot)\|$ which is a C^*-norm on $\ell_r^1(S)$, the restricted reduced C^*-algebra and its completion with the norm $\|\cdot\|_{\Sigma_r} := \sup\{\|\tilde{\pi}(\cdot)\|\ , \pi \in \Sigma(S_r)\}$ the restricted full C^*-algebra and show it by $C_r^*(S)$. The dual space of C^*-algebra $C_r^*(S)$ is a unital Banach algebra which is called the restricted Fourier-Stieltjes algebra and is denoted by $B_{r,e}(S)$. The closure of the set of finitely support functions in $B_{r,e}(S)$ is called the restricted Fourier algebra and is denoted by $A_{r,e}(S)$[2].

In [10], Figà-Talamanca introduced a natural generalization of the Fourier algebra, for a compact abelian group G, by replacing $L_2(G)$ by $L_p(G)$. In [11], Herz extended the notion to an arbitrary group, to get the commutative Banach algebra $A_p(G)$, called the Figà–Talamanca–Herz algebra. Figà–Talamanca–Herz algebra and Eymard's Fourier algebra have very similar behavior. For example, Leptin's theorem is valid: G is amenable if and only if $A_p(G)$ has a bounded approximate identity [12]. The p-analog, $B_p(G)$ of the Fourier–Stieltjes algebra is defined as the multiplier algebra of $A_p(G)$, by some authors, as mentioned in [5] and [19]. Runde in [20] defined and studied $B_p(G)$, the p-analog of the Fourier–Stieltjes algebra on the locally compact group G. He developed the theory of representations and defined the suitable coefficient functions on them.

For $p \in (1, \infty)$, Medghalchi and Pourmahmood Aghababa developed the theory of restricted representations on $\ell_p(S)$ and defined the Banach algebra of p-pseudomeasures $PM_p(S)$ and the Figa- Talamanca-Herz algebras $A_p(S)$. They showed that $A_q(S)^* = PM_p(S)$ for dual pairs p, q. They characterized $PM_p(S)$ and $A_p(S)$ for Clifford semigroups, in the sense of p-pseudomeasures and Figa–Talamanca–Herz algebras of maximal semigroups of S, respectively[17].

Amini also worked on quantum version of Fourier transforms in [1].

In this paper we will combine what Medghalchi–Pourmahmood Aghababa and Runde have done. We will define the restricted representations on QSL_p-spaces and the p-analog of the Fourier-Stieltjes algebra on the restricted inverse semigroup.

Section 2 is a review of the theory of QSL_p-spaces. In Section 3 we define the restricted representations on QSL_p-spaces and study their tensor product. In Sections 4 and 5 we construct the p-analog of the restricted Fourier–Stieltjes algebra and study its order structure. The last section will be about Clifford semigroups and the p-analog of their restricted Fourier–Stieltjes algebra. Some new results which improves the results of [17] and [22] will be given in Section 6.

2. Review of the Theory of QSL_p-spaces

This section is a review of the paper of Runde [20].

Definition 2.1. A Banach space $\mathcal{E}$ is called

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(i) an L_p-space if it is of the form $L_p(X)$, for some measure space X.

(ii) a QSL_p-space if it is isometrically isomorphic to a quotient of a subspace of an L_p-space (or equivalently, a subspace of a quotient of an L_p-space [20, Section 1, Remark 1]).

If E is a QSL_p-space and if $p' \in (1, \infty)$ is such that $\frac{1}{p} + \frac{1}{p'} = 1$, the dual space E^* is an $QSL_{p'}$-spaces. In particular, every QSL_p-space is reflexive.

By [14, Theorem 2], the QSL_p-spaces are precisely the p-spaces in the sense of [11], i.e. those Banach spaces E such that for any two measure spaces X and Y the amplification map

$$B(L_p(X), L_p(Y)) \to B(Lp(X, E), Lp(Y, E)), T \to T \otimes id_E$$

is an isometry. In particular, an L_q-space is a QSL_p-space if and only if $2 \leq q \leq p$ or $p \leq q \leq 2$. Consequently, if $2 \leq q \leq p$ or $p \leq q \leq 2$, then every QSL_q-space is a QSL_p-space.

Runde equipped the algebraic tensor product of two QSL_p-spaces with a suitable norm, which comes in the following.

Theorem 2.2. [20, Theorem 3.1] *Let E and F be QSL_p-spaces. Then there exists a norm $\|\cdot\|_p$ on the algebraic tensor product $E \otimes F$ such that:*

(i) *$\|\cdot\|_p$ dominates the injective norm;*

(ii) *$\|\cdot\|_p$ is a cross norm;*

(iii) *the completion $E\tilde{\otimes}_p F$ of $E \otimes F$ with respect to $\|\cdot\|_p$ is a QSL_p-space. The Banach space $E\tilde{\otimes}_p F$ will be called p-projective tensor product of E and F.*

3. Restricted Representation on a Banach space

In this section we give an analog of the theory of group representations on a Hilbert space for the restricted representations for an inverse semigroup on a QSL_p-space.

Definition 3.1. A representation of a discrete inverse semigroup S on a Banach space E is a pair (π, E) consisting of a map $\pi : S \to B(E)$ satisfying $\pi(x)\pi(y) = \pi(xy)$, for $x, y \in S$ and $\|\pi(x)\| \leq 1$, for all $x \in S$.

Definition 3.2. A restricted representation of a discrete inverse semigroup S on a Banach space E is a pair (π, E) consisting of a map $\pi : S \to B(E)$ satisfying

$$\pi(x)\pi(y) = \begin{cases} \pi(xy) & \text{if } x^*x = yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in S$, and $\|\pi(x)\| \leq 1$, for all $x \in S$.

Definition 3.3. Let S be an inverse semigroup, and let (π, E) and (ρ, F) be restricted representations of S, then these restricted representations are said to be equivalent if there exists a surjective isometry $T : E \to F$ such that

$$T\pi(x)T^{-1} = \rho(x), \quad (x \in S).$$

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For any inverse semigroup S and $p \in (1,\infty)$, we denote by $\Sigma_{p,r}(S)$ the collection of all (equivalence classes) of restricted representations of S on a QSL_p-space.

Remark 3.4. By [17] for $p \in (1,\infty)$ the restricted left regular representation $\lambda_p : S \longrightarrow B(\ell^p(S))$

$$\lambda_p(s)(\delta_t) = \begin{cases} \delta_{st} & \text{if } s^*s = tt^*, \\ 0 & \text{otherwise} \end{cases}$$

for $s,t \in S$ is a restricted representation so it belongs to $\Sigma_{p,r}(S)$.

The following propositions are easy to check, similar to [2].

Proposition 3.5. *For an inverse semigroup S and its related restricted semigroup S_r, each restricted representation of S on a Banach space is a representation on S_r which is zero on $0 \in S_r$, i.e. it is multiplicative with respect to the restricted multiplication.*

Proposition 3.6. *For an inverse semigroup S, each restricted representation π of S on a Banach space lifts to a representation of $\ell^1_r(S)$, via*

$$\tilde{\pi}(f) = \sum_{x\in S} f(x)\pi(x).$$

4. Banach Algebra $B_{p,r}(S)$

In this section we define the p-analog of the Fourier–Stieltjes algebra on a inverse semigroup. We show that for $p = 2$ we get the known algebra $B_{r,e}(S)$, defined in [2].

Theorem 4.1. *Let $(\pi, E), (\rho, F) \in \Sigma_{p,r}(S)$ then $(\pi \otimes \rho, E\tilde{\otimes}_p F) \in \Sigma_{p,r}(S)$.*

Proof. By the definition of $\pi \otimes \rho$ we have $\pi \otimes \rho(x)(\xi \otimes \eta) = \pi(x)\xi \otimes \rho(x)\eta$. For $x, y \in S$, $x^*x = yy^*$,

$$\begin{aligned} \pi \otimes \rho(xy)(\xi \otimes \eta) &= \pi(xy)\xi \otimes \rho(xy)\eta \\ &= \pi(x)\pi(y)\xi \otimes \rho(x)\rho(y)\eta \\ &= \pi(x)(\pi(y)\xi) \otimes \rho(x)(\rho(y)\eta) \\ &= \pi \otimes \rho(x)(\pi(y)\xi \otimes \rho(y)\eta) \\ &= \pi \otimes \rho(x)\pi \otimes \rho(y)(\xi \otimes \eta) \end{aligned}$$

when $x^*x \neq yy^*$

$$\begin{aligned} \pi \otimes \rho(x)\pi \otimes \rho(y)(\xi \otimes \eta) &= \pi \otimes \rho(x)(\pi(y)\xi \otimes \rho(y)\eta) \\ &= \pi(x)\left(\pi(y)\xi\right) \otimes \rho(x)\left(\rho(y)\eta\right) \end{aligned}$$

which is equal to zero. Now it is enough to show that $\pi(x) \in B(E)$ and $\rho(y) \in B(F)$, $\pi(x) \otimes \rho(y)$ could be extend to $E\tilde{\otimes}_p F$. This is shown as in the group case [20, Therem 3.1]. $\square$

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Definition 4.2. Let S be an invesre semigroup, and let $(\pi, E) \in \Sigma_{p,r}(S)$. A coefficient function of (π, E) is a function $f : S \longrightarrow \mathbb{C}$ of the form

$$f(x) =< \pi(x)\xi, \phi > \quad (x \in S),$$

where $\xi \in E$ and $\phi \in E^*$.

Definition 4.3. Let S be an inverse semigroup, let $p \in (1, \infty)$, and let $q \in (1, \infty)$ be the dual scalar to p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. We define

$$B_{p,r}(S) := \{f : S \to \mathbb{C} : \, f \, \text{is a coefficient function of some} \, (\pi, E) \in \Sigma_{q,r}(S)\}.$$

Proposition 4.4. *Let S be an inverse semigroup, let $p \in (1, \infty)$, and let $q \in (1, \infty)$ be the dual scalar to p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, and let $f : S \longrightarrow \mathbb{C}$ defined by*

$$f(x) = \sum_{n=1}^{\infty} < \pi_n(x)\xi_n, \phi_n >, \quad (x \in S),$$

where $((\pi_n, E_n))_{n=1}^{\infty}$, $(\xi_n)_{n=1}^{\infty}$, and $(\phi_n)_{n=1}^{\infty}$ are sequences with $(\pi_n, E_n) \in \Sigma_{q,r}(S)$, $\xi_n \in E_n$, and $\phi_n \in E_n^*$, for $n \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\| < \infty.$$

Then f lies in $B_{p,r}(S)$.

Proof. The proof is similar to [20]. Without loss of generality, we may suppose that

$$\sum_{i=1}^{\infty} \|\xi_n\|_q < \infty, \quad \text{and} \sum_{i=1}^{\infty} \|\phi_n\|_p < \infty.$$

Then $E := \ell_q - \oplus_{n=1}^{\infty} E_n$ is a QSL_q-space and for $\xi := (\xi_1, \xi_2, ...)$ and $\phi := (\phi_1, \phi_2, ...)$, we have $\xi \in E$ and $\phi \in E^*$. Now the map $\pi : S \longrightarrow B(E)$ with $\pi(x)\eta = (\pi_1(x)\eta, \pi_2(x)\eta, ...)$ is a restricted representation of S on E, and f is the coefficient function of π. □

Definition 4.5. [17, Definition 3.1]. Let S be an inverse semigroup and let $p, q \in (1, \infty)$ be dual pairs. The space $A_q(S)$ consists of those $u \in c_0(S)$ such that there exist sequences $(f_n)_{n=1}^{\infty} \subseteq \ell_q(S)$ and $(g_n)_{n=1}^{\infty} \subseteq \ell_p(S)$ with $\sum_{n=1}^{\infty} \|f_n\|_q \|g_n\|_q \leq \infty$ and $u = \sum_{n=1}^{\infty} f_n \bullet \check{g_n}$. For $u \in A_q(S)$, let

$$\|u\| = \inf \left\{ \sum_{n=1}^{\infty} \|f_n\|_q \|g_n\|_p : u = \sum_{n=1}^{\infty} f_n \bullet \check{g_n} \right\}$$

Proposition 4.6. [17, Proposition 3.2]. *Let S be an inverse semigroup and let $p \in (1, \infty)$, then $A_p(S)$ is a Banach space and is the closure of finite support functions on S.*

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Proposition 4.7. *Let S be an inverse semigroup, let $p \in (1,\infty)$. Then $B_{p,r}(S)$ is a linear subspace of $c_b(S)$ containing $A_p(S)$. Moreover, if $2 \leq q \leq p$ or $p \leq q \leq 2$, we have $B_{q,r}(S) \subseteq B_{p,r}(S)$.*

Proof. Every thing is easy to check, and is similar to [20]. □

Definition 4.8. Let S be an inverse semigroup, and let (π, E) be a restricted representation of S on the Banach space E. Then (π, E) is called cyclic if there exists $x \in E$ such that $\pi(\ell_r^1(S))x$ is dense in E. For $p \in (1,\infty)$, we set $Cyc_{p,r}(S) := \{(\pi, E) : (\pi, E) \text{ is a cyclic restricted representation on a } QSL_p\text{- space } E\}$.

Definition 4.9. Let S be an inverse semigroup, let $p, q \in (1,\infty)$ be the dual scalars, and let $f \in B_{p,r}(S)$. We define $\|f\|_{B_{p,r}(S)}$ as the infimum over all expressions $\sum_{n=1}^{\infty} \|\xi_n\|\|\phi_n\|$, where, for each $n \in \mathbb{N}$, there is $(\pi_n, E_n) \in Cyc_{q,r}(S)$ with $\xi_n \in E_n$ and $\phi_n \in E_n^*$ such that $\sum_{n=1}^{\infty} \|\xi_n\|\|\phi_n\| < \infty$ and

$$f(x) = \sum_{n=1}^{\infty} < \pi_n(x)\xi_n, \phi_n >, \quad (x \in S).$$

The proof of the following theorem is similar to the group case.

Theorem 4.10. *Let S be an inverse semigroup, let $p \in (1,\infty)$, and let $f, g : S \longrightarrow \mathbb{C}$ be coefficient functions of (π, E) and (ρ, F) in $\Sigma_{p,r}(S)$, respectively. Then the pointwise product of f and g is a coefficient function of $(\pi \otimes \rho, E\tilde{\otimes}_p F)$.*

In the next theorem we give some result about our new constructed space and also the relation between semigroup restricted Herz algebra and our new space.

Theorem 4.11. *Let S be an inverse semigroup, let $p \in (1,\infty)$. Then:*

(i) *$B_{p,r}(S)$ is a commutative Banach algebra.*

(ii) *the inclusion $A_p(S) \subseteq B_{p,r}(S)$ is a contraction.*

(iii) *for $2 \leq p' \leq p$ or $p \leq p' \leq 2$, the inclusion $B_{p',r}(S) \subseteq B_{p,r}(S)$ is a contraction.*

(iv) *for $p = 2$, $B_{r,e}(S)$ is isometrically isomorphic to $B_{p,r}(S)$ as Banach algebras.*

Proof. (i) Let $\frac{1}{p} + \frac{1}{q} = 1$. The space $B_{p,r}(S)$ is the quotient space of complete q-projective tensor product of $E\tilde{\otimes}_q E^*$, for the universal restricted representation (π, E), on QSL_q-space E. Also Theorem 4.10 shows it is an algebra. The submultplicutive property for norm of $B_{p,r}(S)$ is similar to the group case in [20] and it is only based on characteristic property of infimum.

(ii) By the definition of semigroup Herz algebra in [17] for conjugate numbers p, q, each $f \in A_p(S)$ is a coefficient function of the restricted left regular representation on the ℓ_q-space, $\ell_q(S)$. So $A_p(S) \subseteq B_{p,r}(S)$. By the definition of the norm of $f \in B_{p,r}(S)$, the infimum is taken on all expressions of f as the coefficient function of some restricted representation on a QSL_q-space, and

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the norm on the $A_p(S)$ is the infimum only on expressions of f as the coefficient function of restricted left regular representation, so the inclusion map is a contraction.

(iii) For $2 \leq p' \leq p$ or $p \leq p' \leq 2$ and q, q' conjugate scalars to p and p' respectively. Then each restricted representation on a $QSL_{q'}$-space is a restricted representation on a QSL_q-space.

(iv) By the definition, each element of $B_{r,e}(S)$ is a coefficient function of a 2-restricted representation [3].

□

Remark 4.12. A very natural question is that when $A_p(S)$ is an ideal in $B_{p,r}(S)$. Even in $p = 2$ this question is not studied. If we want to go along the proof of the group case, a difficulty to prove this is that in general for $p \in (1, \infty)$, and $(\pi, E) \in \Sigma_{p,r}(S)$, the representations $(\lambda_p \otimes \pi, \ell_p(S, E))$ and $(\lambda_p \otimes id_E, \ell_p(S, E))$ are not equivalent. In fact we can not find a suitable substitution for representation $id : S \to B(E), \quad id(s) = id_E$ in the class of restricted representations. But in a special case, such as Clifford semigroups, we can give a better result.

5. Order Structure of the p-Analog of the Semigroup Fourier–Steiltjes Algebras $B_{p,r}(S)$

Studying the ordered spaces and order structures has a long history. The natural order structure of the Fourier-Stieltjes algebras was favorite in 80s. In [21] the authors studied the order structure of Figà–Talamanca– Herz algebra and generalized results on Fourier algebras. In this section, we consider the p-analog of the restricted Fourier–Stieltjes algebra, $B_{p,r}(S)$, introduced in Section 4, and study its order structure given by the p-analog of positive definite continuous functions.

A *compatible couple* of Banach spaces in the sense of interpolation theory (see [3]) is a pair $(\mathcal{E}_0, \mathcal{E}_1)$ of Banach spaces such that both $\mathcal{E}_0$ and $\mathcal{E}_1$ are embedded continuously in some (Hausdorff) topological vector space. In this case, the intersection $\mathcal{E}_0 \cap \mathcal{E}_1$ is again a Banach space under the norm $\| \cdot \|_{(\mathcal{E}_0, \mathcal{E}_1)} = \max\{\| \cdot \|_{\mathcal{E}_0}, \| \cdot \|_{\mathcal{E}_1}\}$. For example , for a locally compact group G, the pairs $(A_p(G), A_q(G))$ and $(L_p(G), L_q(G))$ are compatible couples.

Definition 5.1. Let (π, E) be a restricted representation of S on a Banach space E, such that $(\mathcal{E}, \mathcal{E}^*)$ is a compatible couple. We mean by a π_r-positive definite function on S, a function which has a representation as $f(x) = \langle \pi(x)\xi, \xi \rangle$, $(x \in S)$, where $\xi \in \mathcal{E} \cap \mathcal{E}^*$. For dual scalars $p, q \in (1, \infty)$, we call each element in the closure of the set of all π_r-positive definite functions on S in $B_{p,r}(S)$, where π is a restricted representation of S on an L_q-space, a restricted p-positive definite function on S and the set of all restricted p-positive definite functions on S, will be denoted by $P_{p,r}(S)$.

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It follows from [21] and the definition of $P_{p,r}(S)$, that for each $f \in P_{p,r}(S)$, associated to a representation (π, E), for a QSL_q-space E, there exists a sequence $(\pi_n, \mathcal{E}_n)_{n=1}^{\infty}$ of cyclic restricted representations of S on closed subspaces E_n of $E \cap E^*$, and $\{\xi_n\}$ in $\mathcal{E}_n$, such that

$$f(x) = \sum_{n=1}^{\infty} \langle \pi_n(x)\xi_n, \xi_n \rangle \quad (x \in S).$$

Proposition 5.2. *The linear span of all finite support elements in $P_{p,r}(S)$ is dense in $A_p(S)$, and $A_p(S)$ is an ordered space.*

Proof. From [17, Proposition 3.2] $A_p(S)$ is a norm closure of the set of elements of the form $\sum_{i=1}^{n} f_i \bullet \check{g}_i$ where f_i, g_i are finite support functions on S, $i = 1, ...n$. Also $f_i \bullet \check{g}_i(x) = \langle \lambda_r(x^*) f_i, g_i \rangle$. Now by Polarization identity, we have the statement. $\square$

Since $A_p(S)$ is the set of coefficient functions of the restricted left regular representation of S on $\ell_p(S)$, we define the positive cone of $A_p(S)$ as the closure in $A_p(S)$, of the set of all function of the form $f = \sum_{i=1}^{n} \xi_i \bullet \tilde{\xi}_i$, for a sequence (ξ_i) in $\ell_p(S) \cap \ell_q(S)$, and denote it by $A_p(S)_+$.

This order structure, in the case where $p = 2$, is the same as the order structure of $A_{r,e}(S)$, induced by the set $P_{r,e}(S) \cap A_{r,e}(S)$, as a positive cone. Because in the case $p = 2$, the extensible restricted positive definitive functions are exactly the closed linear span of $h \bullet \tilde{h}$, for $h \in \ell^2(S)$.

6. p-Analog of the Fourier–Stiletjes Algebras on Clifford Semigroups

Let S be a semigroup. Then, by [13, Chapter 2], there is an equivalence relation $\mathcal{D}$ on S by $s\mathcal{D}t$ if and only if there exists $x \in S$ such that

$$Ss \cup \{s\} = Sx \cup \{x\} \text{ and } \quad tS \cup \{t\} = xS \cup \{x\}.$$

If S is an inverse semigroup, then by [13, Proposition 5.1.2(4)], $s\mathcal{D}t$ if and only if there exists $x \in S$ such that $s^*s = xx^*$ and $t^*t = x^*x$.

Proposition 6.1. [17, Proposition 4.1]. *Let S be an inverse semigroup,*

(i) *and let D be a $\mathcal{D}$-class of S. Then $\ell_p(D)$ is a closed $\ell_r^1(S)$-submodule of $\ell_p(S)$.*

(ii) *and let $\{D_\lambda; \lambda \in \Lambda\}$ be the family of $\mathcal{D}$-classes of S indexed by some set Λ. Then there is an isometric isomorphism of Banach $\ell_r^1(S)$-bimodules*

$$\ell^p(S) \cong \ell^p - \bigoplus_{\lambda \in \Lambda} \ell_p(D_\lambda). \tag{6.1}$$

Corollary 6.2. *Let S be an inverse semigroup, and let $\{D_\lambda; \lambda \in \Lambda\}$ be the family of $\mathcal{D}$-classes of S indexed by some set Λ. Then for a QSL_p-space E of functions on S, there is a family of QSL_p-spaces $\{E_\lambda\}_{\lambda \in \Lambda}$, where for each $\lambda \in \Lambda$, E_λ consists of functions on D_λ, and $E \cong \ell^p - \oplus_{\lambda \in \Lambda} E_\lambda$.*

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Proof. This is clear by the definition of a QSL_p-space, and the fact that the isomorphism 6.1 is compatible with taking quotients and subspaces of $\ell_p(D_\lambda)$s. $\square$

An inverse semigroup S is called a Clifford semigroup if $s^*s = ss^*$ for all $s \in S$. For $e \in E(S)$ define $G_e := \{s \in S | s^*s = ss^* = e\}$. Then G_e is a group with identity e. Here each $\mathcal{D}$-class D contains a single idempotent (say e) and we have $D = G_e$

We modified the isometrical isomorphism derived in [17, Section 5.3] in the following theorem.

Theorem 6.3. *Let S be a Clifford semigroup with the family of $\mathcal{D}$-classes $\{G_e\}_{e\in E(S)}$, and let $p \in (1, \infty)$. Then*

$$B_{p,r}(S) \cong \ell^1 - \bigoplus_{e\in E(S)} B_p(G_e)$$

Proof. Let p, q are conjugate scalars. Fix $e \in E(S)$, assume that $G_e = \{x \in S; x^*x = e\}$. Define $\pi : S \to B(\ell_q(G_e))$

$$\pi(s)(\delta_t) = \begin{cases} \delta_t & \text{if } s^*s = e, \\ 0 & \text{otherwise} \end{cases}$$

for $s \in S$. Then π is a restricted representation and $\chi_{G_e}(s) = \langle \pi(s)\delta_t, \delta_t\rangle$. Hence χ_{G_e} is in $B_{p,r}(S)$, and indeed χ_{G_e} is a restricted positive definite function. Now for each $u \in B_{p,r}(S)$, $u \cdot \chi_{G_e}$ is in $B_{p,r}(S)$. In fact the set $\{u \in B_{p,r}(S); u(s) = 0 \text{ for all } s \in S \setminus G_e\}$ is a closed subspace of $B_{p,r}(S)$ and it is isometrically isomorphic to $B_p(G_e)$. This follows from the fact that, each coefficient function of a restricted representation of S on a QSL_q-space that is zero on ${G_e}^c$, is a coefficient function of a representation on a QSL_q-space of G_e, using Corollary 6.2.

Let $u \in B_{p,r}(S)$, then we could decompose u to $(u_e)_{e\in E(S)}$, for some $u_e \in B_p(G_e)$, by the above paragraph. Now for all $e \in E(S)$ and all explanations of u_e as $u_e = \langle \pi_e(\cdot)\xi_e, \eta_e\rangle$, where $\pi_e \in \Sigma_q(G_e)$, ξ_e in some QSL_q and η_e in some QSL_p-space for dual scalars p, q we have $\|u_e\| \leq \|\xi_e\|\|\eta_e\|$ and also $u = (u_e)_{e\in E(S)} = \langle \oplus\pi_e(\cdot) \oplus \xi_e, \oplus\eta_e\rangle$ and then $\oplus\pi_e$ is a restricted representation of S on a QSL_q-space and $\sum_{e\in E(S)} \|u_e\| \leq \sum_{e\in E(S)} \|\xi_e\|\|\eta_e\| \leq (\sum_{e\in E(S)} \|\xi_e\|^q)^{\frac{1}{q}} (\sum_{e\in E(S)} \|\eta_e\|^p)^{\frac{1}{p}} = \|(\xi_e)\|\|(\eta_e)\|$, the last equality comes from Proposition 6.1. Now we have:

$$\sum_{e\in E(S)} \|u_e\| \leq \|(u_e)_{e\in E(S)}\| = \| \sum_{e\in E(S)} u_e\|$$

by the definition of the norm in the Fourier–Stieltjes algebras. So we have an isometric isomorphism of Banach algebras. $\square$

The following corollary improves [22, Proposition 2.6].

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Corollary 6.4. *Let S be a Clifford semigroup with the family of $\mathcal{D}$-classes $\{G_e\}_{e\in E(S)}$, and let $p \in (1,\infty)$. Then $A_p(S)$ is an ideal of $B_{p,r}(S)$, and $B_{p,r}(S)$ is amenable if and only if $B_p(G_e)$ is amenable for all $e \in E(S)$.*

Theorem 6.5. *Let S be a Clifford semigroup with the family of amenable $\mathcal{D}$-classes $\{G_e\}_{e\in E(S)}$ and let $p \in (1,\infty)$. Then $A_p(S)$ is equal to the closure of $B_{p,r}(S) \cap F(S)$ in the norm of $A_p(S)$.*

Proof. Since for each $e \in E(S)$ the group G_e is amenable, the natural embedding $i_e : A_p(G_e) \to B_p(G_e)$ is an isometry by [20, Corollary 5.3]. Now let $f \in B_{p,r}(S) \cap F(S)$. Then by Theorem 6.3, $f = \sum_{e\in E(S)} f_e$, where $f_e \in B_p(G_e) \cap F(G_e)$, so f_e belongs to $A_p(G_e)$ by [20]. Now since

$$A_p(S) \cong \ell_1 - \bigoplus_{e\in E(S)} A_p(G_e),$$

[17, Equation 5.1], we conclude that $f \in A_p(S)$. On the other hand let $f \in A_p(S) \cap F(S)$. Then for each $e \in E(S)$ the function f_e, which has been defined in Theorem 6.3, belongs to $A_p(G_e) \cap F(G_e)$ and so to $B_p(G_e) \cap F(G_e)$ with the same norm because of amenability of G_e. Now, by Theorem 6.3 we have $f \in B_{p,r}(S) \cap F(S)$. Since $F(S)$ is dense in $A_p(S)$ with norm of $A_p(S)$, [17, Proposition 3.2 vi], result follows.

$\square$

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References

1. M. Amini, Quantum Error–correction Codes on Abelian Groups, *Iranian Journal of Mathematical Sciences and Informatics*, **5**(1), (2010), 55–67.
2. M. Amini, A. R. Medghalchi, Restricted Algebras on Inverse Semigroups I, *Math. Nachr.*, **279**(16), (2006), 1-10.
3. M. Amini, A. R. Medghalchi, Restricted Algebras on Inverse Semigroups II, Positive Definite Functions, *Int. J. Math. Math. Sci.*, (2011).
4. M. Amini, A. R. Medghalchi, Spectrum of the Fourier–Stieltjes algebra of a Semigroup, *Iranian Journal of Mathematical Sciences and Informatics*, **1**(2), (2006), 1–8.
5. M. Cowling, An Application of Littlewood-Paley Theory in Harmonic Analysis. *Math. Ann.*, **241**, (1979), 83-96.
6. H. G. Dales, *Banach Algebras and Automatic Continuity*, Clarendon Press, Oxford, 2000.
7. J. Duncan, A. L. T. Paterson, Amenability for Discrete Convolution Semigroup Algebras, *Math. Scand.*, **66**, (1990), 141-146.
8. C. F. Dunkl, D. E. Rumirez, L^∞-Representations of Commutative Semitopological semigroups, *Semigroup Forum*, **7**, (1974), 180–199.
9. P. Eymard, Ĺalgebra de Fourier d́un Groupe Localment Compact, *Bull. Soc. Math. France*, **92**, (1964), 181–236.

[DOI: 10.7508/ijmsi.2015.02.006]

10. A. Figà-Talamanca, Translation Invariant Operators in L^p, *Duke Math. J.*, **32**, (1965), 495–501.
11. C. Herz, The Theory of p-Spaces with Applications to Convolution Operators, *Trans. Amer. Math. Soc.*, **154**, (1971), 69-82.
12. C. Herz, Harmonic Synthesis for Subgroups, *Ann. Inst. Fourier (Grenoble)*, **23**, (1973), 91-123.
13. J. M. Howie, *Fundamentals of Semigroup Theory*, in: London Mathematical Society Monographs, vol. 12, Clarendon Press, Oxford, 1995.
14. S. Kwapien, On Operators Factoring Through Lp-space, *Bull. Soc. Math. France, Mem.*, **3132**, (1972), 215-225.
15. A. T. M. Lau, The Fourier Stieltjes algebra of a Topological Semigroup with Involution, *Pac. J. Math.*, **77**, (1978), 165–181.
16. M. V. Lawson, *Inverse Semigroups, the Theory of Partial Symmetries*, World Scientific, Singapore, 1998.
17. A. R. Medghalchi, H. Pourmahmood Aghababa, Figa–Talamanca-Herz Algebras for Restricted Inverse Semigroups and Clifford Semigroups, *J. Math. Anal. Appl.*, **395**, (2012), 473-485.
18. A. T. Paterson, *Groupoids, Inverse Semigroups, and their Operator Algebras*, Progr. Math. vol. 170, Birkhuser, Boston, 1999.
19. J. P. Pier, *Amenable Locally Compact Groups*, Wiley–Interscience, 1984.
20. V. Runde, Representations of Locally Compact Groups on QSL_p-spaces and a p-Analog of the Fourier-Stieltjes Algebra, *Pacific J. Math.*, **221**, (2005), 379–397.
21. M. Shams Yousefi, M. Amini, F. Sady, Order structure of the Figà-Talamanca- Herz algebra, *ISRN Geometry*, (2011), Article ID 531023, 13 pages.
22. M. Shams Yousefi, M. Amini, Amenability of the Restricted Fourier algebras, *ISRN Algebra*, (2012), Article ID 782953, 8 pages.

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