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On Diameter of Line Graphs

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Abstract. The diameter of a connected graph G, denoted by $diam(G)$, is the maximum distance between any pair of vertices of G. Let $L(G)$ be the line graph of G. We establish necessary and sufficient conditions under which for a given integer $k \geq 2$, $diam(L(G)) \leq k$.

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1. Introduction

Let G be a simple connected graph on n vertices. Let the vertices of G be labeled as $v_1, v_2, \ldots, v_n$. The *distance* between the vertices v_i and v_j in G is equal to the length of a shortest path joining v_i and v_j, and is denoted by $d_G(v_i, v_j)$. The *diameter* of G, denoted by $diam(G)$ is the maximum distance between any pair of vertices of G.

The above distance provides the simplest and most natural metric in graph theory, and is one of the popular areas of research in discrete mathematics. Details on distance in graph theory can be found in the books [3, 5, 8] and the papers [1, 6, 7, 15, 16] published in this journal.

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As usual, by K_n, P_n, and $K_{1,n-1}$ we denote, respectively, the complete graph, the path, and the star on n vertices.

The *line graph* $L(G)$ of G is the graph whose vertices correspond to the edges of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. The second line graph of G is $L^2(G) = L(L(G))$.

Metric properties of line graphs have been much studied in the mathematical literature [2,4,9,12,14,17–20], and found remarkable applications in chemistry [10,11,13,14].

We first recall some known established properties of line graphs, needed for the considerations that follow.

Lemma 1.1. [17] *If G_1 is an induced subgraph of G then $L(G_1)$ is an induced subgraph of $L(G)$.*

Theorem 1.2. [19] *If diam$(G) \leq 2$ and if none of the three graphs F_1, F_2, and F_3 depicted in Fig. 1 are induced subgraphs of G, then diam$(L(G)) \leq 2$.*

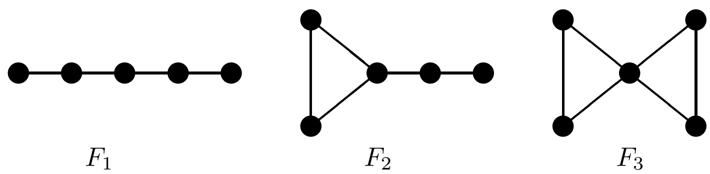


Fig. 1. The graphs mentioned in Theorem 1.2

In this paper we establish structural conditions for the graph G, under which for a given integer k, $k \geq 2$, the diameter of $L(G)$ does not exceed k. We also establish conditions under which for a given integer k, $k \geq 3$, the diameter of $L(G)$ is not less than k.

2. Main results

Let F_1^k be the path on $(k+3)$ vertices, $k \geq 2$. The vertices of F_1^k are $v_1, v_2, \ldots, v_{k+3}$, labeled so that v_i is adjacent to v_{i+1}, $i = 1, 2, \ldots, k+2$.

Let F_2^k be the graph obtained from F_1^k by adding to it an edge between the vertices v_1 and v_3. Let F_3^k be the graph obtained from F_1^k by adding to it edges between v_1 and v_3 and between v_{k+1} and v_{k+3} (see Fig. 2).

Theorem 2.1. *Let $k \geq 2$. For a connected graph G, diam$(L(G)) \leq k$, if and only if none of the three graphs F_1^k, F_2^k and F_3^k, depicted in Fig. 2, are an induced subgraph of G.*

Proof. The result can be easily verified for graphs of order $n \leq 4$. We thus assume that $n > 4$.

Let $k \geq 2$ and let $diam(L(G)) \leq k$. Suppose that F_1^k is an induced subgraph of G. By Lemma 1.1, $L(F_1^k)$ is an induced subgraph of $L(G)$. It is straightforward to check that $diam(L(F_1^k)) = diam(P_{k+2}) = k+1 > k$. Hence $diam(L(G)) > k$, a contradiction. Therefore F_1^k is not an induced subgraph of G.

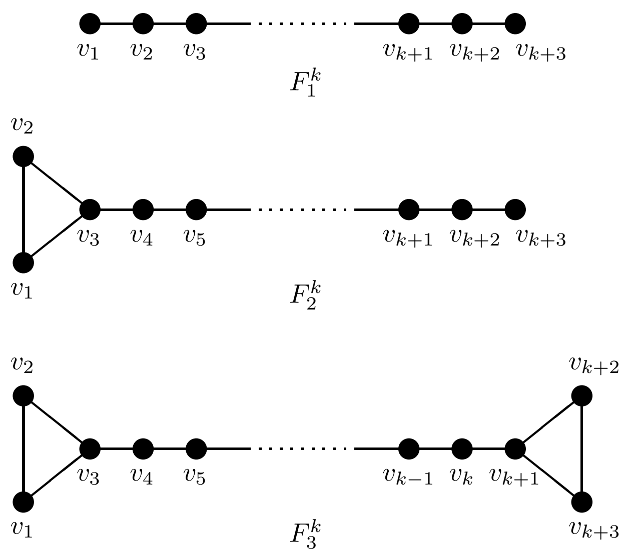


Fig. 2 The graphs mentioned in Theorems 2.1 and 2.3

Similarly we can show that F_2^k and F_3^k are also not induced subgraphs of G.

Conversely, suppose that $k \geq 2$ and that $diam(L(G)) > k$. Then G must possess two independent edges, say $e_i = (uv)$ and $e_j = (xy)$, such that neither u nor v are adjacent to either x or y. If so, then because the diameter of $L(G)$ is greater than k, there must exist $k-1$ vertices, say $u_1, u_2, \ldots, u_{k-1}$ such that u is adjacent to u_1, u_i is adjacent to u_{i+1}, $i = 1, 2, \ldots, k-2$, and u_{k-1} is adjacent to x. If u_i , $i = 1, 2, \ldots, k-1$ are not adjacent to either v or y, then G has F_1^k as an induced subgraph (spanned by the vertices $v, u, u_1, u_2, \ldots, u_{k-1}, x, y$). If u_1 is adjacent to v (or u_{k-1} is adjacent to y), then G has F_2^k as an induced subgraph. If u_1 is adjacent to v and u_{k-1} is adjacent to y, then G has F_3^k as an induced subgraph, a contradiction. Hence $diam(L(G)) \leq k$. □

Theorem 1.2 is a special case of Theorem 2.1, for $k = 2$. From Theorem 2.1, we observe that the condition $diam(G) \leq 2$, in Theorem 1.2 was not necessary.

Theorem 2.2. *Let G be a connected graph with $n \geq 3$ vertices. Then $diam(L(G)) = 1$ if and only if $G \cong K_3$ or $G \cong K_{1,n-1}$.*

Proof. If $G \cong K_3$, then $L(K_3) = K_3$ and $diam(L(K_3)) = diam(K_3) = 1$. If $G \cong K_{1,n-1}$, then all the edges of $K_{1,n-1}$ are incident to a common vertex. Therefore all vertices are adjacent to each other in $L(K_{1,n-1})$ and thus $L(K_{1,n-1} \cong K_{n-1}$. Hence $diam(L(K_{1,n-1})) = 1$.

Conversely, let $diam(L(G)) = 1$. Suppose that $G \ncong K_3$, $K_{1,n-1}$. Then in G there exists at least two independent edges, say $e_i = (uv)$ and $e_j = (xy)$. Therefore $d_{L(G)}(e_i, e_j) > 1$. Thus $diam(L(G)) > 1$, a contradiction. Hence it must be $G \cong K_3$ or $G \cong K_{1,n-1}$. □

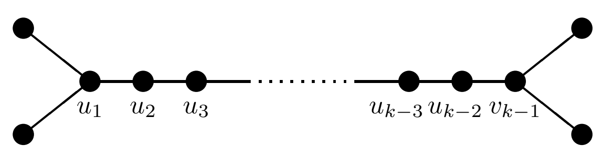
Evidently, the diameter of $L(G)$ is zero if and only if $G \cong K_1$ or $G \cong K_2$.

A statement equivalent to Theorem 2.1 is:

Theorem 2.3. *Let G be a connected graph with $n \geq 3$ vertices. Let $k \geq 2$. Then $diam(L(G)) > k$, if and only if either F_1^k or F_2^k or F_3^k, depicted in Fig. 2, is an induced subgraph G.*

3. A result for second line graph

Let P_{k-1} be the path with vertices $u_1, u_2, \ldots, u_{k-1}$, where u_i is adjacent to u_{i+1}, $i = 1, 2, \ldots, k-2$, $k \geq 3$. Let F_4^k be the graph obtained from P_{k-1} by joining two vertices to u_1 and another two vertices to u_{k-1} (see Fig. 3). F_4^k has $k+3$ vertices and $k+2$ edges.



F_4^k

Fig. 3. The graph mentioned in Theorem 3.1

Theorem 3.1. *Let $k \geq 3$. If F_4^k is an induced subgraph of G, then $diam(L^2(G)) \geq k-1$.*

Proof. Let $k \geq 3$. Let F_4^k be the induced subgraph of G. Then $L(F_4^k)$ is isomorphic to F_3^{k-1}, and by Lemma 1.1, $L(F_4^k)$ is an induced subgraph of $L(G)$. Therefore F_3^{k-1} is an induced subgraph of $L(G)$. Hence by Theorem 2.3, $diam(L(L(G))) = diam(L^2(G)) > k-1$. □

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