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Approximation of Jordan Homomorphisms in Jordan Banach Algebras

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Abstract. In this paper, we investigate the generalized Hyers-Ulam stability of Jordan homomorphisms in Jordan Banach algebras for the functional equation

$$\sum_{k=2}^{n}\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n} f\left(\sum_{i=1, i\neq i_1,\cdots,i_{n-k+1}}^{n} x_i - \sum_{r=1}^{n-k+1} x_{i_r}\right) + f\left(\sum_{i=1}^{n} x_i\right) - 2^{n-1} f(x_1) = 0,$$

where n is an integer greater than 1.

Keywords: Generalized Hyers-Ulam stability, Jordan homomorphism, Jordan algebra.

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1. Introduction

A classical question in the theory of functional equations is that "when is it true that a function which approximately satisfies a functional equation $\mathcal{E}$ must be somehow close to an exact solution of $\mathcal{E}$". Such a problem was formulated by Ulam [34] in 1940 and solved in the next year for the Cauchy functional equation

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by Hyers [14]. It gave rise to the *stability theory* for functional equations. The result of Hyers was generalized by Aoki [3] for approximate additive functions and by Th.M. Rassias [30] for approximate linear functions. The stability phenomenon that was proved by Th.M. Rassias is called the *Hyers-Ulam-Rassias stability* or the *generalized Hyers-Ulam stability* of functional equations. In 1994, a generalization of the Th.M. Rassias' theorem was obtained by Găvruta [12] as follows: Suppose that $(G,+)$ is an abelian group and E is a Banach space and that the so-called admissible control function $\varphi : G \times G \to \mathbb{R}$ satisfies

$$\tilde{\varphi}(x,y) := \sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y) < \infty$$

for all $x, y \in G$. If $f : G \to E$ is a mapping with

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x,y)$$

for all $x, y \in G$, then there exists a unique mapping $T : G \to E$ such that $T(x+y) = T(x) + T(y)$ and $\|f(x) - T(x)\| \le \tilde{\varphi}(x,x)$ for all $x, y \in G$. If moreover G is a real normed space and $f(tx)$ is continuous in t for each fixed x in G, then T is a linear function.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4]–[8], [10], [11], [13], [15], [17]–[29], [31]– [33]).

Recently, Eshaghi Gordji et al. [9] defined the following n-dimensional additive functional equation

$$D_f(x_1,\cdots,x_n) := \sum_{k=2}^{n}\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n} f\left(\sum_{i=1, i\neq i_1,\cdots,i_{n-k+1}}^{n} x_i - \sum_{r=1}^{n-k+1} x_{i_r}\right) + f\left(\sum_{i=1}^{n} x_i\right) - 2^{n-1} f(x_1) = 0, \tag{1.1}$$

where n is an integer greater than 1, and investigated the functional equation (1.1) in random normed spaces via fixed point method.

Note that a unital algebra A, endowed with the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$ on A, is called a Jordan algebra. A $\mathbb{C}$-linear mapping L of a Jordan algebra A into a Jordan algebra B is called a Jordan homomorphism if $L(x \circ y) = (L(x) \circ L(y))$ holds for all $x, y \in A$.

Throughout this paper, let A be a Jordan Banach algebra with norm $\|\cdot\|$ and unit e, and B a Jordan Banach algebra with norm $\|cdot\|$.

In this paper, we prove the generalized Hyers-Ulam stability of Jordan homomorphisms in Jordan Banach algebras for the functional equation (1.1).

2. Main results

We need the following lemma in the proof of our main theorem.

Lemma 2.1. ([9]) *A mapping* $f : \mathcal{A} \to \mathcal{B}$ *with* $f(0) = 0$ *satisfies* (1.1) *if and only if* $f : \mathcal{A} \to \mathcal{B}$ *is additive.*

We are going to prove the main result.

Theorem 2.2. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^{n+2} \to [0, \infty)$ such that*

$$\tilde{\varphi}(x_1, \cdots x_n, z, w) := \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x_1, \cdots 2^j x_n, 2^j z, 2^j w) < \infty, \tag{2.1}$$

$$\Bigg\| \sum_{k=2}^{n} \sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^{n} h\left(\sum_{i=1, i \neq i_1, \cdots, i_{n-k+1}}^{n} \mu x_i - \sum_{r=1}^{n-k+1} \mu x_{i_r} \right)$$
$$+ \mu h\left(\sum_{i=1}^{n} x_i \right) - \mu 2^{n-1} h(x_1) + h(z \circ w) - h(z) \circ h(w) \Bigg\| \tag{2.2}$$
$$\leqslant \ \varphi(x_1, \dots x_n, z, w)$$

for all $\mu \in T^1 := \{\lambda \in \mathbb{C} |\ |\lambda| = 1\}$ and $x_1, \dots x_n, z, w \in \mathcal{A}$. Then there exists a unique Jordan homomorphism $L : \mathcal{A} \to\ \mathcal{B}$ such that

$$\|h(x) - L(x)\| \leq \frac{1}{2^{n-1}} \tilde{\varphi}(x, x, \underbrace{0 \dots 0}_{n-times}) \tag{2.3}$$

for all $x \in \mathcal{A}$.

Proof. Let $\mu = 1$. Using the relation

$$1 + \sum_{k=1}^{n-k} \binom{n-k}{k} = \sum_{k=0}^{n-k} \binom{n-k}{k} = 2^{n-k} \tag{2.4}$$

for all $n > k$ and putting $x_1 = x_2 = x$ and $x_i = z = w = 0$ $(i = 3, \dots, n)$ in (2.2), we obtain

$$\|2^{n-2} h(2x) - 2^{n-1} h(x)\| \leq \varphi(x, x, \underbrace{0, \dots, 0}_{n-times}) \tag{2.5}$$

for all $x \in \mathcal{A}$. So

$$\left\| \frac{h(2x)}{2} - h(x) \right\| \leq \frac{1}{2^{n-1}} \varphi(x, x, \underbrace{0, \dots, 0}_{n-times}) \tag{2.6}$$

for all $x \in \mathcal{A}$. By induction on m, we can show that

$$\left\| \frac{h(2^m x)}{2^m} - h(x) \right\| \leq \frac{1}{2^{n-1}} \sum_{j=0}^{m-1} \frac{1}{2^j} \varphi(2^j x, 2^j x, \underbrace{0, \dots, 0}_{n-times}) \tag{2.7}$$

for all $x \in \mathcal{A}$. It follows from (2.1) and (2.7) that the sequence $\left\{ \frac{h(2^m x)}{2^m} \right\}$ is a Cauchy sequence for all $x \in \mathcal{A}$. Since $\mathcal{A}$ is complete, the sequence $\left\{ \frac{h(2^m x)}{2^m} \right\}$ converges. Thus one can define the mapping $L : \mathcal{A} \to \mathcal{B}$ by

$$L(x) := \lim_{m \to \infty} \frac{h(2^m x)}{2^m}$$

for all $x \in \mathcal{A}$. Let $z = w = 0$ and $\mu = 1$ in (2.2). By (2.1),

$$\begin{aligned} \|D_f(x_1, ..., x_n)\| &= \lim_{j\to\infty} \frac{1}{2^j} \left\|D_f\left(2^j x_1, ..., 2^j x_n\right)\right\| \\ &\le \lim_{j\to\infty} \frac{1}{2^j} \varphi\left(2^j x_1, ..., 2^j x_n, 0, 0\right) = 0 \end{aligned}$$

for all $x_1, \cdots, x_n \in \mathcal{A}$. So $D_f(x_1, \cdots, x_n) = 0$. By Lemma 2.1, the mapping $L : \mathcal{A} \to \mathcal{B}$ is additive. Moreover, passing the limit $m \to \infty$ in (2.7), we get the inequality (2.3).

Now, let $L' : \mathcal{A} \to \mathcal{B}$ be another additive mapping satisfying (1.1) and (2.3). Then

$$\begin{aligned} \|L(x) - L'(x)\| &= \frac{1}{2^n} \|L(2^n x) - L'(2^n x)\| \\ &\le \frac{1}{2^m} (\|L(2^n x) - h(2^n x)\| + \|L'(2^n x) - h(2^n x)\|) \\ &\le \frac{2}{2^m 2^{n-1}} \widetilde{\varphi}(2^m x, 2^m x, \underbrace{0, \ldots, 0}_{n-times}) \end{aligned}$$

which tends to zero as $m \to \infty$ for all $x \in \mathcal{A}$. So we can conclude that $L(x) = L'(x)$ for all $x \in \mathcal{A}$. This proves the uniqueness of L.

Let $\mu \in \mathbb{T}^1$. Set $x_1 = x$ and $z = w = x_i = 0$ $(i = 2, ..., n)$ in (2.2). Then by (2.1), we get

$$\|2^{n-1} h(\mu x) - 2^{n-1} \mu h(x)\| \le \varphi(x, 0, ..., 0, 0, 0) \tag{2.8}$$

for all $x \in \mathcal{A}$. So

$$\|2^{-m}(h(2^m \mu x) - \mu h(2^m x))\| \le \frac{2}{2^{n+m}} \varphi(2^m x, 0, ..., 0, 0, 0)$$

for all $x \in \mathcal{A}$. Since the right hand side of the above inequality tends to zero as $m \to \infty$, we have

$$L(\mu x) = \lim_{m\to\infty} \frac{h(2^m \mu x)}{2^m} = \lim_{m\to\infty} \frac{\mu h(2^m x)}{2^m} = \mu L(x) \tag{2.9}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$.

Now let $\lambda \in \mathbb{C} (\lambda \ne 0)$ and M an integer greater than $4|\lambda|$. Then $|\lambda/M| < 1/4 < 1 - 2/3 = 1/3$. By Theorem 1 of [16], there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. And $L(x) = L\left(3 \cdot \frac{1}{3}x\right) = 3L\left(\frac{1}{3}x\right)$ for all $x \in \mathcal{A}$. So $L\left(\frac{1}{3}x\right) = \frac{1}{3}L(x)$ for all $x \in \mathcal{A}$. Thus by (2.9)

$$\begin{aligned} L(\lambda x) =& L\left(\frac{M}{3} \cdot 3\frac{\lambda}{M}x\right) = M \cdot L\left(\frac{1}{3} \cdot 3\frac{\lambda}{M}x\right) = \frac{M}{3} L\left(3\frac{\lambda}{M}x\right) \\ &= \frac{M}{3} L(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}(L(\mu_1 x) + L(\mu_2 x) + L(\mu_3 x)) \\ &= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3) L(x) = \frac{M}{3} \cdot 3\frac{\lambda}{M} L(x) = \lambda L(x) \end{aligned}$$

for all $x \in \mathcal{A}$. Hence

$$L(\zeta x_1 + \eta x_2) = L(\zeta x_1) + L(\eta x_2) = \zeta L(x_1) + \eta L(x_2)$$

for all $\zeta, \eta \in \mathbb{C}$ $(\zeta, \eta \neq 0)$ and all $x_1, x_2 \in \mathcal{A}$. And $L(0x) = 0 = 0L(x)$ for all $x \in \mathcal{A}$.
So $L : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

Let $x_i = 0$ $(i \geq 0)$ in (2.2). Then we get

$$\|h(z \circ w) - h(z) \circ h(w)\| \leq \varphi(\underbrace{0, \cdots, 0}_{n-times}, z, w)$$

for all $z, w \in \mathcal{A}$. Since

$$\frac{1}{2^{2m}} \varphi(\underbrace{0, \cdots, 0}_{n-times}, 2^m z, 2^m w) \leq \frac{1}{2^m} \varphi(\underbrace{0, \cdots, 0}_{n-times}, 2^m z, 2^m w),$$

$$\begin{aligned} \frac{1}{2^{2m}} \|h(2^m z \circ 2^m w) - h(2^m z) \circ h(2^m w)\| &\leq \frac{1}{2^{2m}} \varphi(\underbrace{0, \ldots, 0}_{n-times}, z, w) \\ &\leq \frac{1}{2^m} \varphi(\underbrace{0, \ldots, 0}_{n-times}, z, w), \end{aligned}$$

which tends to zero as $m \to \infty$ for all $z, w \in \mathcal{A}$. Hence

$$\begin{aligned} L(z \circ w) &= \lim_{m \to \infty} \frac{h\left(2^{2m}(z \circ w)\right)}{2^{2m}} \\ &= \lim_{m \to \infty} \frac{h(2^m z \circ 2^m w)}{2^{2m}} \\ &= \lim_{m \to \infty} \frac{1}{2^{2m}} (h(2^m z) \circ h(2^m w)) \\ &= \lim_{m \to \infty} \left(\frac{h(2^m z)}{2^m} \circ \frac{h(2^m w)}{2^m} \right) \\ &= L(z) \circ L(w) \end{aligned}$$

for all $z, w \in \mathcal{A}$. So the $\mathbb{C}$-linear mapping $L : \mathcal{A} \to \mathcal{B}$ is a Jordan homomorphism satisfying (2.3). $\square$

Corollary 2.3. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping with $h(0) = 0$ for which there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} &\Bigg\| \sum_{k=2}^{n} \sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_n-k+1=i_{n-k}+1}^{n} h\left(\sum_{i=1, i \neq i_1, \cdots, i_{n-k+1}}^{n} \mu x_i - \sum_{r=1}^{n-k+1} \mu x_{i_r} \right) \\ &+ \mu h\left(\sum_{i=1}^{n} x_i \right) - \mu 2^{n-1} h(x_1) + h(z \circ w) - h(z) \circ h(w) \Bigg\| \\ &\leq \epsilon(\|x_1\|^p + \cdots + \|x_n\|^p + \|z\|^p + \|w\|^p) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x_1, x_2, ..., x_n, z, w \in \mathcal{A}$. *Then there exists a unique Jordan homomorphism* $L : \mathcal{A} \to \mathcal{B}$ *such that*

$$\|h(x) - L(x)\| \leq \frac{\epsilon}{2^n(1 - 2^{p-1})}\|x\|^p$$

for all $x \in \mathcal{A}$.

Proof. Define $\varphi(x_1, \cdots x_n, z, w) = \epsilon(\|x_1\|^p + \cdots + \|x_n\|^p + \|z\|^p + \|w\|^p)$ and apply Theorem 2.2. Then we get the desired result. □

Corollary 2.4. *Suppose that* $h : \mathcal{A} \to \mathcal{B}$ *is mapping with* $h(0) = 0$ *satisfying* (2.2). *If there exists a function* $\varphi : \mathcal{A}^{n+2} \to [0, \infty)$ *such that*

$$\tilde{\varphi}(x_1, \cdots x_n, z, w) := \sum_{j=0}^{\infty} 2^j \varphi(2^{-j}x_1, \cdots 2^{-j}x_n, 2^{-j}z, 2^{-j}w) < \infty$$

for all $z, w, x_i \in \mathcal{A}$ $(i = 1, ..., n)$, *then there exists a unique Jordan homomorphism* $L : \mathcal{A} \to \mathcal{B}$ *such that*

$$\|h(x) - L(x)\| \leq \frac{1}{2^{n-1}}\widetilde{\varphi}(x, x, \underbrace{0 \ldots 0}_{n-times})$$

for all $x \in \mathcal{A}$.

Proof. By the same method as in the proof of Theorem 2.2, one can obtain that

$$L(x) = \lim_{m\to\infty} \frac{h(2^m x)}{2^m}$$

for all $x \in \mathcal{A}$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 2.5. *Let* $h : \mathcal{A} \to \mathcal{B}$ *be a mapping with* $h(0) = 0$ *for which there exists a function*
$\varphi : \mathcal{A}^{n+2} \to [0, \infty)$ *satisfying* (2.1) *such that*

$$\begin{aligned} &\Bigg\| \sum_{k=2}^{n} \sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^{n} h\left(\sum_{i=1, i\neq i_1, \cdots, i_{n-k+1}} \mu x_i - \sum_{r=1}^{n-k+1} \mu x_{i_r} \right) \\ &\quad + \mu h\left(\sum_{i=1}^{n} x_i\right) - \mu 2^{n-1} h(x_1) + h(z \circ w) - h(z) \circ h(w) \Bigg\| \qquad (2.10) \\ &\quad \leqslant \ \varphi(x_1, \cdots x_n, z, w) \end{aligned}$$

for $\mu = 1, i$ *and all* $x_1, \cdots, x_n, z, w \in \mathcal{A}$. *If* $h(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in \mathcal{A}$, *then there exists a unique Jordan homomorphism* $L : \mathcal{A} \to \mathcal{B}$ *satisfying* (2.3).

Proof. Put $z = w = 0$ in (2.10). By the same reasoning as in the proof of Theorem 2.2, there exists a unique additive mapping $L : \mathcal{A} \to \mathcal{B}$ satisfying (2.3). The additive mapping $L : \mathcal{A} \to \mathcal{B}$ is given by

$$L(x) = \lim_{m\to\infty} \frac{h(2^m x)}{2^m}$$

for all $x \in \mathcal{A}$. By the same reasoning as in the proof of Theorem 2.2 the additive mapping $L : \mathcal{A} \to \mathcal{B}$ is $\mathbb{R}$-linear.

Putting $x_i = z = w = 0$ $(i = 2, \cdots, n)$ and $\mu = i$ in (2.10), we get

$$\|h(ix) - ih(x)\| \le \varphi(x, \underbrace{0, \cdots, 0}_{(n+1)-times})$$

for all $x \in \mathcal{A}$. So

$$\frac{1}{2^n}\|h(2^m ix) - ih(2^m x)\| \le \frac{1}{2^n}\varphi(2^n x, \underbrace{0, \ldots, 0}_{(n+1)-times}),$$

which tends to zero as $m \to \infty$. Hence

$$L(ix) = \lim_{m\to\infty} \frac{h(2^m ix)}{2^m} = \lim_{m\to\infty} \frac{ih(2^m x)}{2^m} = iL(x)$$

for all $x \in \mathcal{A}$.

For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$L(\lambda x) = L(sx+itx) = sL(x)+tL(ix) = sL(x)+itL(x) = (s+it)L(x) = \lambda L(x)$$

for all $x \in \mathcal{A}$. So

$$L(\zeta x_1 + \eta x_2) = L(\zeta x_1) + L(\eta x_2) = \zeta L(x_1) + \eta L(x_2)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x_1, x_2 \in \mathcal{A}$. Hence the additive mapping $L : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 2.2. $\square$

Corollary 2.6. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping with $h(0) = 0$ for which there exist constants $\epsilon \ge 0$ and $p > 1$ such that*

$$\Bigg\| \sum_{k=2}^{n} \sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^{n} h\left(\sum_{i=1, i\ne i_1, \cdots, i_{n-k+1}}^{n} \mu x_i - \sum_{r=1}^{n-k+1} \mu x_{i_r} \right)$$
$$+\mu h\left(\sum_{i=1}^{n} x_i\right) - \mu 2^{n-1} h(x_1) + h(z \circ w) - (h(z) \circ h(w)) \Bigg\|$$
$$\le \epsilon(\|x_1\|^p + \ldots + \|x_n\|^p + \|z\|^p + \|w\|^p)$$

for all $z, w, x_i \in \mathcal{A}$ $(i = 1, 2, \cdots, n)$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique Jordan homomorphism $L : \mathcal{A} \to \mathcal{B}$ such that

$$\|h(x) - L(x)\| \le \frac{\epsilon}{2^n(2^{1-p} - 1)}\|x\|^p$$

for all $x \in \mathcal{A}$.

Proof. Define $\varphi(x_1, \cdots x_n, z, w) = \epsilon(\|x_1\|^p + \cdots + \|x_n\|^p + \|z\|^p + \|w\|^p)$ and apply Theorem 2.2. Then we get the desired result. $\square$

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