Iranian Journal of Mathematical Sciences and Informatics
Vol.1, No.2 (2006),pp 31-51
DOI: 10.7508/ijmsi.2006.02.003

RICCI CURVATURE OF SUBMANIFOLDS OF A SASAKIAN SPACE FORM

SUNGPYO HONG AND MUKUT MANI TRIPATHI

DEPARTMENT OF MATHEMATICS, POHANG UNIVERSITY OF SCIENCE AND TECHNOLOGY (POSTECH), SAN 31 HYOJA DONG, NAM-GU POHANG 790-784, REPUBLIC OF KOREA
DEPARTMENT OF MATHEMATICS AND ASTRONOMY, LUC-KNOW UNIVERSITY LUCKNOW 226 007, INDIA

EMAIL: SUNGPYO@POSTECH.AC.KR
EMAIL: MMTRIPATHI66@YAHOO.COM

Abstract. Involving the Ricci curvature and the squared mean curvature, we obtain basic inequalities for different kind of submanifolds of a Sasakian space form tangent to the structure vector field of the ambient manifold. Contrary to already known results, we find a different necessary and sufficient condition for the equality case. We also give very simple proofs (1) for a basic inequality for Ricci curvature of C-totally real submanifolds of a Sasakian space form, and (2) of the fact that if a C-totally real submanifold of maximum dimension satisfies the equality case, then it must be minimal. Two basic inequalities for submanifolds of any Riemannian manifold, one involving scalar curvature and the squared mean curvature and the other involving the invariant θ_k and the squared mean curvature are also obtained. These results are applied to get corresponding results for submanifolds of Sasakian space forms.

Keywords: Einstein manifold, Sasakian space form, invariant submanifold, semi-invariant submanifold, almost semi-invariant submanifold, CR-submanifold, slant submanifold, C-totally real submanifold, Ricci curvature, k-Ricci curvature, scalar curvature.

2000 Mathematics subject classification: 53C40, 53C15, 53C25.

1. Introduction

One of the basic interests in the submanifold theory is to establish simple relationships between the main intrinsic invariants and the main extrinsic invariants of a submanifold. The main intrinsic invariants include the classical curvature invariants namely the Ricci curvature and the scalar curvature. For a unit vector X in an n-dimensional submanifold M of a real space form $R^m(c)$, B. Y. Chen [9] proved the following basic inequality

$$\|H\|^2 \geq \frac{4}{n^2}\left\{\mathrm{Ric}(X) - (n-1)\,c\right\} \tag{1.1}$$

involving the Ricci curvature Ric and the squared mean curvature $\|H\|^2$ of the submanifold. The inequality (1.1) drew attention of several authors and they established same kind of inequalities for different kind of submanifolds in ambient manifolds possessing different kind of structures. The submanifolds include mainly invariant, anti-invariant and slant submanifolds, while ambient manifolds include mainly real, complex and Sasakian space forms.

Recently in [15], the present authors made a general theory for a submanifold of Riemannian manifolds by proving a basic inequality, involving the Ricci curvature and the squared mean curvature of the submanifold. The goal was achieved by use of the concept of k-Ricci curvature $(2 \leq k \leq n)$ in an n-dimensional Riemannian manifold introduced by B.-Y. Chen [9]. In this paper, we apply this general theory to study submanifolds of Sasakian space forms. In section 2, we recall the notion of Ricci curvature, scalar curvature and k-Ricci curvature and give a brief account of submanifolds. Then we recall the general basic inequality from [15] (see Theorem 2.1) for a submanifold of Riemannian manifolds, involving the Ricci curvature and the squared mean curvature of the submanifold. The equality case is also discussed. Section 3 contains a brief discussion about Sasakian manifolds and Sasakian space forms. In section 4, by applying Theorem 2.1 we obtain a basic inequality for submanifolds of a Sasakian space form such that the structure vector field of the ambient manifold is tangent to the submanifold. Equality cases are also discussed. Next, in section 5, we obtain a basic inequality for Ricci curvature of C-totally real submanifolds of a Sasakian space form. We also give a simple proof of the fact that if a C-totally real submanifold of maximum dimension satisfies the equality case, then it must be minimal. In section 6, first we give a basic inequality for submanifolds of any Riemannian manifold involving scalar curvature and the squared mean curvature. Then, we obtain one more basic inequality involving the invariant θ_k and the squared mean curvature. Then, we apply these results to get corresponding results for submanifolds of Sasakian space forms.

2. Ricci curvature of submanifolds

Let M be an n-dimensional Riemannian manifold equipped with a Riemannian metric g. The inner product of the metric g is denoted by $\langle\, ,\rangle$. We denote the set of unit vectors in T_pM by T_p^1M; thus

$$T_p^1M = \{X \in T_pM \mid \langle X, X\rangle = 1\}.$$

Let $\{e_1, \ldots, e_k\}$, $2 \leq k \leq n$, be an orthonormal basis of a k-plane section Π_k of T_pM. If $k = n$ then $\Pi_n = T_pM$; and if $k = 2$ then Π_2 is a plane section of T_pM. For a fixed $i \in \{1, \ldots, k\}$, a *k-Ricci curvature* of Π_k at e_i, denoted $\mathrm{Ric}_{\Pi_k}(e_i)$, is defined by [9]

$$\mathrm{Ric}_{\Pi_k}(e_i) = \sum_{j\neq i}^{k} K_{ij}, \tag{2.1}$$

where K_{ij} is the sectional curvature of the plane section spanned by e_i and e_j. In fact, a k-Ricci curvature is a $(k-1)$-Ricci curvature in the sense of H. Wu [32]. We note that an *n-Ricci curvature* $\mathrm{Ric}_{T_pM}(e_i)$ is the usual *Ricci curvature* of e_i, denoted $\mathrm{Ric}\,(e_i)$. Thus for any orthonormal basis $\{e_1, \ldots, e_n\}$ for T_pM and for a fixed $i \in \{1, \ldots, n\}$, we have

$$\mathrm{Ric}_{T_pM}(e_i) \equiv \mathrm{Ric}\,(e_i) = \sum_{j\neq i}^{n} K_{ij}.$$

The scalar curvature $\tau\,(\Pi_k)$ of the k-plane section Π_k is given by

$$\tau\,(\Pi_k) = \sum_{1\leq i<j\leq k} K_{ij}. \tag{2.2}$$

Geometrically, $\tau(\Pi_k)$ is the scalar curvature of the image $\exp_p(\Pi_k)$ of Π_k at p under the exponential map at p. We define the normalized scalar curvature $\tau_N\,(\Pi_k)$ of Π_k by

$$\tau_N\,(\Pi_k) = \frac{2\tau\,(\Pi_k)}{k\,(k-1)}. \tag{2.3}$$

The normalized scalar curvature at p is defined as [8]

$$\tau_N\,(p) = \frac{2\tau\,(p)}{n\,(n-1)}. \tag{2.4}$$

Then, we see that $\tau_N\,(p) = \tau_N\,(T_pM)$. The scalar curvature $\tau(p)$ of M at p is identical with the scalar curvature of the tangent space T_pM of M at p, that is, $\tau(p) = \tau\,(T_pM)$. If Π_2 is a plane section and $\{e_1, e_2\}$ is any orthonormal basis for Π_2, then

$$\mathrm{Ric}_{\Pi_2}(e_1) = \mathrm{Ric}_{\Pi_2}(e_2) = \tau(\Pi_2) = \tau_N(\Pi_2) = K_{12}.$$

Let M be an n-dimensional submanifold of an m-dimensional Riemannian manifold $\widetilde{M}$ equipped with a Riemannian metric $\widetilde{g}$. We use the inner product notation $\langle , \rangle$ for both the metrics $\widetilde{g}$ of $\widetilde{M}$ and the induced metric g on the submanifold M. The Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) \qquad \text{and} \qquad \widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\widetilde{\nabla}$, ∇ and $\nabla^{\perp}$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\widetilde{M}$, M and the normal bundle $T^{\perp}M$ of M respectively, and σ is the second fundamental form related to the shape operator A by $\langle\sigma(X,Y),N\rangle = \langle A_N X, Y\rangle$. The equation of Gauss is given by

$$\begin{aligned} R(X,Y,Z,W) &= \widetilde{R}(X,Y,Z,W) + \langle\sigma(X,W),\sigma(Y,Z)\rangle \\ &\quad - \langle\sigma(X,Z),\sigma(Y,W)\rangle \end{aligned} \tag{2.5}$$

for all $X, Y, Z, W \in TM$, where $\widetilde{R}$ and R are the Riemann curvature tensors of $\widetilde{M}$ and M respectively. The curvature tensor $R^{\perp}$ of the normal bundle of M is defined by

$$R^{\perp}(X,Y)\,\nu = \nabla^{\perp}_X\nabla^{\perp}_Y\nu - \nabla^{\perp}_Y\nabla^{\perp}_X\nu - \nabla^{\perp}_{[X,Y]}\nu$$

and the covariant derivative of σ by

$$(\nabla'\sigma)(X,Y,Z) = \nabla^{\perp}_X\sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).$$

If $R^{\perp} = 0$, then the normal connection $\nabla^{\perp}$ of M is said to be *trivial* or *flat*. If $\nabla'\sigma = 0$, then the second fundamental form is said to be *parallel*.

The mean curvature vector H is given by $nH = \mathrm{trace}(\sigma)$. The submanifold M is *totally geodesic* in $\widetilde{M}$ if $\sigma = 0$, and *minimal* if $H = 0$. If $\sigma(X,Y) = \langle X,Y\rangle H$ for all $X, Y \in TM$, then M is *totally umbilical*.

The *relative null space* of M at p is defined by [9]

$$\mathcal{N}_p = \{X \in T_pM \mid \sigma(X,Y) = 0 \text{ for all } Y \in T_pM\},$$

which is also known as the *kernel of the second fundamental form* at p [10].

Now, let $\{e_1,\ldots,e_n\}$ be an orthonormal basis of the tangent space T_pM and e_r belongs to an orthonormal basis $\{e_{n+1},\ldots,e_m\}$ of the normal space $T_p^{\perp}M$. We put

$$\sigma^r_{ij} = \langle\sigma(e_i,e_j),e_r\rangle \quad \text{and} \quad \|\sigma\|^2 = \sum_{i,j=1}^{n}\langle\sigma(e_i,e_j),\sigma(e_i,e_j)\rangle.$$

Let K_{ij} and $\widetilde{K}_{ij}$ denote the sectional curvature of the plane section spanned by e_i and e_j at p in the submanifold M and in the ambient manifold $\widetilde{M}$ respectively. Thus, we can say that K_{ij} and $\widetilde{K}_{ij}$ are the "intrinsic" and "extrinsic" sectional curvature of the $\mathrm{Span}\{e_i,e_j\}$ at p. In view of (2.5), we get

$$K_{ij} = \widetilde{K}_{ij} + \sum_{r=n+1}^{m}\left(\sigma^r_{ii}\sigma^r_{jj} - (\sigma^r_{ij})^2\right). \tag{2.6}$$

From (2.6) it follows that

$$2\tau(p) = 2\widetilde{\tau}(T_pM) + n^2\|H\|^2 - \|\sigma\|^2, \tag{2.7}$$

where $\widetilde{\tau}(T_pM)$ denotes the scalar curvature of the n-plane section T_pM in the ambient manifold $\widetilde{M}$. Thus, we can say that $\tau(p)$ and $\widetilde{\tau}\,(T_pM)$ are the "intrinsic" and "extrinsic" scalar curvature of the submanifold at p respectively.

Now, we recall the following

Theorem 2.1. [15] *Let M be an n-dimensional submanifold of a Riemannian manifold $\widetilde{M}$. Then the following statements are true.*

(a) *For $X \in T_p^1M$ we have*

$$\mathrm{Ric}(X) \leq \frac{n^2}{4}\,\|H\|^2 + \widetilde{\mathrm{Ric}}_{(T_pM)}(X), \tag{2.8}$$

where $\widetilde{\mathrm{Ric}}_{(T_pM)}(X)$ is the n-Ricci curvature of T_pM at $X \in T_p^1M$ with respect to the ambient manifold $\widetilde{M}$.

(b) *The equality case of* (2.8) *is satisfied by $X \in T_p^1M$ if and only if*

$$\sigma(X,X) = \frac{n}{2}H(p) \quad \text{and} \quad \sigma(X,Y) = 0 \tag{2.9}$$

for all $Y \in T_pM$ such that $\langle X, Y\rangle = 0$.

(c) *The equality case of* (2.8) *holds for all $X \in T_p^1M$ if and only if either* **(1)** *p is a totally geodesic point or* **(2)** *$n = 2$ and p is a totally umbilical point.*

From Theorem 2.1, we immediately have the following

Corollary 2.2. *Let M be an n-dimensional submanifold of a Riemannian manifold. For $X \in T_p^1M$ any two of the following three statements imply the remaining one.*

(a) *The mean curvature vector $H(p)$ vanishes.*

(b) *The unit vector X belongs to the relative null space $\mathcal{N}_p$.*

(c) *The unit vector X satisfies the equality case of* (2.8)*, namely*

$$\mathrm{Ric}(X) = \frac{1}{4}\,n^2\|H\|^2 + \widetilde{\mathrm{Ric}}_{(T_pM)}(X). \tag{2.10}$$

3. Sasakian space forms

A 1-form η on a differentiable manifold $\widetilde{M}$ of odd dimension $2m+1$ $(m \geq 1)$ is called a *contact form* if $\eta \wedge (d\eta)^m \neq 0$ everywhere on $\widetilde{M}$, and $\widetilde{M}$ equipped with a contact form is a *contact manifold.* In 1953, S. S. Chern [12] proved that the structural group of a $(2m+1)$-dimensional contact manifold can be reduced to $\mathcal{U}(m) \times 1$. A $(2m+1)$-dimensional differentiable manifold $\widetilde{M}$ is called an *almost contact manifold* [14] if its structural group can be reduced to $\mathcal{U}(m) \times 1$. Equivalently, there is an *almost contact structure* (φ, ξ, η) [25] consisting of a tensor field φ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0. \tag{3.1}$$

First and one of the remaining three relations of (3.1) imply the other two relations of (3.1). An almost contact structure is *normal* if the Nijenhuis tensor of φ equals $-2d\eta \otimes \xi$.

Let $\widetilde{g}$ be a compatible Riemannian metric with (φ, ξ, η), that is,

$$\langle X, Y\rangle = \langle \varphi X, \varphi Y\rangle + \eta(X)\eta(Y), \qquad X, Y \in T\widetilde{M}, \tag{3.2}$$

where $\langle\, ,\rangle$ denotes the inner product of the metric $\widetilde{g}$. Then, $\widetilde{M}$ becomes an *almost contact metric manifold* equipped with an *almost contact metric structure* $(\varphi, \xi, \eta, \widetilde{g})$. The equation (3.2) is equivalent to

$$\Phi(X, Y) \equiv \langle X, \varphi Y\rangle = -\langle \varphi X, Y\rangle \quad \text{and} \quad \langle X, \xi\rangle = \eta(X). \tag{3.3}$$

An almost contact metric structure becomes a *contact metric structure* if $\Phi = d\eta$. A contact metric manifold is a *K-contact manifold* if ξ is Killing. A normal contact metric manifold is a *Sasakian manifold*. An almost contact metric manifold is Sasakian if and only if

$$(\widetilde{\nabla}_X \varphi)Y = \langle X, Y\rangle \xi - \eta(Y)X, \qquad X, Y \in T\widetilde{M}. \tag{3.4}$$

A Sasakian manifold is always a K-contact manifold and the converse is true in the dimension three. A compact K-contact Einstein manifold is also Sasakian.

A plane section in $T_p\widetilde{M}$ is called a *φ-section* if there exists a vector $X \in T_p\widetilde{M}$ orthogonal to ξ such that $\{X, \varphi X\}$ span the section. The sectional curvature is called *φ-sectional curvature*. Just as the sectional curvatures of a Riemannian manifold determine the curvature completely and the holomorphic sectional curvatures of a Kaehler manifold determine the curvature completely, on a Sasakian manifold the φ-sectional curvatures determine the curvature completely. Moreover on a Sasakian manifold of dimension ≥ 5 if at each point the φ-sectional curvature is independent of the choice of φ-section at the point, it is constant on the manifold and the curvature tensor is given by

$$\begin{aligned}
\widetilde{R}(X, Y)Z &= \frac{c+3}{4}\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\} \\
&\quad + \frac{c-1}{4}\{\langle X, \varphi Z\rangle \varphi Y - \langle Y, \varphi Z\rangle \varphi X + 2\langle X, \varphi Y\rangle \varphi Z \\
&\qquad + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
&\qquad + \langle X, Z\rangle \eta(Y)\xi - \langle Y, Z\rangle \eta(X)\xi\}
\end{aligned} \tag{3.5}$$

for all $X, Y, Z \in T\widetilde{M}$. A Sasakian manifold of constant φ-sectional curvature c is called a *Sasakian space form* $\widetilde{M}(c)$.

A well known result of Tanno [26] is that a complete simply connected Sasakian manifold of constant φ-sectional curvature c is isometric to one of certain model spaces depending on whether $c > -3$, $c = -3$ or $c < -3$. The model space for $c > -3$ is a sphere with a D-homothetic deformation of the standard structure. For $c = -3$ the model space is $\mathbf{R}^{2n+1}$ with the contact form $\eta = \frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i)$ together with the metric $ds^2 = \eta\otimes\eta + \frac{1}{4}\sum_{i=1}^{n}((dx^i)^2 +$

$(dy^i)^2)$. For $c < -3$ one has a canonically defined contact metric structure on the product $B^n \times \mathbf{R}$ where B^n is a simply connected bounded domain in $\mathbf{C}^n$ with a Kaehler structure of constant negative holomorphic curvature. In particular Sasakian space forms exist for all values of c. For more details we refer to [3].

4. Ricci curvature of submanifolds of Sasakian space forms

A submanifold M of an almost contact metric manifold $\widetilde{M}$ with the structure $(\varphi, \xi, \eta, \widetilde{g})$ is said to be *invariant* if $\varphi(TM) \subseteq TM$. If the structure vector field ξ is tangent to the invariant submanifold M, then M inherits an almost contact metric structure (φ, ξ, η, g) by restriction. If M is an invariant submanifold of a contact metric manifold, then $\xi \in TM$, $\sigma(X, \xi) = 0$ and M is a minimal contact metric manifold equipped with the induced structure (φ, ξ, η, g). If $\widetilde{M}$ is K-contact or Sasakian then the induced structure (φ, ξ, η, g) on M is K-contact or Sasakian respectively. An invariant submanifold of a Sasakian manifold is also known as a Sasakian submanifold. For more details we refer to [3].

Now, we need the following

Lemma 4.1. *Let M be a submanifold of a K-contact manifold such that $\xi \in TM$. If $p \in M$ is a totally umbilical point, then p is a totally geodesic point, and hence $\varphi(T_pM) \subset T_pM$.*

Proof. An almost contact metric manifold $\widetilde{M}$ is K-contact if and only if

$$\widetilde{\nabla}_X \xi = -\varphi X, \qquad X \in T\widetilde{M}, \tag{4.1}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection. Therefore, for a submanifold M of a K-contact manifold such that $\xi \in TM$ we have

$$\nabla_X \xi = -PX \quad \text{and} \quad \sigma(X, \xi) = -FX \tag{4.2}$$

for all $X \in TM$, where PX and FX are the tangential and the normal parts of φX respectively. Let $p \in M$ be a totally umbilical point. Then, we get

$$H = \langle \xi, \xi \rangle H = \sigma(\xi, \xi) = -F\xi = 0,$$

which shows that $\sigma(X, Y) = 0$ for all $X, Y \in T_pM$, that is, p is a totally geodesic point. Since p is a totally geodesic point, therefore we have

$$0 = \sigma(X, \xi) = -FX$$

for all $X \in TM$, which shows that $\varphi(T_pM) \subset T_pM$. $\square$

Consequently, we have the following

Proposition 4.2. *A totally umbilical submanifold of a K-contact manifold, such that $\xi \in TM$, is a totally geodesic invariant submanifold.*

Now, we prove the following result.

Theorem 4.3. *Let M be an n-dimensional submanifold of a Sasakian space form $\widetilde{M}(c)$ such that the structure vector field ξ is tangent to the submanifold M. Then, the following statements are true.*

(a) *For each $X \in T_p^1 M$ we have*

$$4\mathrm{Ric}(X) \leq n^2\|H\|^2 + 4(n-1) + (c-1)\left\{3\|PX\|^2 + (n-2)(1-\eta(X)^2)\right\}. \tag{4.3}$$

(b) *A vector $X \in T_p^1 M$ satisfies the equality case of (4.3) if and only if (2.9) is true. If $H(p) = 0$, then $X \in T_p^1 M$ satisfies the equality case of (4.3) if and only if $X \in \mathcal{N}_p$.*

(c) *The equality case of (4.3) holds for all $X \in T_p^1 M$ if and only if $\varphi(T_pM) \subset T_pM$ and p is a totally geodesic point.*

Proof. If M is an n-dimensional submanifold of a Sasakian space form $\widetilde{M}(c)$ such that $\xi \in TM$, then in view of (3.5) we have

$$\widetilde{\mathrm{Ric}}_{(T_pM)}(X) = \frac{1}{4}(n-1)(c+3) + \frac{1}{4}(c-1)\{3\|PX\|^2 - (n-2)\eta(X)^2 - 1\} \tag{4.4}$$

for all $X \in T_p^1 M$. Using (4.4) in (2.8) yields the inequality (4.3). Proof of **(b)** is as usual. Next, in view of the statement **(c)** of Theorem 2.1 and Lemma 4.1, the statement **(c)** follows easily. □

The above Theorem is an improvement of Theorem 3.2 of [28].

Let M be an n-dimensional invariant submanifold of a Sasakian space form $\widetilde{M}(c)$. Then $\xi \in TM$, $\sigma(X,\xi) = 0$ for all $X \in TM$, and M is minimal [3]. Since M is invariant, we get

$$\|PX\|^2 = 1 - \eta(X)^2, \qquad X \in T_p^1 M. \tag{4.5}$$

Using minimality condition and (4.5) in the inequality (4.3) we get

Theorem 4.4. *Every totally geodesic invariant submanifold of a Sasakian space form $\widetilde{M}(c)$ satisfies*

$$4\mathrm{Ric}(X) = 4(n-1) + (c-1)(n+1)\left(1 - \eta(X)^2\right), \qquad X \in T_p^1 M. \tag{4.6}$$

Next, we prove the following

Theorem 4.5. *A totally geodesic invariant submanifold of a Sasakian space form $\widetilde{M}(c)$ is Einstein if and only if $c = 1$.*

Proof. Let M be an n-dimensional totally geodesic invariant submanifold of a Sasakian space form $\widetilde{M}(c)$. If $c = 1$, from (4.6) we see that M is Einstein. Conversely, if M is Einstein, then for any $X \in T_p^1 M$ orthogonal to ξ, from (4.6) it follows that

$$0 = \mathrm{Ric}(X) - \mathrm{Ric}(\xi) = \frac{1}{4}(c-1)(n+1),$$

which shows that $c = 1$. □

It is well known that ([17], Lemma 1.2) if the second fundamental form σ of an invariant submanifold M of a Sasakian manifold is parallel, then M is totally geodesic. Now, we consider some well known results for invariant submanifolds of a Sasakian space form as follows.

Theorem 4.6. *Let M be a $(2n+1)$-dimensional invariant submanifold of a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$. Then the following statements are true.*

(1) ([35], Proposition 1.3, p. 313) *M is totally geodesic if and only if M is of constant φ-sectional curvature c.*

(2) ([35], Theorem 1.2, p. 318) *If M is compact, then either M is totally geodesic, or $\|\sigma\|^2 = (n+2)(c+3)/3$, or at some point $p \in M$, $\|\sigma\|^2(p) > (n+2)(c+3)/3$.*

(3) ([35], Theorem 1.4, p. 318) *If M is η-Einstein invariant, then either M is totally geodesic or $\|\sigma\|^2 \geq n(c+3)$.*

(4) ([35], Theorem 1.6, p. 319) *If M is of constant φ-sectional curvature k, and if $c > -3$, then either M is totally geodesic or $c + 3 \geq 2(k+3)$.*

(5) ([35], Theorem 1.7, p. 319) *If M is of constant φ-sectional curvature k, and if the second fundamental form σ of M is η*-parallel ([35], p. 314)*, that is,*

$$(\nabla_{\varphi X}\sigma)(\varphi Y, \varphi Z) = 0, \qquad X, Y, Z \in TM;$$

then either M is totally geodesic or $c + 3 = 2(k+3) > 0$.

(6) ([35], Theorem 1.8, p. 319) *If M is of constant φ-sectional curvature k, and $m - n < n(n+1)/2$, then M is totally geodesic (that is, $k = c$).*

(7) ([35], Theorem 1.10, p. 325) *Let M be of codimension 2 with η-parallel Ricci tensor. If $c \leq -3$, then M is totally geodesic. If $c > -3$, then either M is totally geodesic, or an η-Einstein manifold with $\|\sigma\|^2 = n(c+3)$ and hence the scalar curvature $\tau = n(n(c+3)-2)/2$.*

(8) ([35], Theorem 1.11, p. 326) *Let M be compact with codimension 2 and $c > -3$. If the scalar curvature τ of M is constant, then either M is totally geodesic or an η-Einstein manifold with the scalar curvature $\tau = n(n(c+3)-2)/2$.*

(9) ([17], Theorem 2.1) *If M is compact, then either M is totally geodesic or at some point $p \in M$, $\|\sigma\|^2(p) > (c(n+2)+3n)/3$.*

(10) ([17], Proposition 2.1) *If M is with trivial normal connection, then we have $c \leq 1$ with equality condition if and only if M is totally geodesic and η-Einstein.*

(11) ([17], Theorem 3.1) *If codimension of M is greater than two, then the following two statements are equivalent:*

- **(i)** *the normal connection of M is trivial, and*
- **(ii)** *M is totally geodesic and $c = 1$.*

(12) ([31], Theorem 1) *If M is compact with $c > -3$ and φ-sectional curvature greater than $(c-3)/2$, then M is totally geodesic.*

(13) ([31], Theorem 2) *If M is complete $(n \geq 2)$ with $c > -3$ and sectional curvature greater than $(c+3)/8$, then M is totally geodesic.*

In view of Theorem 4.4 and Theorem 4.6, we can state the following

Corollary 4.7. *Let M be a $(2n+1)$-dimensional invariant submanifold of a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$. Then M satisfies the equality* (4.6)*, that is,*

$$4\mathrm{Ric}(X) = 4(n-1) + (c-1)(n+1)\left(1-\eta(X)^2\right), \qquad X \in T_p^1 M,$$

if one of the following statements is true.

- **(1)** *M is of constant φ-sectional curvature c.*
- **(2)** *M is compact and satisfies $\|\sigma\|^2 < (n+2)(c+3)/3$.*
- **(3)** *M is η-Einstein and satisfies $\|\sigma\|^2 < n(c+3)$.*
- **(4)** *M is of constant φ-sectional curvature k such that $0 < c+3 < 2(k+3)$.*
- **(5)** *M is of constant φ-sectional curvature such that $m-n \geq n(n+1)/2$.*
- **(6)** *M is of codimension 2 with η-parallel Ricci tensor and $c \leq -3$.*
- **(7)** *M is compact with codimension 2 and $c > -3$ such that M is not an η-Einstein manifold.*
- **(8)** *M is compact and $\|\sigma\|^2 \leq (c(n+2)+3n)/3$.*
- **(9)** *M is with trivial normal connection and $c = 1$.*
- **(10)** *M is of codimension greater than two and with trivial normal connection.*
- **(11)** *M is compact with $c > -3$ and φ-sectional curvature greater than $(c-3)/2$.*
- **(12)** *M is complete $(n \geq 2)$ with $c > -3$ and sectional curvature greater than $(c+3)/8$.*
- **(13)** *The second fundamental form σ is parallel.*

Now, we recall the definition of almost semi-invariant submanifold [27] of an almost contact metric manifold.

Definition 4.8. A submanifold M of an almost contact metric manifold $\widetilde{M}$ with $\xi \in TM$ is said to be an *almost semi-invariant submanifold* of $\widetilde{M}$ if there are k distinct functions $\lambda_1, \ldots, \lambda_k$ defined on M with values in the open interval $(0,1)$ such that TM is decomposed as P-invariant mutually orthogonal differentiable distributions given by

$$TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_1} \oplus \cdots \oplus \mathcal{D}^{\lambda_k} \oplus \{\xi\},$$

where $\mathcal{D}_p^1 = \ker\left(F|_{\{\xi\}^\perp}\right)_p$, $\mathcal{D}_p^0 = \ker\left(P|_{\{\xi\}^\perp}\right)_p$ and

$$\mathcal{D}_p^{\lambda_i} = \ker\left(P^2|_{\{\xi\}^\perp} + \lambda_i^2(p)\, I\right)_p, \qquad i \in \{1, \ldots, k\}.$$

If in addition, each λ_i is constant, then M is called an *almost semi-invariant* submanifold*.

An almost semi-invariant submanifold becomes

- **(1)** a *semi-invariant submanifold* ([1], [2]) if $k = 0$.
- **(2)** an *invariant submanifold* if $k = 0$ and $\mathcal{D}^0 = \{0\}$.
- **(3)** an *anti-invariant submanifold* [2] if $k = 0$ and $\mathcal{D}_p^1 = \{0\}$.

(4) a *θ-slant submanifold* [5] if $\mathcal{D}^1 = \{0\} = \mathcal{D}^0$, $k = 1$ and λ_1 is constant. In this case, we have $TM = \mathcal{D}^{\lambda_1} \oplus \{\xi\}$ and the slant angle θ is given by $\lambda_1 = \cos\theta$.
(5) a *semi-slant submanifold* [4] if $\mathcal{D}^1 \neq \{0\}$, $\mathcal{D}^0 = \{0\}$, $k = 1$ and λ_1 is constant. In this case, we have $TM = \mathcal{D}^1 \oplus \mathcal{D}^{\lambda_1} \oplus \{\xi\}$, and the slant angle θ of the distribution $\mathcal{D}^{\lambda_1}$ is given by $\lambda_1 = \cos\theta$.
(6) a *bi-slant submanifold* [4] if $\mathcal{D}^1 = \{0\} = \mathcal{D}^0$, $k = 2$ and λ_1, λ_2 are constant. In this case, we have $TM = \mathcal{D}^{\lambda_1} \oplus \mathcal{D}^{\lambda_2} \oplus \{\xi\}$, and the slant angles θ_i of the distributions $\mathcal{D}^{\lambda_i}$ are given by $\lambda_i = \cos\theta_i$.

Thus, the definition of almost semi-invariant submanifold is the most logical generalized definition. If M is an almost semi-invariant submanifold of an almost contact metric manifold $\widetilde{M}$, then for $X \in TM$ we may write

$$X = U^1X + U^0X + U^{\lambda_1}X + \cdots + U^{\lambda_k}X + \eta(X)\xi,$$

where U^1, U^0, $U^{\lambda_1}, \ldots, U^{\lambda_k}$ are orthogonal projection operators of TM on $\mathcal{D}^1$, $\mathcal{D}^0$, $\mathcal{D}^{\lambda_1}, \ldots, \mathcal{D}^{\lambda_k}$ respectively. Then, it follows that

$$\|X\|^2 = \left\|U^1X\right\|^2 + \left\|U^0X\right\|^2 + \left\|U^{\lambda_1}X\right\|^2 + \cdots + \left\|U^{\lambda_k}X\right\|^2 + \eta(X)^2.$$

We also have

$$P^2X = -U^1X - \lambda_1^2\left(U^{\lambda_1}X\right) - \cdots - \lambda_k^2\left(U^{\lambda_k}X\right),$$

which implies that

$$\|PX\|^2 = \langle PX, PX\rangle = -\left\langle P^2X, X\right\rangle = \sum_{\lambda\in\{1,\lambda_1,\ldots,\lambda_k\}} \lambda^2\left\|U^\lambda X\right\|^2. \tag{4.7}$$

In particular, if M is an n-dimensional θ-slant submanifold, then $\lambda_1^2 = \cos^2\theta$ and we have

$$\|PX\|^2 = \cos^2\theta\left\|U^{\lambda_1}X\right\|^2 = \cos^2\theta\left(\|X\|^2 - \eta(X)^2\right). \tag{4.8}$$

If $X \in T_p^1M$, then (4.8) becomes

$$\|PX\|^2 = \cos^2\theta\left(1 - \eta(X)^2\right). \tag{4.9}$$

Moreover, if the unit vector $X \in T_p^1M$ is orthogonal to the structure vector field ξ, then

$$\|PX\|^2 = \cos^2\theta. \tag{4.10}$$

Now, from Theorem 4.3 we immediately have the following Corollary.

Corollary 4.9. *Let M be an n-dimensional submanifold of a Sasakian space form $\widetilde{M}(c)$ such that the structure vector field ξ is tangent to M. Then, the following statements are true.*

(a) *If M is an almost semi-invariant submanifold, then for $X \in T_p^1M$ we have*

$$4\mathrm{Ric}(X) \leq n^2\|H\|^2 + 4(n-1)$$

$$+(c-1)\left(3\sum_{\lambda\in\{1,\lambda_1,\ldots,\lambda_k\}}\lambda^2\left\|U^\lambda X\right\|^2+(n-2)(1-\eta(X)^2)\right), \tag{4.11}$$

where U^1, $U^{\lambda_1},\ldots,U^{\lambda_k}$ are orthogonal projection operators of TM on $\mathcal{D}^1$, $\mathcal{D}^{\lambda_1}$, $\ldots,\mathcal{D}^{\lambda_k}$ respectively.

(b) *If M is a θ-slant submanifold, then for $X\in T_p^1M$ we have*

$$4\mathrm{Ric}(X)\le n^2\|H\|^2+4(n-1)+(c-1)\left(3\cos^2\theta+n-2\right)\left(1-\eta(X)^2\right). \tag{4.12}$$

(c) *If M is an anti-invariant submanifold, then for $X\in T_p^1M$, we have*

$$4\mathrm{Ric}(X)\le n^2\|H\|^2+4(n-1)+(c-1)(n-2)\left(1-\eta(X)^2\right). \tag{4.13}$$

(d) *The equality cases of* (4.11), (4.12) *and* (4.13) *are satisfied by $X\in T_p^1M$ if and only if* (2.9) *is true. If $H(p)=0$, then $X\in T_p^1M$ satisfies the equality cases of* (4.11), (4.12) *and* (4.13) *if and only if $X\in\mathcal{N}_p$.*

Proof. Using (4.7) in the inequality (4.3) we get (4.11). Next, using (4.9) in the inequality (4.3) we get the inequality (4.12). Putting $\theta=\pi/2$ in (4.12) we get (4.13). Rest of the proof is straightforward. □

Next, we have

Theorem 4.10. *Let M be an n-dimensional submanifold of a Sasakian space form $\widetilde{M}(c)$ such that the structure vector field ξ is tangent to the submanifold M. Then, the following statements are true.*

(a) *For each unit vector $X\in\{\xi\}_p^\perp$ we have*

$$4\mathrm{Ric}(X)\le n^2\|H\|^2+(n-1)(c+3)+(3\|PX\|^2-1)(c-1). \tag{4.14}$$

(b) *The equality case of* (4.14) *is satisfied by the unit vector $X\in\{\xi\}_p^\perp$ if and only if* (2.9) *is true. If $H(p)=0$, then the unit vector $X\in\{\xi\}_p^\perp$ satisfies the equality case of* (4.14) *if and only if $X\in\mathcal{N}_p$.*

(c) *The equality case of* (4.14) *holds for all unit vectors $X\in\{\xi\}_p^\perp$ if and only if $\varphi(T_pM)\subset T_pM$ and p is a totally geodesic point.*

Proof. Put $\eta(X)=0$ in (4.3) to get (4.14). Rest of the proof is similar. Of course, one also uses the fact that $\sigma(\xi,\xi)=0$. □

The following is an immediate Corollary of Theorem 4.10.

Corollary 4.11. *Let M be an n-dimensional submanifold of a Sasakian space form $\widetilde{M}(c)$ tangent to ξ. Then, the following statements are true.*

(a) *If M is an almost semi-invariant submanifold, then for a unit vector $X\in\{\xi\}_p^\perp$ we have*

$$\begin{aligned}\mathrm{Ric}(X)\ \le\ & \frac{1}{4}\left\{n^2\|H\|^2+(n-1)(c+3)\right.\\ & \left.+\left(3\sum_{\lambda\in\{1,\lambda_1,\ldots,\lambda_k\}}\lambda^2\left\|U^\lambda X\right\|^2-1\right)(c-1)\right\},\end{aligned} \tag{4.15}$$

where U^1, $U^{\lambda_1}, \ldots, U^{\lambda_k}$ are orthogonal projection operators of TM on $\mathcal{D}^1$, $\mathcal{D}^{\lambda_1}, \ldots, \mathcal{D}^{\lambda_k}$ respectively.

(b) *The equality case of* (4.15) *is satisfied by a unit vector $X \in \{\xi\}_p^{\perp}$ if and only if* (2.9) *is true. If $H(p) = 0$, then a unit vector $X \in \{\xi\}_p^{\perp}$ satisfies the equality cases of* (4.15) *if and only if $X \in \mathcal{N}_p$.*

Proof. By using (4.7) in (4.14) we get (4.15). Rest of the proof is straightforward. □

Now, we prove the following

Theorem 4.12. *Let M be an n-dimensional submanifold of a Sasakian space form $\widetilde{M}(c)$ tangent to ξ, and let $X \in \{\xi\}_p^{\perp}$ be a unit vector. Then, the following statements are true.*

(a) *If M is a proper θ-slant submanifold, then*

$$\mathrm{Ric}(X) < \frac{1}{4}\left\{n^2\|H\|^2 + (n-1)(c+3) + \left(3\cos^2\theta - 1\right)(c-1)\right\}. \tag{4.16}$$

(b) *If M is anti-invariant, then*

$$\mathrm{Ric}(X) < \frac{1}{4}\{n^2\|H\|^2 + (n-1)(c+3) - (c-1)\}. \tag{4.17}$$

(c) *If M is invariant, then*

$$\mathrm{Ric}(X) \leq \frac{1}{4}\{(n-1)(c+3) + 2(c-1)\}. \tag{4.18}$$

Proof. By using (4.10) in (4.14), for a unit vector $X \in \{\xi\}_p^{\perp}$ we get

$$\mathrm{Ric}(X) \leq \frac{1}{4}\left\{n^2\|H\|^2 + (n-1)(c+3) + \left(3\cos^2\theta - 1\right)(c-1)\right\}. \tag{4.19}$$

Putting $\theta = \pi/2$ in the inequality (4.19), for a unit vector $X \in \{\xi\}_p^{\perp}$ we get

$$\mathrm{Ric}(X) \leq \frac{1}{4}\{n^2\|H\|^2 + (n-1)(c+3) - (c-1)\}. \tag{4.20}$$

(4.16). Putting $\theta = \pi/2$ in the inequality (4.19) we get (4.20). If possible, let equality case of (4.19) or (4.20) is satisfied by a unit vector $X \in \{\xi\}_p^{\perp}$. Then, it follows that $\sigma(X, \xi) = 0$, which in view of (4.2), gives $FX = 0$, a contradiction. Thus, (4.16) and (4.17) are proved. Using minimality condition and $\theta = 0$ in (4.19), we get (4.18). □

We also have the following

Corollary 4.13. (Theorem 3.6, [28]) *Let M be an n-dimensional semi-invariant submanifold in a Sasakian space form $\widetilde{M}(c)$. Then, the following statements are true.*

(a) *For each unit vector X belonging to the invariant distribution $\mathcal{D}^1$, we have*

$$\mathrm{Ric}(X) \leq \frac{1}{4}\left\{n^2\|H\|^2 + (n+1)\,c + 3n - 5\right\}. \tag{4.21}$$

(b) *For each unit vector X belonging to the anti-invariant distribution $\mathcal{D}^0$, we have*

$$\mathrm{Ric}(X) < \frac{1}{4}\left\{n^2\|H\|^2 + (n-2)c + 3n - 2\right\}. \tag{4.22}$$

(c) *If $H(p) = 0$, a unit vector X belonging to the invariant distribution $\mathcal{D}^1$ satisfies the equality case of* (4.21) *if and only if $X \in \mathcal{N}_p$.*

Remark 4.14. The inequality (4.14) is the corrected version of the inequality (2.10) of Theorem 2.2 in [23], namely

$$\mathrm{Ric}(X) \leq \frac{1}{4}\left\{n^2\|H\|^2 + (n-1)(c+3) + \frac{1}{2}(3\,\|PX\|^2 - 2)(c-1)\right\}.$$

The inequality (4.16) is the corrected version of the inequality (2.1) of Theorem 2.1 in [13], namely

$$\mathrm{Ric}(X) \leq \frac{1}{4}\left\{n^2\|H\|^2 + (n-1)(c+3) + \frac{1}{2}\left(3\cos^2\theta - 2\right)(c-1)\right\}.$$

The inequality (4.17) is the inequality (2.15) of Corollary 2.4 in [23] (or the inequality (2.16) of Corollary 2.5 in [13]). The inequality (4.18) is the corrected version of the inequality (2.14) of Corollary 2.3 in [23] (or the inequality (2.15) of Corollary 2.4 in [13]), namely

$$\mathrm{Ric}(X) \leq \frac{1}{4}\left\{(n-1)(c+3) + \frac{1}{2}(c-1)\right\}.$$

Remark 4.15. In view of the Statement **(a)** of Theorem 4.12, it follows that the following are not feasible:
(1) Theorem 2.2 and Theorem 2.3 in [24],
(2) the statements (ii) and (iii) of Corollary 2.4 in [23] and
(3) the statements (ii) and (iii) of Theorem 2.1 in [13].

Remark 4.16. The inequalities (4.21) and (4.22) are the corrected version of the inequalities (i) and (ii) of Corollary 2.5 in [23], namely

$$\mathrm{Ric}(X) \leq \frac{1}{4}\left\{n^2\|H\|^2 + (n-1)(c+3) + \frac{1}{2}(c-1)\right\},$$

and

$$\mathrm{Ric}(X) \leq \frac{1}{4}\left\{n^2\|H\|^2 + (n-2)c + 3n - 2\right\}.$$

5. Ricci curvature of C-totally real submanifolds

A submanifold M of an almost contact metric manifold is said to be *anti-invariant* ([34], [35]) if $\varphi(TM) \subseteq T^{\perp}M$. A submanifold M in a contact manifold is called an *integral submanifold* [3] if every tangent vector of M belongs to the contact distribution defined by $\eta = 0$. In this case we have $d\eta\,(X,Y) = 0$; $X, Y \in TM$. An integral submanifold M in a contact metric manifold is called a *C-totally real submanifold* [33]. Thus C-totally real submanifolds of a contact metric manifold are always anti-invariant. In particular, a C-totally real submanifold of a Sasakian manifold is anti-invariant.

Now, we have the following Theorem.

Theorem 5.1. *If M is an n-dimensional C-totally real submanifold of a Sasakian space form $\widetilde{M}(c)$, then the following statements are true.*

(a) *For $X \in T_p^1 M$, we have*

$$4\,\mathrm{Ric}(X) \leq n^2 \|H\|^2 + (n-1)(c+3), \tag{5.1}$$

(b) *The equality case of* (5.1) *is satisfied by $X \in T_p^1 M$ if and only if* (2.9) *is true. If $H(p) = 0$, $X \in T_p^1 M$ satisfies equality in* (5.1) *if and only if $X \in \mathcal{N}_p$.*

(c) *The equality case of* (5.1) *holds for all $X \in T_p^1 M$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Proof. If M is an n-dimensional C-totally real submanifold, then we have

$$4\,\widetilde{\mathrm{Ric}}_{(T_p M)}(X) = (n-1)(c+3), \quad X \in T_p^1 M. \tag{5.2}$$

Using (5.2) in (2.8), we find the inequality (5.1). Rest of the proof is straightforward. □

By polarization, from Theorem 5.1, we derive

Theorem 5.2. *Let M be an n-dimensional C-totally real submanifold of a Sasakian space form $\widetilde{M}(c)$. Then the Ricci tensor S satisfies*

$$S \leq \frac{1}{4}\left(n^2 \|H\|^2 + (n-1)(c+3)\right) g, \tag{5.3}$$

where g is the induced Riemannian metric on M. The equality case of (5.3) *is true if and only if either M is a totally geodesic submanifold or $n = 2$ and M is a totally umbilical submanifold.*

Remark 5.3. The inequality (5.1) is the same as the inequalities (i) of Theorem 1 in [21], (2.1) of Theorem 2.1 in [23] and (2.1) in Theorem 2.1 in [22]. The inequality (5.3) is same as the inequalities (9) in the Theorem 3.1 in [19] or [18], (2.9) in Theorem 2.2 in [22] and the inequality in Theorem 2 in [21]).

The maximum Ricci curvature function on a Riemannian manifold M, denoted $\widehat{\mathrm{Ric}}$, is defined as [10]

$$\widehat{\mathrm{Ric}}(p) = \max\left\{\mathrm{Ric}(X) \mid X \in T_p^1 M\right\}.$$

B.-Y. Chen ([10], Theorem 2) proved an inequality for maximum Ricci curvature $\widehat{\mathrm{Ric}}$ for Lagrangian submanifolds of complex space forms, and proved that in the equality case, the Lagrangian submanifolds must be minimal. Here, we prove the following

Theorem 5.4. *Let M be an n-dimensional C-totally real submanifold of a $(2n+1)$-dimensional Sasakian space form $\widetilde{M}(c)$. If a unit vector of T_pM satisfies the equality case of* (5.1)*, then $H(p) = 0$.*

Proof. Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM such that e_1 satisfies the equality case of (5.1). Then, $\{e_{n+1}, \ldots, e_{2n}, e_{2n+1} = \xi\}$ is an orthonormal basis of $T_p^\perp M$ such that $e_{n+j} = \varphi e_j$, $j \in \{1, \ldots, n\}$. It is known that [34] if M is a C-totally real submanifold of a Sasakian manifold, then $A_\xi = 0$ and $A_{\varphi X}Y = A_{\varphi Y}X$ for $X, Y \in TM$. Using these two facts alongwith (2.9), for any $Y = \sum_{j=1}^n a_j e_{n+j} + a\xi \in T_p^\perp M$, we have

$$\begin{aligned}
\langle \sigma(e_1, e_1), Y\rangle &= a_1 \langle \sigma(e_1, e_1), \varphi e_1\rangle \\
&\quad + \sum_{j=2}^{n} a_j \langle \sigma(e_1, e_1), \varphi e_j\rangle + a \langle \sigma(e_1, e_1), \xi\rangle \\
&= a_1 \left\langle \sum_{j=2}^{n} \sigma(e_j, e_j), \varphi e_1 \right\rangle + \sum_{j=2}^{n} a_j \langle \sigma(e_1, e_1), \varphi e_j\rangle + 0 \\
&= a_1 \sum_{j=2}^{n} \langle \sigma(e_1, e_j), \varphi e_j\rangle + \sum_{j=2}^{n} a_j \langle \sigma(e_1, e_j), \varphi e_1\rangle \\
&= 0 + 0 = 0.
\end{aligned}$$

Hence in view of (2.9), $H(p) = 0$. □

Consequently, we have the following (see also Theorem 4.1 of [19, 18] or Theorem 3.1 of [22]).

Theorem 5.5. *Let M be an n-dimensional C-totally real submanifold of a $(2n+1)$-dimensional Sasakian space form $\widetilde{M}(c)$, then*

$$\widehat{\mathrm{Ric}} \leq \frac{1}{4}\left\{n^2\|H\|^2 + (n-1)(c+3)\right\}. \tag{5.4}$$

If M satisfies the equality case of (5.4) *identically, then M is a minimal submanifold and*

$$\widehat{\mathrm{Ric}} = 4(n-1)(c+3). \tag{5.5}$$

Combining Corollary 3 of [21] and Corollary 3.2 of [22], we have

Theorem 5.6. *Let M be an n-dimensional C-totally real submanifold of a $(2n+1)$-dimensional Sasakian space form $\widetilde{M}(c)$. Then M satisfies the equality case of* (5.4) *identically if and only if* $\dim(\mathcal{N}_p) \geq 1$*. If* $\dim(\mathcal{N}_p)$ *is a positive constant, then $\mathcal{N}_p$ is integrable and its leaves are totally geodesic, that is, $\mathcal{N}_p$ is foliated by totally geodesic submanifolds.*

For minimal C-totally real submanifold of maximum dimension of a Sasakian space form $\widetilde{M}(c)$, the following are also known.

Theorem 5.7. (Theorem 4.2, [19, 18]) *Let M be an n-dimensional minimal C-totally real submanifold of a $(2n+1)$-dimensional Sasakian space form $\widetilde{M}(c)$. Then the following statements are true.*

(1) *The submanifold M satisfies the equality case of* (5.4) *if and only if* $\dim(\mathcal{N}_p) \geq 1$.

(2) *If* $\dim(\mathcal{N}_p)$ *is a positive constant d, then $\mathcal{N}_p$ is completely integral distribution and M is d-ruled, that is, for each $p \in M$, M contains a d-dimensional totally geodesic submanifold M' of $\widetilde{M}(c)$ passing through p.*

(3) *If the submanifold M is also ruled, then it satisfies the equality case of* (5.4) *identically if and only if, for each ruling M' in M, the normal bundle $T^{\perp}M$ restricted to M' is a parallel normal subbundle of the normal bundle $T^{\perp}M'$ along M'.*

6. Scalar curvature of submanifolds

We begin with the following

Proposition 6.1. *For an n-dimensional submanifold M of a Riemannian manifold at each point $p \in M$, we have*

$$\tau(p) \leq \frac{1}{2} n^2 \|H\|^2 + \widetilde{\tau}(T_p M) \tag{6.1}$$

with equality if and only if p is a totally geodesic point.

Proof. The proof follows from (2.7). □

Now, we state the following algebraic Lemma without proof.

Lemma 6.2. *If $a_1, \ldots, a_n$ are n $(n>1)$ real numbers then*

$$\frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2 \leq \sum_{i=1}^{n} a_i^2, \tag{6.2}$$

with equality holding if and only if $a_1 = a_2 = \cdots = a_n$.

Then using Lemma 6.2 we improve the inequality (6.1). In fact, we have

Theorem 6.3. *For an n-dimensional submanifold M in an m-dimensional Riemannian manifold, at each point $p \in M$, we have*

$$\tau(p) \leq \frac{n(n-1)}{2} \|H\|^2 + \widetilde{\tau}(T_p M) \tag{6.3}$$

with equality if and only if p is a totally umbilical point.

Proof. We choose an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_m\}$ at p such that $e_1, ..., e_n$ are tangential to M at p, $\{e_{n+1}, ..., e_m\}$ normal to M at p and e_{n+1} is parallel to the mean curvature vector $H(p)$ and $e_1, ..., e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then the shape operators take the forms

$$A_{e_{n+1}} = \operatorname{diag}\left(\sigma_{11}^{n+1}, \sigma_{22}^{n+1}, \ldots, \sigma_{nn}^{n+1}\right), \tag{6.4}$$

$$A_{e_r} = \left(\sigma_{ij}^r\right), \quad \operatorname{trace} A_{e_r} = \sum_{i=1}^{n} \sigma_{ii}^r = 0 \tag{6.5}$$

for all $i, j = 1, ..., n$ and $r = n+2, ..., m$; and from (2.7), we get

$$2\tau(p) = 2\widetilde{\tau}(T_pM) + n^2 \|H\|^2 - \sum_{i=1}^{n} \left(\sigma_{ii}^{n+1}\right)^2 - \sum_{r=n+2}^{m} \sum_{i,j=1}^{n} (\sigma_{ij}^r)^2. \tag{6.6}$$

Using Lemma 6.2, we get

$$n\|H\|^2 \leq \sum_{i=1}^{n} \left(\sigma_{ii}^{n+1}\right)^2. \tag{6.7}$$

In view of (6.6) and (6.7), we have

$$\tau(p) \leq \frac{n(n-1)}{2} \|H\|^2 + \widetilde{\tau}(T_pM) - \frac{1}{2} \sum_{r=n+2}^{m} \sum_{i,j=1}^{n} (\sigma_{ij}^r)^2, \tag{6.8}$$

which implies (6.3). If the equality in (6.3) holds, then from Lemma 6.2 and (6.8) it follows that

$$\sigma_{11}^{n+1} = \sigma_{22}^{n+1} = \cdots = \sigma_{nn}^{n+1} \quad \text{and} \quad A_{e_r} = 0, \quad r = n+2, ..., m.$$

Therefore, p is a totally umbilical point. The converse is straightforward. $\square$

For each integer k, $2 \leq k \leq n$, the Riemannian invariant θ_k on an n-dimensional Riemannian manifold M is defined by [11]

$$\theta_k(p) = \left(\frac{1}{k-1}\right) \inf_{\Pi_k, X} \operatorname{Ric}_{\Pi_k}(X), \quad p \in M, \tag{6.9}$$

where Π_k runs over all k-plane sections in T_pM and X runs over all unit vectors in Π_k. We denote by $\Pi_{i_1 \ldots i_k}$ the k-plane section spanned by $e_{i_1}, ..., e_{i_k}$. From (2.1) and (2.2), it follows that

$$\tau(L_{i_1 \ldots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \ldots, i_k\}} \operatorname{Ric}_{L_{i_1 \ldots i_k}}(e_i), \tag{6.10}$$

and

$$\tau(p) = \frac{1}{C_{k-2}^{n-2}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \tau(L_{i_1 \ldots i_k}). \tag{6.11}$$

Combining (6.9), (6.10) and (6.11), one obtains

$$\tau(p) \geq \frac{n(n-1)}{2} \theta_k(p). \tag{6.12}$$

In view of the equations (6.3) and (6.12), we have the following relationship between the Riemannian invariant θ_k and the squared mean curvature for submanifolds of a Riemannian manifold.

Theorem 6.4. *Let M be an n-dimensional submanifold of a Riemannian manifold. Then, for each integer k, $2 \leq k \leq n$, and every point $p \in M$, we have*

$$\theta_k(p) \leq \|H\|^2 + \widetilde{\tau}_N (T_pM), \tag{6.13}$$

where $\widetilde{\tau}_N (T_pM)$ is the normalized scalar curvature of the n-plane section T_pM in the ambient submanifold.

Now, we study scalar curvature of submanifolds of Sasakian space forms. In fact, we have the following

Theorem 6.5. *Let M be an n-dimensional submanifold of a Sasakian space form $\widetilde{M}(c)$, such that the structure vector field ξ is tangent to M. Then at each point $p \in M$, we have*

$$\tau(p) \leq \frac{n(n-1)}{2} \|H\|^2 + \frac{1}{8} \left\{ 3(c-1) \|P\|^2 + (n-1) ((n-2)c + 3n + 2) \right\}. \tag{6.14}$$

with equality if and only if p is a totally umbilical point.

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM. The squared norm of P at $p \in M$ is defined to be $\|P\|^2 = \sum_{i,j=1}^{n} \langle Pe_i, e_j \rangle^2$. Then, using $2\widetilde{\tau} (T_pM) = \sum_{i=1}^{n} \widetilde{\mathrm{Ric}}_{(T_pM)} (e_i)$ in (4.4), we get

$$\widetilde{\tau} (T_pM) = \frac{1}{8} \left\{ 3(c-1) \|P\|^2 + (n-1)((n-2)c + 3n + 2) \right\}. \tag{6.15}$$

Using (6.15) in (6.3) gives (6.14). □

In view of (6.15), the equation (2.7) becomes

$$2\tau(p) = n^2 \|H\|^2 - \|\sigma\|^2 + \frac{1}{4} \left\{ 3(c-1) \|P\|^2 + (n-1)((n-2)c + 3n + 2) \right\}. \tag{6.16}$$

Next, using (6.15) in (6.13) gives the following

Theorem 6.6. *Let M be an n-dimensional submanifold of a Sasakian space form $\widetilde{M}(c)$ such that $\xi \in TM$. Then, for each integer k, $2 \leq k \leq n$, and every point $p \in M$, we have*

$$\theta_k (p) \leq \|H\|^2 + \frac{3(c-1)}{4n(n-1)} \|P\|^2 + \frac{1}{4n} \{(n-2)c + 3n + 2\}. \tag{6.17}$$

Now, we give following two Corollaries.

Corollary 6.7. *Let M be an n-dimensional invariant submanifold of a Sasakian space form $\widetilde{M}(c)$. Then the following statements are true.*

(a) *At each point $p \in M$ it follows that*

$$\tau(p) \leq \frac{n-1}{8} \{3(c-1) + (n-2)c + 3n + 2\}. \tag{6.18}$$

with equality if and only if p is a totally umbilical point.

(b) *For each integer k, $2 \leq k \leq n$, and every point $p \in M$, we have*

$$\theta_k(p) \leq \frac{1}{4n}\{3(c-1)+(n-2)c+3n+2\}. \tag{6.19}$$

Proof. Using $\|P\|^2 = n-1$ and $\|H\|^2 = 0$ in (6.14) and (6.17) gives (6.18) and (6.19) respectively. □

Corollary 6.8. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\widetilde{M}(c)$ such that ξ is tangent to M. Then the following statements are true.*

(a) *At each point $p \in M$ it follows that*

$$\tau(p) \leq \frac{n(n-1)}{2}\|H\|^2 + \frac{1}{8}\{(n-2)c+3n+2\} \tag{6.20}$$

with equality if and only if p is a totally umbilical point.

(b) *For each integer k, $2 \leq k \leq n$, and every point $p \in M$, we have*

$$\theta_k(p) \leq \|H\|^2 + \frac{1}{4n}\{(n-2)c+3n+2\}. \tag{6.21}$$

Proof. Using $P = 0$ in (6.14) and (6.17) gives (6.20) and (6.21) respectively. □

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