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# Weak $\phi$-Ideals and Weak $c^{\#}$-Ideals

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Abstract. In 2021, Ciloglu and Towers introduced the notion of a weak c-ideal of a Lie algebra, and used it give some characterizations of solvable and supersolvable Lie algebras. In this paper, analogously, we introduce the notions of weak $\phi$-ideals and weak $c^{\#}$-ideals, and obtain some new characterizations of solvable and supersolvable Lie algebras by using these notions.

**Keywords:** Lie algebra, Solvable, Supersolvable, Weak $\phi$-ideal, Weak $c^{\#}$-ideal.

**2000 Mathematics subject classification:** 17B05, 17B20, 17B30.

## 1. Introduction

Throughout, all Lie algebras will be of finite dimension over a fixed field $F$. We denote the largest ideal of $L$ contained in all the maximal subalgebras of $L$, the Frattini ideal of $L$, by $\phi(L)$. For a subalgebra $H$ of $L$, the core of $H$ with respect to $L$, $H_L$, is the largest ideal of $L$ contained in $H$. Also vector spaces direct sum will be denoted by $\dotplus$. We say the factor algebra $A/B$ is a chief factor of $L$ if $B$ is an ideal of $L$ and $A/B$ is a minimal ideal of $L/B$. Let $H$ be a subalgebra of $L$. Then $H$ is a subideal of $L$, if there exists a chain of subalgebras of $L$, $H = H_0 < H_1 < \cdots < H_n = L$ such that $H_i$ is an ideal of $H_{i+1}$, for all $0 \leq i \leq n-1$. Also, a Lie algebra $L$ is called supersolvable, if there is a chain of ideals $\{0\} \subseteq L_1 \subseteq L_2 \subseteq \cdots \subseteq L_n = L$ such that $\dim L_i = i$. In 2009, Towers [9] introduced the notion of a $c$-ideal of a finite dimensional

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Lie algebra, as follows:
A subalgebra $H$ of $L$ is a $c$-ideal of $L$, if there is an ideal $K$ of $L$ such that $L = H + K$ and $H \cap K \leq H_L$. He obtained some properties of $c$-ideals and used them to give some characterizations of solvable and supersolvable Lie algebras. Also, he [10] defined the notions of covering and avoiding properties and CAP-subalgebras of Lie algebras, as follows:
Let $L$ be a Lie algebra and $H$ a subalgebra of $L$ and $A/B$ a chief factor of $L$. We say that
$(i)$ $H$ covers $A/B$, if $H + A = H + B$; and
$(ii)$ $H$ avoids $A/B$, if $H \cap A = H \cap B$.
A subalgebra $H$ of $L$ is called a $CAP$-subalgebra of $L$, if $H$ either covers or avoids every chief factor of $L$. These concepts have been explored in many articles. In particular, in [4] and [3], $\phi$-ideals and $c^{\#}$-ideals were introduced and considered by using these concepts. In 2021, Ciloglu Sahin and Towers [2] introduced the concept of weak $c$-ideals of a Lie algebra, which encouraged us to define new concepts weak $\phi$-ideals and weak $c^{\#}$-ideals of a Lie algebra.

**Definition 1.1.** $(i)$ The subalgebra $H$ of $L$ is called a weak $\phi$-ideal of $L$, if there is a subideal $K$ of $L$ such that $L = H + K$ and $H \cap K \leq \phi(H)$.
$(ii)$ The subalgebra $H$ of $L$ is called a weak $c^{\#}$-ideal of $L$, if there is a subideal $K$ of $L$ such that $L = H + K$ and $H \cap K$ is a $CAP$-subalgebra of $L$.

Clearly, if $H$ is a $\phi$-ideal of $L$, then $H$ is also a weak $\phi$-ideal of $L$. But the converse is not necessarily true. For example, let $L$ be three-dimensional Heisenberg algebra (with basis $a, b, c$ and products given by $[a, b] = c, [a, c] = [b, c] = 0$). In [4, Example 3.3], it was shown that if $M$ is a maximal subalgebra of $L$, then it is not a $\phi$-ideal of $L$. Now, we consider the maximal subalgebra $M = span\{a, c\}$. Clearly, $L = M + span\{b\}$ and $M \cap span\{b\} = \{0\} = \phi(M)$. We have $span\{b\} \lhd span\{b, c\} \lhd L$. It shows that $M$ is a weak $\phi$-ideal of $L$.
In this paper, by the above notions, we obtain some characterizations for solvability and supersolvability of finite dimensional Lie algebras.

## 2. Preliminary results

This section is devoted to some basic results which are needed in our investigation.

**Lemma 2.1.** *Let $L$ be a Lie algebra, $H$ and $K$ subalgebras of $L$ and $N$ an ideal of $L$.*
*$(i)$ If $H$ is a weak $\phi$-ideal of $L$ and $H \leq K \leq L$, then $H$ is a weak $\phi$-ideal of $K$.*
*$(ii)$ If $N \leq H$ and $H$ is a weak $\phi$-ideal of $L$, then $H/N$ is a weak $\phi$-ideal of $L/N$. Furthermore, if $N \leq \phi(H)$, then the converse is true.*
*$(iii)$ If $N \leq H$, then $H$ is a weak $c^{\#}$-ideal of $L$ if and only if $H/N$ is a weak $c^{\#}$-ideal of $L/N$.*

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*Proof.* $(i)$ Let $H$ be a weak $\phi$-ideal of $L$. Then there is a subideal $C$ of $L$ with $L = H + C$ and $H \cap C \leq \phi(H)$. So we have $K = K \cap (H + C) = H + (K \cap C)$. Since $C$ is a subideal of $L$, so there exists a chain of subalgebras of $L$, as follows:

$$C = C_0 < C_1 < ... < C_n = L,$$

such that for $0 \leq i \leq n-1$, $C_i \lhd C_{i+1}$. Thus there is a chain of subalgebras of $K$,

$$C \cap K = C_0 \cap K < C_1 \cap K < ... < C_n \cap K = K.$$

Clearly $C_i \cap K \lhd C_{i+1} \cap K$. Therefore $C \cap K$ is a subideal of $K$. Also $H \cap (C \cap K) = (H \cap C) \cap K \leq \phi(H) \cap K = \phi(H)$.
$(ii)$ Let $H$ be a weak $\phi$-ideal of $L$. Then there is a subideal $C$ of $L$ with $L = H + C$ and $H \cap C \leq \phi(H)$. Hence we have $L/N = (H + C)/N = H/N + (C + N)/N$ such that $(C + N)/N$ is a subideal of $L/N$ and by [8, Proposition 4.3],

$$H/N \cap (C + N)/N = ((H \cap C) + N)/N \leq (\phi(H) + N)/N \leq \phi(H/N).$$

Therefore $H/N$ is a weak $\phi$-ideal of $L/N$. If $N \leq \phi(H)$, then $\phi(H/N) = \phi(H)/N$, thanks to [8, Proposition 4.3] and so the converse clearly holds.
$(iii)$ We suppose that $H$ is a weak $c^{\#}$-ideal of $L$. Then there is a subideal $C$ of $L$ with $L = H + C$ and $H \cap C$ is a $CAP$-subalgebra of $L$. So $L/N = H/N + (C + N)/N$ and $H/N \cap (C + N)/N = ((H \cap C) + N)/N$. Now, since $H \cap C$ is a $CAP$-subalgebra of $L$, then $(H \cap C) + N$ is a $CAP$-subalgebra of $L$, by [10, Lemma 2.5], so $((H \cap C) + N)/N$ is a $CAP$-subalgerbra of $L/N$, thanks to [10, Lemma 2.1$(v)$]. Therefore $H/N$ is a weak $c^{\#}$-ideal of $L/N$.
Conversely, if $H/N$ is a weak $c^{\#}$-ideal of $L/N$, then there is a subideal $C/N$ of $L/N$ with $L/N = H/N + C/N = (H + C)/N$ and $H/N \cap C/N = (H \cap C)/N$ is a $CAP$-subalgebra of $L/N$. Therefore $L = H + C$ and $H \cap C$ is a $CAP$-subalgebra of $L$, by [10, Lemma 2.1$(v)$]. □

**Lemma 2.2.** *Let $L$ be a Lie algebra, $N$ an abelian minimal ideal of $L$ and $M$ a maximal subalgebra of $N$. If $M$ is a weak $\phi$-ideal or weak $c^{\#}$-ideal of $L$, then $dim N = 1$.*

*Proof.* Since $M$ is a weak $\phi$-ideal or a weak $c^{\#}$-ideal of $L$, there is an subideal $C$ of $L$ such that $L = M + C$ and $M \cap C \leq \phi(M)$ or $M \cap C$ is a $CAP$-subalgebra of $L$. Now by using [2, Lemma 4.2], we have $N \subseteq C$ or $[N, L] = 0$. In the former case, $M \subseteq N \subseteq C$ and so $M = M \cap C \leq \phi(M)$, which is impossible, or $M = M \cap C$ is a $CAP$-subalgebra of $L$ and so covers or avoids $N/\{0\}$. But $M$ cannot cover $N$. Therefore $M \cap N = M \cap \{0\}$ which concludes that $dim N = 1$.
If $[N, L] = 0$, then the result holds, trivially. □

**Theorem 2.3.** *If $K$ is a subalgebra of $L$ and $H$ is a weak $c^{\#}$-ideal of $L$ with $H \leq \phi(K)$, then $H$ is a $CAP$-subalgebra of $L$.*

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*Proof.* Since $H$ is a weak $c^{\#}$-ideal of $L$, there exists a subideal $N$ of $L$ such that $L = H+N$ and $H \cap N$ is a $CAP$-subalgebra of $L$. Also, $K = H+(K \cap N)$. Now, by using of [8, Lemma 2.1], we conclude that $K = K \cap N$ and so $H \subseteq K \subseteq N$. Hence $L = N$ and $H = H \cap N$ is a $CAP$-subalgebra of $L$. □

**Theorem 2.4.** *Let $L$ be a Lie algebra. Then the one-dimensional subalgebra $Fx$ of $L$ is a weak $\phi$-ideal of $L$ if and only if it is a $\phi$-ideal of $L$.*

*Proof.* Let $Fx$ be a weak $\phi$-ideal of $L$. Then there is a subideal $K$ of $L$ with $L = Fx + K$ and $Fx \cap K \leq \phi(Fx) = 0$. Thus $K$ has codimension one in $L$. Therefore $K$ is an ideal of $L$ and so $Fx$ is a $\phi$-ideal of $L$. The coverse follows from the fact that every $\phi$-ideal of $L$ is a weak $\phi$-ideal of $L$. □

In [2, Corollary 3.3], Ciloglu and Towers proved that over a field of characteristic zero, $L$ is solvable if and only if every maximal subalgebra of $L$ is a weak $c$-ideal. Is the same theorem true for weak $\phi$-ideals? In the following, we give an example of a solvable Lie algebra that has a maximal subalgebra that is not a weak $\phi$-ideal.

Example 2.5. Consider the real Lie algebra $A_{5,12} = span\{e_1, e_2, e_3, e_4, e_5\}$, with nonzero multiplications $[e_1, e_5] = e_1$, $[e_2, e_5] = e_1 + e_2$, $[e_3, e_5] = e_2 + e_3$, $[e_4, e_5] = e_3 + e_4$. (See [5]). If we put $M = span\{e_1, e_2, e_3, e_4\}$, then $M$ is a maximal subalgebra of $A_{5,12}$ and $M$ is not a weak $\phi$-ideal of $L$.

## 3. Main results

**Lemma 3.1.** *(*[9, Lemma 4.1]*) Let $L$ be a Lie algebra over any field $F$, let $A$ be an ideal of $L$, and let $U/A$ be a maximal nilpotent subalgebra of $L/A$. Then $U = C + A$, where $C$ is a maximal nilpotent subalgebra of $L$.*

**Proposition 3.2.** *Let $L$ be a Lie algebra, in which all maximal subalgebras of each maximal nilpotent subalgebra of $L$ are weak $\phi$-ideals or weak $c^{\#}$-ideals of $L$. If $A$ is a minimal ideal of $L$, then all maximal subalgebras of each maximal nilpotent subalgebra of $L/A$ are weak $\phi$-ideals or weak $c^{\#}$-ideals of $L/A$.*

*Proof.* We suppose that $U/A$ is a maximal nilpotent subalgebra of $L/A$. Then $U = C + A$, where $C$ is a maximal nilpotent subalgebra of $L$, by Lemma 3.1. If $B/A$ is a maximal subalgebra of $U/A$, then $B = B \cap (C + A) = (B \cap C) + A = D + A$, where $D$ is a maximal subalgebra of $C$ and $B \cap C \leq D$. Now
$(i)$ If $D$ is a weak $\phi$-ideal of $L$, then there exists a subideal $K$ of $L$ with $L = D + K$ and $D \cap K \leq \phi(D)$. If $A \leq K$, $L/A = (D + A)/A + K/A = B/A + K/A$ and $(D + A)/A \cap K/A = ((D \cap K) + A)/A \leq (\phi(D) + A)/A \cong \phi(B/A)$. So the result holds.
If $A \nleq K$, then $[L, A] = 0$ by [2, Lemma 4.2]. Hence $A \subseteq Z(L)$ and so $A \leq C$ and $B = D$. Consequently, we have $L = B + K$ and $B \cap K \leq \phi(B)$. Hence

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$L/A = B/A + (K + A)/A$ and

$$B/A \cap (K + A)/A = ((B \cap K) + A)/A \leq (\phi(B) + A)/A \leq \phi(B/A).$$

Therefore $B/A$ is a weak $\phi$-ideal of $L/A$.
$(ii)$ If $D$ is a weak $c^{\#}$-ideal of $L$, then there exists a subideal $K$ of $L$ with $L = D + K$ and $D \cap K$ is a $CAP$-subalgebra of $L$. Therefore $D \cap K$ covers or avoids $A/\{0\}$. If $D \cap K + A = D \cap K$, then $A \subseteq D \cap K \subseteq D$ and so $B = D$. It follows from Lemma 2.1$(iii)$ that $B/A$ is a weak $c^{\#}$-ideal of $L/A$ and so the result holds. If $D \cap K \cap A = 0$, then by [2, Lemma 4.2] we consider two cases:
1. $A \leq K$: In this case, $L/A = (D + A)/A + K/A = B/A + K/A$ and $(D + A)/A \cap K/A = ((D \cap K) + A)/A$. Since $D \cap K$ is a $CAP$-subalgebra of $L$ and $A$ is an ideal of $L$, then by [10, Lemma 2.5], $(D \cap K) + A$ is a $CAP$-subalgebra of $L$ and so $((D \cap K) + A)/A$ is a $CAP$-subalgebra of $L/A$, thanks to [10, Lemma 2.1$(v)$]. Thus $B/A$ is a weak $c^{\#}$-ideal of $L/A$.
2. $[A, L] = 0$: In this case, $A \subseteq Z(L)$. Consequently, $U = C + A$ is a nilpotent subalgebra of $L$ and so we must have $C + A = C$. Therefore $A \leq C$ and $B = D$ and $B/A$ is a weak $c^{\#}$-ideal of $L/A$, thanks to Lemma 2.1$(iii)$. □

Finally, we obtain a condition implying a Lie algebra $L$ to be supersolvable.

**Theorem 3.3.** *Let $L$ be a solvable Lie algebra, in which all maximal subalgebras of each maximal nilpotent subalgebra of $L$ are a weak $\phi$-ideal or a weak $c^{\#}$-ideal of $L$. Then $L$ is supersolvable.*

*Proof.* Let $L$ be a minimal counterexample and $A$ be a minimal ideal of $L$. Then by the previous proposition, $L/A$ satisfies the hypothesis of this theorem and so $L/A$ is supersolvable. It is sufficient to show that $dimA = 1$. If there is another ideal $I$ of $L$, then $A \cong (A + I)/I \leq L/I$ and so $dimA = 1$ and $L$ is supersolvable, a contradiction.
Therefore, we suppose that $A$ is a unique minimal ideal of $L$. If $A \leq \phi(L)$, then $L/\phi(L)$ is supersolvable and so $L$ is supersolvable by [1, Theorem 7], a contradiction. If $A \nleq \phi(L)$, then there is a maximal subalgebra $M$ of $L$ such that $L = A \dot{+} M$. Now, if $C$ is a maximal nilpotent subalgebra of $L$ with $A \leq C$, then we consider two cases:
1. $A = C$: In this case, $A$ is a maximal nilpotent subalgebra of $L$ and so by the assumption, every maximal subalgebra of $A$ is a weak $\phi$-ideal or weak $c^{\#}$-ideal of $L$. Hence $dimA = 1$, thanks to Lemma 2.2.
2. $A < C$: In this case, we have $C = A + (C \cap M)$. Now, let $B$ be a maximal subalgebra of $C$ that contains $C \cap M$. Then if $B$ is a weak $c^{\#}$-ideal of $L$, then there is a subideal $K$ of $L$ such that $L = B + K$ and $B \cap K$ is a $CAP$-subalgebra of $L$. Therefore $B \cap K$ covers or avoids $A/\{0\}$. If $(B \cap K) + A = B \cap K$, then $A \leq B \cap K \leq B$ and so $C \leq B$, a contradiction.
If $B \cap K \cap A = \{0\}$, then by noting to [2, Lemma 4.2], if $A \subseteq K$, then $B \cap A = \{0\}$ and $C = B \dot{+} A$. Thus $C/B \cong A$ and consequently $dimA = 1$

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and $L$ is supersolvable, a contradiction. If $[L, A] = 0$, then clearly $dimA = 1$, a contradiction.
Now, if $A < C$ and $B$ is a maximal subalgebra of $C$ that contains $A$ and if we suppose that $B$ is a weak $\phi$-ideal of $L$, then there is a subideal $K$ of $L$ with $L = B + K$ and $B \cap K \leq \phi(B)$. Since $A$ is a minimal ideal of $L$, so $A \subseteq K$ or $[L, A] = 0$, by [2, Lemma 4.2]. If $A \subseteq K$, then $A \subseteq B \cap K \leq \phi(B)$. Thus we have

$$B = B \cap L = B \cap (A + M) = A + (B \cap M) = B \cap M.$$

Therefore $A \subseteq B \subseteq M$, a contradiction. If $[L, A] = 0$, then $dimA = 1$ and $L$ is supersolvable, a contradiction. □

**Theorem 3.4.** *Let $L$ be a Lie algebra over a field of characteristic zero.*
*(i) If every maximal subalgebra of $N(L)$ is a weak $\phi$-ideal of $L$, then $L$ is solvable.*
*(ii) If every maximal subalgebra of $N(L)$ is a weak $c^{\#}$-ideal of $L$, then $L$ is supersolvable.*

*Proof.* $(i)$ We suppose that $L$ be a minimal counterexample. If $\phi(L) \neq 0$, then we can easily show that $L$ is solvable, a contradiction. If $\phi(L) = 0$, then $N(L) = A_1 \oplus A_2 \oplus ... \oplus A_n$ such that $A_i$'s are minimal abelian ideals of $L$, by [7, Theorem 3]. If $A_i^*$ is a maximal subalgebra of $A_i$, then $M = A_i^* + (A_1 \oplus ... \oplus \hat{A}_i \oplus ... \oplus A_n)$ is a maximal subalgebra of $N(L)$. So by the assumption, there is a subideal $C$ of $L$ such that $L = M + C$ and $M \cap C \leq \phi(M) = 0$. Therefore $M$ is a weak $c$-ideal of $L$. Consequently $L$ is solvable, by [2, Theorem 3.6].
$(ii)$ Again, we suppose that $L$ be a minimal counterexample. If $\phi(L) \neq 0$, then we can easily show that $L$ is supersolvable, a contradiction. Now, we suppose thet $\phi(L) = 0$. Then $N(L) = A_1 \oplus A_2 \oplus ... \oplus A_n$ such that $A_i$'s are minimal abelian ideals of $L$, by [7, Theorem 3]. If $A_i^*$ is a maximal subalgebra of $A_i$, then $M = A_i^* + (A_1 \oplus ... \oplus \hat{A}_i \oplus ... \oplus A_n)$ is a maximal subalgebra of $N(L)$. So by the assumption, there is a subideal $K$ of $L$ such that $L = M + K$ and $M \cap K$ is a $CAP$-subalgebra of $L$. Hence, $M \cap K$ covers or avoids $\frac{A_i}{\{0\}}$. If $M \cap K + A_i = M \cap K$, then $A_i \subseteq M \cap K \subseteq M$, a contradiction. If $M \cap K \cap A_i = 0$, then $A_i^* \cap K = 0$. Since $A_i$ is a minimal ideal of $L$, either $A_i \subseteq K$ or $[L, A_i] = 0$, by [2, Lemma 4.2]. If $A_i \subseteq K$, then $A_i^* = 0$ and so $\dim A_i = 1$. But, if $[L, A_i] = 0$, then $[L, A_i^*] = 0$. Thus, $A_i^*$ is an ideal of $L$ and so $A_i^* = 0$. Therefore, $\dim A_i = 1$.
Now, we put $K_i = A_1 \oplus A_2 \oplus \cdots \oplus A_i$, $i = 1, \cdots, n$. It follows that $N(L)$ has a series $0 \leq K_1 \leq \cdots \leq K_n = N(L)$ such that $\dim(K_i/K_{i-1}) = 1$, for $i = 1, \cdots, n$. Therefore, $L$ is supersolvable, by [6, Corollary 2.9], a contradiction. □

Finally, we bring an example that shows that the converse of the first part of the above theorem is not valid.

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EXAMPLE 3.5. Consider the Lie algebra $L = A_{5,7}^{a,b,c} = span\{e_1, e_2, e_3, e_4, e_5\}$, with nonzero multiplications $[e_1, e_5] = e_1$, $[e_2, e_5] = ae_2$, $[e_3, e_5] = be_3$ and $[e_4, e_5] = ce_4$, where $abc \neq 0$, $-1 \leq c \leq b \leq a \leq 1$ (See [5]). There is a maximal subalgebra $M = span\{e_1, e_2, e_3\}$ of $N(L) = span\{e_1, e_2, e_3, e_4\}$ such that $M$ is not a weak $\phi$-ideal, while $L$ is a solvable Lie algebra.

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