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NORMED HYPERVECTOR SPACES

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Abstract. The main purpose of this paper is to study normed hypervector spaces. We generalize some definitions such as basis, convexity, operator norm, closed set, Cauchy sequences, and continuity in such spaces and prove some theorems about them.

Keywords and Phrases: Norm; Linear form; Hypervector space; Basis.

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1. Introduction

Let $P(X)$ be the power set of a set X, $P^\star(X) = P(X)\setminus\{\varnothing\}$, and K a field. A *hypervector space* over K that is defined in [7], is a quadruplet $(X, +, \circ, K)$ such that $(X, +)$ is an abelian group and

$$\circ : K \times X \longrightarrow P^\star(X)$$

is a mapping that for all a, $b \in K$ and x, $y \in X$ the following properties holds:

(i) $(a + b) \circ x \subseteq (a \circ x) + (b \circ x)$,
(ii) $a \circ (x + y) \subseteq (a \circ x) + (a \circ y)$,
(iii) $a \circ (b \circ x) = (ab) \circ x$, where $a \circ (b \circ x) = \{\, a \circ y \;:\; y \in b \circ x \,\}$,
(iv) $(-a) \circ x = a \circ (-x)$
(v) $x \in 1 \circ x$.

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A hypervector space is called *strongly left distributive* (respectively, *strongly right distributive*), if equality holds in (i) (respectively, in (ii)) and is called *good*, if for every $\lambda \in K$, $\lambda \circ 0 = \{0\}$.

Note that every vector space is a hypervector space that is strongly left and right distributive and specially, every field is a hypervector space over itself.

A non-empty subset H of a hypervector space X over a field K is called a *subspace* of X if the following holds:

(i) $H - H \subseteq H$,
(ii) $a \circ H \subseteq H$, for every $a \in K$,

where $H - H = \{a - b \, : \, a, b \in H\}$. Let $(X, +, \circ, K)$ be a hypervector space. Suppose that for every $a \in K$, $|a|$ denoted the valuation of a in K. A *pseudonorm* on X that is defined in [8], is a mapping

$$|| \, . \, || : X \longrightarrow \mathbb{R}$$

that for all $a \in K$ and $x, y \in X$ has the following properties:

(i) $||0|| = 0$,
(ii) $||x + y|| \leqslant ||x|| + ||y||$,
(iii) $\sup ||a \circ x|| = |a| \, ||x||$.

A pseudonorm on X is called a *norm*, if:

$$||x|| = 0 \quad \Longleftrightarrow \quad x = 0.$$

Let $(X, +, \circ, K)$ and $(Y, +', \circ', K)$ be two hypervector spaces. As defined in [7], a *strong homomorphism* between X and Y is a mapping

$$f : X \longrightarrow Y$$

such that for all $a \in K$ and $x, y \in X$ the following hold:

(i) $f(x + y) = f(x) +' f(y)$,
(ii) $f(a \circ x) = a \circ' f(x)$.

A *linear form* is a strong homomorphism $f : X \longrightarrow K$. A strong homomorphism is called *good*, if $\ker f$ is a subspace of X. Clearly, every linear form is good.

Theorem 1.1. [7, Theorem 2]. *Let X and Y be two hypervector spaces and f a strong homomorphism between them. Then* $\ker f$ *is a subspace of X if and only if Y is a good hypervector space.*

This shows that the kernel of every linear form is a subspace of X.

For a field K and a natural number n, the set of all $n \times 1$ matrices over K is denoted by K^n, and for every $X \in K^n$, we denote the transpose of X by X^t.

The concepts of hyperstructures were introduced by a lot of mathematicians in many branches of mathematics such as algebra, geometry and analysis (see [4-6]). Most of them tried to generalize definitions and proved some famous theorems by replacing new definitions by classical ones. In this paper we consider normed hypervector spaces, and generalize some definitions such as basis, dimension, convexity, operator norm, closed set, Cauchy sequences, and continuity in such spaces and prove some important theorems about hypervector spaces.

2. Basis for Hypervector Spaces

The basis of a vector space has a very important role in linear algebra. We are interested to define a basis for a hypervector space.

Definition 2.1. *Let $(X, +, \circ, K)$ be a strongly left distributive hypervector space. A subset $A = \{x_\lambda\}_{\lambda \in \Lambda}$ of X is said to be an independent set if for every $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in K$, $x_1, \dots, x_n \in X$, and $x \in X$, the following hold:*

(i) $0 \circ x \subseteq \lambda_1 \circ x_1 + \cdots + \lambda_n \circ x_n$ *implies that* $\lambda_1 = \cdots = \lambda_n = 0$,
(ii) $\lambda_1 \circ x_1 + \cdots + \lambda_n \circ x_n \subseteq 0 \circ x$ *implies that* $\lambda_1 = \cdots = \lambda_n = 0$.

An independent subset A of X is called a *basis* if for every $x \in X$, there are $n \in \mathbb{N}$, non-zero elements $\lambda_1, \dots, \lambda_n \in K$, and $x_1, \dots, x_n \in A$ such that $1 \circ x \subseteq \lambda_1 \circ x_1 + \cdots + \lambda_n \circ x_n$. The hypervector space is said *finite dimensional* if it has a finite basis.

Remark 2.2. *Since X is a strongly left distributive hypervector space, it is easily seen that for every $x \in X$, this representation is unique.*

Definition 2.3. *Let $(X, +, \circ, K)$ and $(Y, +', \circ', K)$ be two hypervector spaces. A strong homomorphism $f : X \longrightarrow Y$ is called one to one, if for every x, $y \in X$, $x \neq y$ implies that $f(x) \neq f(y)$.*

Theorem 2.4. *Let $(X, +, \circ, K)$, $(Y, +', \circ', K)$ be two strongly left distributive hypervector spaces and $f : X \longrightarrow Y$ a strong homomorphism. If f is one to one and onto, and $B = \{x_\alpha\}_{\alpha \in A}$ is a basis for X, then $B' = \{f(x_\alpha)\}_{\alpha \in A}$ is a basis for Y.*

Proof. Let $\lambda_1, \dots, \lambda_n \in K$, $f(x_1), \dots, f(x_n) \in B'$. Let $y \in Y$ be such that $0 \circ' y \subseteq \lambda_1 \circ' f(x_1) + \cdots + \lambda_n \circ' f(x_n)$. Since f is onto, there is $x \in X$ such that $f(x) = y$. So we have

$$f(0 \circ x) = 0 \circ' f(x) = 0 \circ' y \subseteq f(\lambda_1 \circ x_1) + \cdots + f(\lambda_n \circ x_n) = f(\lambda_1 \circ x_1 + \cdots + \lambda_n \circ x_n).$$

Since f is one to one, we have $0 \circ x \subseteq \lambda_1 \circ x_1 + \cdots + \lambda_n \circ x_n$. Hence $\lambda_1 = \cdots = \lambda_n = 0$. Similarly, it is shown that if $\lambda_1 \circ' f(x_1) + \cdots + \lambda_n \circ' f(x_n) \subseteq 0 \circ' y$, then $\lambda_1 = \cdots = \lambda_n = 0$. So B' is an independent set.

On the other hand B is a basis for X, there are non-zero elements $\lambda_1, \dots, \lambda_n \in K$ and $x_1, \dots, x_n \in B$ such that $1 \circ x \subseteq \lambda_1 \circ x_1 + \cdots + \lambda_n \circ x_n$. So

$$y \in 1 \circ y = 1 \circ f(x) = f(1 \circ x) \subseteq \lambda_1 \circ' f(x_1) + \cdots + \lambda_n \circ' f(x_n).$$

Therefore B' is a basis for Y and the proof is complete. $\square$

Theorem 2.5. *Let $(X,+,\circ,K)$ be a strongly left distributive hypervector space and $\{x_1,\ldots,x_n\}$ a basis for X. If $\{y_1,\ldots,y_m\}$ is an independent set in X, then $m \leqslant n$.*

Proof. By contradiction, suppose that $m > n$. For every j, $1 \leqslant j \leqslant m$, there are c_{ij}, $1 \leqslant i \leqslant n$ such that $1 \circ y_j \subseteq c_{1j} \circ x_1 + \cdots + c_{nj} \circ x_n$. If $C = (c_{ij})$, then C is an $n \times m$ matrix with $m > n$ and so the equation $CX = 0$ has a non-zero solution $(\lambda_1, \ldots, \lambda_m)^t \in K^m$. Therefore

$$\lambda_1 \circ y_1 + \cdots + \lambda_m \circ y_m \subseteq \lambda_1 \circ (1 \circ y_1) + \cdots + \lambda_m \circ (1 \circ y_m)$$
$$\subseteq \lambda_1 \circ (c_{11} \circ x_1 + \cdots + c_{n1} \circ x_n) + \cdots + \lambda_m \circ (c_{1m} \circ x_1 + \cdots + c_{nm} \circ x_n)$$
$$= (\lambda_1 c_{11} + \cdots + \lambda_m c_{1m}) \circ x_1 + \cdots + (\lambda_1 c_{n1} + \cdots + \lambda_m c_{nm}) \circ x_n$$
$$= 0 \circ x_1 + \cdots + 0 \circ x_n = 0 \circ (x_1 + \cdots x_n).$$

Since $\{y_1,\ldots,y_m\}$ is an independent set, then $\lambda_1 = \cdots = \lambda_m = 0$, a contradiction. So $m \leqslant n$ and the proof is complete. $\square$

Corollary 2.6. *If $(X,+,\circ,K)$ is a finite dimensional strongly left distributive hypervector space, then every two bases of X have the same cardinality.*

Definition 2.7. *Let X be a finite dimensional strongly left distributive hypervector space, the dimension of X is the cardinality of its bases and is denoted by* $\dim X$.

Corollary 2.8. *Let $(X,+,\circ,K)$ be a finite dimensional strongly left distributive hypervector space and $\dim X = n$. Then every subset of X with at least $n+1$ elements is not an independent set.*

3. Linear Forms

We begin by the following definition.

Definition 3.1. *Let $(X,+,\circ,K)$ be a hypervector space. A proper subspace M of X is called a hyperplane if for every $x \in X$, there is $\lambda \in K$ such that $x \in \lambda \circ x_0 + M$, for every $x_0 \in X$ that $1 \circ x_0 \cap M = \varnothing$.*

Lemma 3.2. *Let $(X,+,\circ,K)$ be a strongly left distributive hypervector space, M a hyperplane of X, and $x_0 \in X$ such that $1 \circ x_0 \cap M = \varnothing$. Then for every $x \in X$, there is a unique $\lambda \in K$ such that $x \in \lambda \circ x_0 + M$.*

Proof. Suppose there are λ_1, $\lambda_2 \in K$ such that $\lambda_1 \neq \lambda_2$ and $x \in \lambda_1 \circ x_0 + M$ and $x \in \lambda_2 \circ x_0 + M$. So there are a, $b \in M$, $z \in \lambda_1 \circ x_0$, and $w \in \lambda_2 \circ x_0$ such that $x = z + a$ and $x = w + b$. Therefore $z - w = b - a$. So

$$z - w \in \lambda_1 \circ x_0 - \lambda_2 \circ x_0 = (\lambda_1 - \lambda_2) \circ x_0.$$

Since M is a subspace, then $(\lambda_1 - \lambda_2)^{-1}(b - a) \subseteq M$. Hence

$$(\lambda_1 - \lambda_2)^{-1} \circ (z - w) = (\lambda_1 - \lambda_2)^{-1}(b - a) \subseteq M \cap 1 \circ x_0,$$

a contradiction. This completes the proof. $\square$

Theorem 3.3. *Let $(X, +, \circ, K)$ be a hypervector space. We have the following:*

(i) *If $f : X \longrightarrow K$ is a linear form, then $\ker f$ is a hyperplane.*

(ii) *If X is a strongly left distributive hypervector space and M a hyperplane of X, then there is a linear form, f, over X such that $M \subseteq \ker f$.*

Proof. (i) Let $f : X \longrightarrow K$ be a linear form. By Theorem , $\ker f$ is a subspace of X. Suppose x_0 is in X such that $1 \circ x_0 \cap \ker f \neq \varnothing$ and $x \in X$. Clearly, $f(x_0) \neq 0$. It suffices to prove that $x \in \frac{f(x)}{f(x_0)} \circ x_0 + \ker f$. Let $y \in x - \frac{f(x)}{f(x_0)} \circ x_0$, then

$$f(y) \in f(x - \frac{f(x)}{f(x_0)} \circ x_0) = \left\{ f(x-z) \, : \, z \in \frac{f(x)}{f(x_0)} \circ x_0 \right\}$$

$$= f(x) - \left\{ f(z) \, : \, z \in \frac{f(x)}{f(x_0)} \circ x_0 \right\} = f(x) - f\left(\frac{f(x)}{f(x_0)} \circ x_0 \right)$$

$$= f(x) - \frac{f(x)}{f(x_0)} f(x_0) = \{0\}.$$

So $y \in \ker f$. Since $x - y \in \frac{f(x)}{f(x_0)} \circ x_0$, then $x = (x-y) + y \in \frac{f(x)}{f(x_0)} \circ x_0 + \ker f$ and $\ker f$ is a hyperplane.

(ii) Let X be a strongly left distributive hypervector space and M a hyperplane. Suppose x_0 is in X such that $1 \circ x_0 \cap M \neq \varnothing$. By Lemma , for every $x \in X$, there is a unique $\lambda_x \in K$ such that $x \in \lambda_x \circ x_0 + M$. Define $f : X \longrightarrow K$ by $f(x) = \lambda_x$, then f is a linear form. Because suppose $x, y \in X$ and $\alpha \in K$. Then $x \in \lambda_x \circ x_0 + M$ and $y \in \lambda_y \circ x_0 + M$. So we have

$$x + y \in \lambda_x \circ x_0 + M + \lambda_y \circ x_0 + M \subseteq (\lambda_x + \lambda_y) \circ x_0 + M.$$

Therefore $f(x+y) = \lambda_x + \lambda_y = f(x) + f(y)$. Also,

$$\alpha \circ x \subseteq \alpha \circ (\lambda_x \circ x_0 + M) = (\alpha\lambda_x) \circ x_0 + \alpha \circ M \subseteq (\alpha\lambda_x) \circ x_0 + M.$$

Hence $f(\alpha \circ x) = \alpha\lambda_x = \alpha f(x)$, and f is a linear form.

Now, let $x \in M$. Since X is strongly left distributive hypervector space, then

$$0 \in 0 \circ x_0 - 0 \circ x_0 \subseteq 0 \circ x_0.$$

So $x \in 0 \circ x_0 + M$. It means that $f(x) = 0$ and the proof is complete. □

Note that if f is a linear form and $\lambda \in K$, then λf, is also a linear form, where $(\lambda f)(x) = \lambda f(x)$, for every $x \in X$.

Theorem 3.4. *Let $(X, +, \circ, K)$ be a hypervector space and f_1, f_2 two linear forms over X such that $\ker f_1 = \ker f_2$. Then $f_2 = kf_1$ for some $k \in K$.*

Proof. It is trivial if $f_1 = 0$, otherwise let $x_0 \in X$ be such that $f_1(x_0) \neq 0$. Therefore $f_2(x_0) \neq 0$, too. Put $k = \frac{f_2(x_0)}{f_1(x_0)}$ and let $x \in X$. It is enough to prove that $f_2(x) = kf_1(x)$. Let $\alpha = \frac{f_1(x)}{f_1(x_0)}$, then $f_1(x) = \alpha f_1(x_0) = f_1(\alpha \circ x_0)$. So for every $y \in \alpha \circ x_0$, we have $f_1(x-y) = 0$. Hence $x - \alpha \circ x_0 \subseteq \ker f_1 = \ker f_2$, and it means that

$$f_2(x) = f_2(\alpha \circ x_0) = \alpha f_2(x_0) = \alpha k f_1(x_0) = kf_1(x).$$

This completes the proof. □

4. Continuous Strong Homomorphisms

In this section we define bounded strong homomorphisms in order to construct a norm on them

Definition 4.1. *Let $(X, +, \circ, ||\ ||, K)$ and $(Y, +', \circ', ||\ ||', K)$ be two normed hypervector spaces. A strong homomorphism $f : X \longrightarrow Y$ is called bounded if there exists $M \geqslant 0$ such that $||f(x)||' \leqslant M||x||$, for every $x \in X$.*

Theorem 4.2. *Let K be a field, $(X, +_1, \circ_1, ||\ .\ ||_1, K)$, $(Y, +_2, \circ_2, ||\ .\ ||_2, K)$ two normed hypervector spaces, and $f : X \longrightarrow Y$ a strong homomorphism. Then the following are equivalent:*

(i) *f is continuous,*
(ii) *f is continuous at $x_0 \in X$,*
(iii) *f is bounded.*

Proof. Clearly, (i) implies (ii). To prove (ii) implies (iii) let f be continuous at $x_0 \in X$. Then for $\epsilon = 1$, there is $\delta > 0$ such that $||f(z)f(x_0)||_2 < 1$, whenever $||z - x_0||_1 < \delta$. Now, let $y \in X$ and put $A = x_0 + (\frac{\delta}{||y||_1} \circ y)$. Hence if $w \in A$, then $w = x_0 + v$, for some $v \in \frac{\delta}{||y||_1} \circ y$. Therefore

$$||w - x_0||_1 = ||x_0 + v - x_0||_1 = ||v||_1 \leqslant \sup ||\frac{\delta}{||y||_1} \circ y||_1 = \frac{\delta}{||y||_1}||y||_1 = \delta,$$

and so $||f(w) - f(x_0)||_2 < 1$. It means that $||f(v)||_2 = ||f(w - x_0)||_2 < 1$. Hence $||f(v)||_2 < 1$, for every $v \in (\frac{\delta}{||y||_1} \circ y)$. So $\sup ||f(\frac{\delta}{||y||_1} \circ y)||_2 < 1$ and it implies that

$$(\frac{\delta}{||y||_1})||f(y)||_2 = \sup ||\frac{\delta}{||y||_1} \circ' f(y)||_2 < 1.$$

Therefore $||f(y)||_2 < \frac{1}{\delta}||y||_1$. Since $\frac{1}{\delta} > 0$, we conclude that f is bounded.

To show that (iii) implies (i), let $x_0 \in X$ and $\epsilon > 0$. Since f is bounded, there is $M > 0$ such that $||f(x)||_2 \leqslant M||x||_1$, for every $x \in X$. Suppose that $\delta < \frac{\epsilon}{M}$. For every $y \in X$, $||y - x_0||_1 < \delta$ implies that

$$||f(y) - f(x_0)||_2 = ||f(y - x)||_2 \leqslant M||y - x_0||_1 < M\frac{\epsilon}{M} = \epsilon.$$

So f is continuous at x_0, and the proof is completed. □

Definition 4.3. *Let $(X, +_1, \circ_1, ||\ .\ ||_1, K)$, $(Y, +_2, \circ_2, ||\ .\ ||_2, K)$ be two normed hypervector spaces. For a bounded strong homomorphism $f : X \longrightarrow Y$, we define the norm of f by*

$$||f||^\star = \sup \left\{ \sup ||f(\frac{1}{||x||} \circ x)||' \ :\ 0 \neq x \in X \right\}.$$

Remark 4.4. *Note that*

$$\begin{aligned} ||f||^{\star} &= \sup\left\{ \sup ||\frac{1}{||x||_1} \circ_2 f(x)||_2 \ : \ 0 \neq x \in X \right\} \\ &= \sup\left\{ \frac{||f(x)||_2}{||x||_1} \ : \ 0 \neq x \in X \right\} \\ &= \inf\{ M \ : \ ||f(x)||_2 \leqslant M||x||_1 \textit{ for all } x \in X\}. \end{aligned}$$

Definition 4.5. *For a normed hypervector space $X = (X, +, \circ, ||\ .\ ||, K)$, if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X, then*

$$\lim_{n\to\infty} x_n = x \quad \Longleftrightarrow \quad \lim_{n\to\infty} ||x_n - x|| = 0.$$

In [8], it is proved that if X is a normed hypervector space, then for every $x \in X$, we have

(i) $|| - x|| = ||x||$,
(ii) $||x|| \geqslant 0$.

Remark 4.6. *Since for every pair x, $y \in X$, we have $||x + y|| \leqslant ||x|| + ||y||$ and $|| - x|| = ||x||$, it is easily checked that $|\ ||x|| - ||y||\ | \leqslant ||x - y||$. It shows that if $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} ||x_n|| = ||x||$.*

Lemma 4.7. *Let $(X, +, \circ, ||\ .\ ||, K)$ be a normed hypervector space, $\lambda \in K$, and $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ two sequences in X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then the following hold:*

i) $\lim_{n\to\infty}(x_n + y_n) = (x + y)$,
ii) *If $\lim_{n\to\infty} x_n = 0$, then $\lim_{n\to\infty} \lambda \circ x_n = 0$ (in the sense that for every sequence $\{y_n\}$ that $y_n \in \lambda \circ x_n$, $\lim_{n\to\infty} y_n = 0$).*

Proof. i) This can be deduced simply from the inequality:

$$||\,(x_n + y_n) - (x + y)\,|| \leqslant ||x_n - x|| + ||y_n - y||.$$

ii) It is easily proved, since $\sup ||\lambda \circ x_n|| = |\lambda|\,||x_n||$. The proof is complete. □

Definition 4.8. *Let $(X, +, \circ, ||\ .\ ||, K)$ be a normed hypervector space. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $||x_n - x_m|| < \epsilon$, for every m, $n \geqslant N$.*

Theorem 4.9. *Let $(X, +_1, \circ_1, ||\ .\ ||_1, K)$, $(Y, +_2, \circ_2, ||\ .\ ||_2, K)$ be two normed hypervector spaces and $f : X \longrightarrow Y$ a strong homomorphism. Then the following are equivalent:*

(i) *f is continuous,*
(ii) *f sends Cauchy sequences in X to Cauchy sequences in Y,*

(iii) *f sends convergent sequences in X to convergent sequences in Y.*

Proof. First, we show that (i) implies (ii). Suppose $\{x_n\}$ is a Cauchy sequence in X. Then

$$||f(x_n) - f(x_m)||_2 = ||f(x_n - x_m)||_2 \leqslant ||f||^\star ||x_n - x_m||_1,$$

for m, $n \in \mathbb{N}$. Since $||f||^\star < \infty$, then $\{f(x_n)\}$ is a Cauchy sequence in Y.

Next, to prove (i) implies (iii), let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = x$. Then

$$||f(x_n) - f(x)||_2 = ||f(x_n - x)||_2 \leqslant ||f||^\star ||x_n - x||_1.$$

Since $||f||^\star < \infty$, it means that $\lim_{n\to\infty} f(x_n) = f(x)$.

Last, we show that (ii) implies (i) and (iii) implies (i). So assume that f is not continuous and therefore by Theorem , it is not bounded and for every $n \in \mathbb{N}$, there is $x_n \in X$ such that

$$\sup ||f(\frac{1}{||x_n||_1} \circ x_n)||_2 = \sup ||\frac{1}{||x_n||_1} \circ' f(x_n)||_2 > n.$$

Hence there is $y_n \in \frac{1}{||x_n||_1} \circ x_n$ such that $||f(y_n)||_2 > n$, for every $n \in \mathbb{N}$. Also

$$||y_n||_1 \leqslant \sup ||\frac{1}{||x_n||_1} \circ x_n||_1 = \frac{1}{||x_n||_1} ||x_n||_1 = 1.$$

If $W_n = \frac{1}{\sqrt{n}} \circ y_n$, then

$$\sup ||f(W_n)||_2 = \sup ||f(\frac{1}{\sqrt{n}} \circ y_n)||_2 =$$

$$\sup ||\frac{1}{\sqrt{n}} \circ' f(y_n)||_2 = \frac{1}{\sqrt{n}} ||f(y_n)||_2 > \frac{n}{\sqrt{n}} = \sqrt{n}.$$

So there is $w_n \in W_n$ such that $||f(w_n)||_2 > \sqrt{n}$. Clearly, $\{f(w_n)\}$ is not convergent. On the other hand

$$\sup ||W_n||_1 = \sup ||\frac{1}{\sqrt{n}} \circ y_n||_1 \leqslant \frac{1}{\sqrt{n}} ||y_n||_1 \leqslant \frac{1}{\sqrt{n}},$$

and therefore $||w_n||_1 \leqslant \frac{1}{\sqrt{n}}$, for every $n \in \mathbb{N}$. So $\{w_n\}$ is a Cauchy sequence that $\lim_{n\to\infty} w_n = 0$, and the proof is complete. □

Definition 4.10. *Let $(X, +, \circ, || \ ||, K)$ be a normed hypervector space. The subset $A \subseteq X$ is called closed if for every sequence $\{x_n\}$ in X, $\lim_{n\to\infty} x_n = x$ implies $x \in X$.*

Theorem 4.11. *Let $(X, +, \circ, || \ ||, K)$ be a normed hypervector spaces and $f : X \longrightarrow K$ a linear form. Then $\ker f$ is a closed subspace of X if and only if f is continuous.*

Proof. First, suppose that $\ker f$ is a closed subspace of X. By contradiction, assume that f is not continuous. So it is not bounded and there is $x_n \in X$ such that

$$\sup |f(\frac{1}{||x_n||} \circ x_n)| = \frac{|f(x_n)|}{||x_n||} > n,$$

for every $n \in \mathbb{N}$. Therefore there is $y_n \in \frac{1}{||x_n||} \circ x_n$ such that $|f(y_n)| > n$. Clearly, for every $n \in \mathbb{N}$,

$$||y_n|| \leqslant \sup ||\frac{1}{||x_n||} \circ x_n|| = \frac{1}{||x_n||}||x_n|| = 1.$$

Now, let $Z_n = y_1 - \frac{f(y_1)}{f(y_n)} \circ y_n$. If $z \in Z_n$, then there is $w \in \frac{f(y_1)}{f(y_n)} \circ y_n$ such that $z = y_1 - w$ and

$$f(z) = f(y_1) - f(w) \in f(y_1) - f(\frac{f(y_1)}{f(y_n)} \circ y_n) = f(y_1) - \frac{f(y_1)}{f(y_n)} f(y_n) = \{0\}.$$

For every $n \in \mathbb{N}$, suppose that $z_n \in Z_n$. We have

$$||z_n - y_1|| \leqslant \sup ||y_1 - \frac{f(y_1)}{f(y_n)} \circ y_n - y_1|| =$$

$$\sup ||\frac{f(y_1)}{f(y_n)} \circ y_n|| = |\frac{f(y_1)}{f(y_n)}|||y_n|| < \frac{|f(y_1)|}{n}.$$

So $\lim_{n\to\infty} z_n = y_1$. But $y_1 \notin \ker f$, a contradiction. So f is continuous.

Last, suppose that f is a continuous linear form, and $\{x_n\}$ a sequence in $\ker f$. So $f(x_n) = 0$, for every $n \in \mathbb{N}$. Now, for an arbitrary $\epsilon > 0$, there is $n \in \mathbb{N}$ such that

$$|f(x) - 0| = |f(x) - f(x_n)| < \epsilon.$$

Therefore $x \in \ker f$ and this completes the proof. □

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