THE EFFECT OF PURE SHEAR ON THE REFLECTION OF PLANE WAVES AT THE BOUNDARY OF AN ELASTIC HALF-SPACE

W. HUSSAIN

DEPARTMENT OF MATHEMATICS, FORMAN CHRISTIAN COLLEGE (A CHARTERED UNIVERSITY) FEROZEPUR ROAD, LAHORE, PAKISTAN

EMAIL: WASIQHUSSAIN@FCCOLLEGE.EDU.PK

ABSTRACT. This paper is concerned with the effect of *pure shear* on the reflection from a plane boundary of infinitesimal plane waves propagating in a half-space of incompressible isotropic elastic material.

For a special class of constitutive laws it is shown that an incident plane harmonic wave propagating in the considered plane gives rise to a surface wave in addition to a reflected wave (with angle of reflection equal to the angle of incidence) although its amplitude may vanish at certain discrete angles but is independent of the state of deformation. Reflected wave amplitude is exactly equal to one in this case.

For a second class of constitutive laws similar behavior is found for certain combinations of angle of incidence, material properties and deformations, but additional possibilities also arise. In particular, there may be two reflected waves instead of one reflected wave and a surface wave. Here surface wave amplitude depends upon the pure shear and the reflected wave amplitude is not equal to one in general.

The dependence of the amplitudes of the reflected, and surface waves on the angle of incidence, the states of deformation is illustrated graphically.

Key words: Elastic waves; pure shear; reflection; non-linear elasticity.

AMS Subject Classifications: 74Bxx; 74Jxx.

1. Introduction

Wave propagation characteristics of elastic materials are used extensively for the determination of material properties. The effect of finite deformation on the propagation of surface waves in elastic solids was first discussed by Hayes and Rivlin [8] and Biot [1]. Surface waves in pre-stressed compressible elastic solids were also examined by Chadwick and Jarvis [2], while aspects of surface wave propagation and their connection with stability of the finite deformation were analyzed by Dowaikh and Ogden [6] and Connor and Ogden [3] for incompressible materials and by Dowaikh and Ogden [7] for compressible materials. These papers contain detailed references to other contributions.

Some basic aspects of plane wave propagation and reflection in pre-stressed solids have been examined in the paper by Sidhu and Singh [13], while Norris [10] has pointed to errors in several earlier papers by a number of authors. Inhomogeneous 'longitudinal' plane waves in a deformed elastic solid is done by Destrade and Hayes [4], and recently Destrade and Ogden [5] studied surface waves in a stretched and sheared incompressible elastic materials. Santosa and Symes [12] gave the analysis of dispersive effective medium for wave propagation in periodic composites.

In [9] the effect of finite strain on wave reflections by considering a half-space of incompressible isotropic elastic material subject to simple shear is studied. The direction of shear is taken to be parallel to the half-space boundary and the effect of principal axes orientation on waves and deformations in a half-space has been exemplified. Here one principal axes of primary deformation is no longer normal to the boundary and in general Cartesian axes are not coincident with the principal axes of deformation.

The purpose of this paper is to analyze the effect of pure shear deformation on the reflection of plane waves from the plane boundary of a half-space. The considered finite deformation is a pure homogeneous strain so that the orientation of the principal axes of strain is fixed whatever the magnitude of the strain. In general, the principal axes of deformation are aligned (one normal to the boundary) with the Cartesian axes. Specifically, we illustrate the results which typify the influence of finite strain by considering *incompressible materials*, so that plane waves are necessarily transverse. We restrict attention to a finite homogeneous deformation which corresponds to pure shear and to the propagation of plane waves in a principal plane of deformation with polarization in that plane. The influence of the finite deformation on the reflection of plane waves at a plane boundary is the primary concern. It is shown, in particular, that for a given incident SV wave two separate reflected SV waves with different speeds (and directions) are in general required to satisfy the boundary conditions. The reflection coefficients of two waves are obtained. Conditions under which (a) one of the reflected waves is replaced by a surface wave, and (b) two plane harmonic waves may be reflected when the angle of incidence lies within certain ranges of values (which depend on the pure shear deformation)

are found. Outside this range there is in general a single reflected wave, and a surface wave is generated.

The required equations and notations are summarized in Section 2. In Section 3 the propagation of plane harmonic waves is discussed with reference to the *slowness curves* in respect of two distinct classes of strain-energy functions.

In Section 4 the reflection coefficients (or amplitudes in case of a surface wave) for the two categories of strain-energy functions are calculated. For a certain class of constitutive laws it is shown that for each angle of incidence a single reflected wave, with angle of reflection equal to the angle of incidence, is generated when a homogeneous plane (SV) wave is incident on the boundary of the half-space, and it is accompanied by a surface wave (whose amplitude may vanish for certain discrete angles of incidence). For a second class of constitutive laws, two reflected (homogeneous plane SV) waves may be generated instead of one reflected and one surface wave.

The theory in Section 4 is illustrated in Section 5 using graphical results to show the dependence of the amplitudes of the waves on the angle of incidence for representative values of the material and deformation parameters.

2. Basic equations

We identify the undeformed configuration of the material, $\mathcal{B}_0$ say, and let a material particle in $\mathcal{B}_0$ be labelled by its position vector $\mathbf{X}$. Let $\mathbf{x}$ be the position vector of the same particle in the deformed configuration, $\mathcal{B}$ say. We write the deformation of the material from $\mathcal{B}_0$ to $\mathcal{B}$, $\boldsymbol{\chi}$ say, as

$$(2.1) \qquad \mathbf{x} = \boldsymbol{\chi}(\mathbf{X}), \quad \mathbf{X} \in \mathcal{B_0}.$$

The deformation gradient tensor $\mathbf{A}$ is defined as

$$(2.2) \qquad \mathbf{A} = \mathrm{Grad}\boldsymbol{\chi},$$

where Grad denotes the gradient with respect to $\mathbf{X}$, and is subject to the usual condition

$$(2.3) \qquad \det \mathbf{A} > 0.$$

The polar decomposition theorem enables $\mathbf{A}$ to be written as

$$(2.4) \qquad \mathbf{A} = \mathbf{VR},$$

where $\mathbf{R}$ is a proper orthogonal tensor and $\mathbf{V}$ is the symmetric and positive definite *left stretch tensor*.

For a volume preserving deformation we have

$$(2.5) \qquad \det \mathbf{A} \equiv \lambda_1 \lambda_2 \lambda_3 = 1,$$

where $\lambda_i \, (> 0) \, (i = 1, 2, 3)$ are the *principal stretches*.

Let $\mathbf{S}$ denote the nominal stress tensor. Then, the equilibrium equation, in the absence of body forces, is

$$(2.6) \qquad \mathrm{Div}\, \mathbf{S} = \mathbf{0},$$

where Div is the divergence operator in the reference configuration. For a (homogeneous) elastic material with strain-energy function $W = W(\mathbf{A})$ per unit volume, subject to the incompressibility constraint (2.5), we have

$$(2.7) \qquad \mathbf{S} = \frac{\partial W}{\partial \mathbf{A}} - p\mathbf{A}^{-1},$$

where p is a Lagrange multiplier. In addition, if the material is isotropic, W depends symmetrically on $\lambda_1, \lambda_2, \lambda_3$ subject to (2.5) and we write $W(\lambda_1, \lambda_2, \lambda_3)$.

Since the material is isotropic, the principal Cauchy stresses are given by

$$(2.8) \qquad \sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, \qquad i \in \{1, 2, 3\}.$$

For (plane strain) deformations confined to the $(1, 2)$-plane, we may set $\lambda_3 = 1$, so that (2.5) reduces to

$$(2.9) \qquad \lambda_1 \lambda_2 = 1.$$

Homogeneous *pure shear* deformation is defined by

$$(2.10) \qquad \lambda_1 = \lambda \neq 1, \ \ \lambda_2 = \lambda^{-1}, \ \lambda_3 = 1 \text{ with } \sigma_1 \neq 0, \ \sigma_2 = 0,$$

where a non-vanishing stress σ_3 is required to maintain $\lambda_3 = 1$.

Superimposed on the deformation just described we consider incremental motions in the (x_1, x_2)-plane with displacement vector $\mathbf{v}$ having components

$$v_1(x_1, x_2, t), \quad v_2(x_1, x_2, t), \quad v_3 = 0.$$

The (linearized) incremental incompressibility condition div $\mathbf{v} = 0$ enables v_1, v_2 to be expressed in terms of a scalar function, $\psi(x_1, x_2, t)$ say, so that

$$(2.11) \qquad v_1 = \psi_{,2}, \qquad v_2 = -\psi_{,1},$$

where $,i$ denotes $\partial/\partial x_i$, $i \in \{1, 2\}$.

The incremental nominal stress tensor is denoted by $\boldsymbol{\Sigma}$ when referred to the deformed configuration. Its components are given by

$$(2.12) \qquad \Sigma_{ji} = \mathcal{A}_{0jilk} v_{k,l} + p v_{j,i} - \pi \delta_{ij},$$

where π is the increment in p and $\mathcal{A}_{0jilk}$ are the components of the fourth-order tensor $\boldsymbol{\mathcal{A}_0}$ of instantaneous elastic moduli (see, for example, Ogden [11]).

The components of $\boldsymbol{\mathcal{A}_0}$ in terms of the derivatives of the strain-energy function W are given by

$$\mathcal{A}_{0iijj} = \lambda_i \lambda_j W_{ij},$$

$$\mathcal{A}_{0ijij} = \frac{(\lambda_i W_i - \lambda_j W_j)\lambda_i^2}{(\lambda_i^2 - \lambda_j^2)} \qquad i \neq j, \quad \lambda_i \neq \lambda_j,$$

$$\mathcal{A}_{0ijij} = \frac{1}{2}(\mathcal{A}_{0iiii} - \mathcal{A}_{0iijj} + \lambda_i W_i) \qquad i \neq j, \quad \lambda_i = \lambda_j,$$

$$(2.13) \qquad \mathcal{A}_{0ijji} = \mathcal{A}_{0jiij} = \mathcal{A}_{0ijij} - \lambda_i W_i \qquad i \neq j,$$

where $W_i = \partial W/\partial \lambda_i$, $W_{ij} = \partial^2 W/\partial \lambda_i \partial \lambda_j$ and there is no summation over repeated indices. Here, the components $\mathcal{A}_{0jilk}$ are constants because the deformation under consideration is homogeneous.

The equation of motion is given by

(2.14) $$\mathcal{A}_{0jilk}v_{k,jl} - \pi_{,i} = \rho\ddot{v}_i, \qquad i \in \{1,2\},$$

where ρ is the mass density of the material and there is summation from 1 to 2 over repeated indices.

The equations of motion (2.14) yield, on restriction to the considered plane motion,

(2.15) $$(\mathcal{A}_{01111} - \mathcal{A}_{01122} + p)v_{1,11} - \pi_{,1} + \mathcal{A}_{02121}v_{1,22} + (\mathcal{A}_{02121} - \sigma_2)v_{2,12} = \rho\ddot{v}_1,$$

(2.16) $$(\mathcal{A}_{02222} - \mathcal{A}_{02211} + p)v_{2,22} - \pi_{,2} + \mathcal{A}_{01212}v_{2,11} + (\mathcal{A}_{02121} - \sigma_2)v_{1,12} = \rho\ddot{v}_2,$$

where a superposed dot indicates the material time derivative.

Elimination of π between (2.15) and (2.16), and use of (2.11) yields an equation for ψ, namely

(2.17) $$\alpha\psi_{,1111} + 2\beta\psi_{,1122} + \gamma\psi_{,2222} = \rho(\ddot{\psi}_{,11} + \ddot{\psi}_{,22}),$$

as given in [3], where the constants α, β, γ are defined by

(2.18) $$\alpha = \mathcal{A}_{01212}, \quad \gamma = \mathcal{A}_{02121}, \quad 2\beta = \mathcal{A}_{01111} + \mathcal{A}_{02222} - 2\mathcal{A}_{01122} - 2\mathcal{A}_{01221}.$$

From (2.12), on use of (2.11), the shear and normal components of the incremental nominal traction Σ_{21}, Σ_{22} on a plane $x_2 = $ constant are expressible in terms of ψ through

(2.19) $$\Sigma_{21} = \gamma\psi_{,22} - (\gamma - \sigma_2)\psi_{,11},$$

(2.20) $$-\Sigma_{22,1} = (2\beta + \gamma - \sigma_2)\psi_{,112} + \gamma\psi_{,222} - \rho\,\ddot{psi}_{,2},$$

in the latter of which the incremental hydrostatic pressure π has been eliminated by differentiating Σ_{22} with respect to x_1 and then using (2.15).

3. PLANE WAVES

We consider time-harmonic homogeneous plane waves of the form

(3.1) $$\psi = A\exp[ik(x_1\cos\theta + x_2\sin\theta - ct)],$$

where A is a constant, c (> 0) the wave speed, k (> 0) the wave number and $(\cos\theta, \sin\theta)$ the direction cosines of the direction of propagation of the wave in the (x_1, x_2)-plane. Substitution of (3.1) into (2.17) gives

(3.2) $$\alpha\cos^4\theta + 2\beta\sin^2\theta\cos^2\theta + \gamma\sin^4\theta = \rho c^2.$$

Equation (3.2) is a relationship between the wave speed and the propagation direction in the (x_1, x_2)-plane and is called the *propagation condition*. The material constants are taken to satisfy the strong ellipticity inequalities

(3.3) $$\alpha > 0, \quad \gamma > 0, \quad \beta > -\sqrt{\alpha\gamma},$$

and it is clear from (3.2) that $\rho c^2 > 0$ if and only if (3.3) hold.

Similarly, from (2.17), for an inhomogeneous plane wave of the form

(3.4) $$\psi = \hat{A}\exp[ik'(x_1 - imx_2 - c't)],$$

(a) (b) (c)

FIGURE 1. Slowness curves in $(\bar{s}_1, \bar{s}_2)$-space for $2\beta = \alpha + \gamma$
with the following values of λ: (a), 1; (b), 1.6; (c), 2.2.

we obtain

$$(3.5) \qquad \alpha - 2\beta m^2 + \gamma m^4 = \rho(1 - m^2)c'^2,$$

which relates the wave speed c' to the 'inhomogeneity factor' m. Note that the wave decays exponentially as $x_2 \to -\infty$ provided m has positive real part.

We now consider a half-space of incompressible isotropic elastic material. The half-space is subjected to *pure shear deformation* in such a way that the principal directions of strain are aligned, one direction being normal to the boundary. In rectangular Cartesian coordinates we take the boundary to be $x_2 = 0$.

Let $\lambda_1, \lambda_2, \lambda_3$ be the stretches associated with the half-space $x_2 < 0$ with strain-energy function W, and the material constants α, β, γ defined by (2.13) with (2.18).

Two distinct strain-energy functions are now examined since these exemplify the range of possible behavior encountered. For these either $2\beta = \alpha + \gamma$ or $2\beta \neq \alpha + \gamma$, in $x_2 < 0$.

3.1. **Case A: $2\beta = \alpha + \gamma$.**   For this case equations (3.2) and (3.5) reduce to

$$(3.6) \qquad \alpha \cos^2 \theta + \gamma \sin^2 \theta = \rho c^2$$

and

$$(3.7) \qquad (m^2 - 1)(\alpha - \gamma m^2 - \rho c'^2) = 0$$

respectively.

In terms of the *slowness vector* (s_1, s_2) defined by

$$(3.8) \qquad (s_1, s_2) = (\cos \theta, \sin \theta)/c$$

equation (3.6) becomes the *slowness* curve

$$(3.9) \qquad \lambda^4 s_1{}^2 + s_2{}^2 = \bar{\rho}, \ x_2 < 0,$$

in the (s_1, s_2)-space, where $\bar{\rho}$ is defined by

$$(3.10) \qquad \bar{\rho} = \rho/\gamma,$$

and $\alpha/\gamma = \lambda^4$ follows from (2.13) and (2.18).

By using the dimensionless notation defined $(\bar{s}_1, \bar{s}_2)$ defined by

$$(3.11) \qquad (\bar{s}_1, \bar{s}_2) \equiv (s_1, s_2)/\sqrt{\bar{\rho}},$$

we can write (3.9) as

$$(3.12) \qquad \lambda^4 \bar{s}_1{}^2 + \bar{s}_2{}^2 = 1, \quad x_2 < 0.$$

Slowness curves for $x_2 < 0$ are shown in Figure 1 for illustrative values of λ.

(a) (b) (c)

FIGURE 2. Slowness curves in $(\bar{s}_1, \bar{s}_2)$-space for $\beta = \sqrt{\alpha\gamma}$ with the following values of λ: (a), 0.9; (b), 1.4; (c), 2.

3.2. **Case B:** $2\beta \neq \alpha + \gamma$.   In this case we take the strain-energy function to satisfy $\beta = \sqrt{\alpha\gamma}$ which was used by Hussain and Ogden in [9]. Then (3.2) takes the form

$$(3.13) \qquad [\sqrt{\alpha}\cos^2\theta + \sqrt{\gamma}\sin^2\theta]^2 = \rho c^2$$

and (3.5) becomes

$$(3.14) \qquad (\sqrt{\alpha} - \sqrt{\gamma}m^2)^2 = \rho(1 - m^2)c'^2.$$

The slowness curve corresponding to (3.13) is given by

$$(3.15) \qquad [\lambda^2\bar{s}_1^2 + \bar{s}_2^2]^2 = \bar{s}_1^2 + \bar{s}_2^2, \qquad x_2 < 0$$

in dimensionless form with the notation (3.11) and $\bar{\rho}$ defined by (3.10).

As in Case A we illustrate the slowness curves for particular values of λ for $x_2 < 0$ in Figure 2.

4. REFLECTION FROM A PLANE BOUNDARY

We now consider a wave incident on the boundary $x_2 = 0$ from the region $x_2 < 0$ with direction of propagation $(\cos\theta, \sin\theta)$ in the (x_1, x_2)-plane and speed c. Because of the symmetry with respect to the normal direction to the boundary we henceforth, without loss of generality, restrict attention to the values of θ in the interval $[0, \pi/2]$.

We write the solution comprising the incident wave, a reflected wave (with angle of reflection equal to the angle of incidence) and a surface wave in $x_2 < 0$ as

$$\psi = A\exp[ik(x_1\cos\theta + x_2\sin\theta - ct)] + AR\exp[ik(x_1\cos\theta - x_2\sin\theta - ct)]$$

$$(4.1) \qquad\qquad + AR'\exp[ik'(x_1 - imx_2 - c't)],$$

where R is the reflection coefficient and R' measures the amplitude of the surface wave. The notations k', m, c' are as used in (3.4) and m has positive real part, so that the surface wave, decays as $x_2 \to -\infty$.

According to Snell's law we have $k\cos\theta = k'$ or, equivalently,

$$(4.2) \qquad \cos\theta/c = 1/c'.$$

The coefficients R, R' will be determined by application of the boundary conditions $\Sigma_{21} = 0$ and $\Sigma_{22,1} = 0$ on $x_2 = 0$, where expressions for Σ_{21} and $\Sigma_{22,1}$ are given by Eqs. (2.19) and (2.20). Again we consider Cases A and B separately.

4.1. **Case A:** $2\beta = \alpha + \gamma$.   In this case we see from Eq.(3.7) that $m = \pm 1$, which yields a surface wave in the half-space $x_2 < 0$ for $m = 1$. The zeros of the other quadratic factor in (3.7) correspond to $m = i\tan\theta$ and $m = -i\tan\theta$, which are associated, respectively, with the incident and reflected waves in $x_2 < 0$. Thus, in $x_2 < 0$ the solution (4.1) applies with $m = 1$.

With the specialization $2\beta = \alpha + \gamma$, for the case of *pure shear*, use of Eqs. (2.19) and (2.20) enables the boundary conditions $\Sigma_{21} = 0$ and $\Sigma_{22,1} = 0$ on $x_2 = 0$ to be expressed in the form

(4.3) $$\psi_{,11} = \psi_{,22} \quad \text{on } x_2 = 0,$$

(4.4) $$(\lambda^4 + 2)\psi_{,112} + \psi_{,222} - \bar{\rho}\ddot{\psi}_{,2} = 0, \quad \text{on } x_2 = 0,$$

where $\bar{\rho}$ is given by (3.10).

Using (4.1) (with $m = 1$)appropriately specialized in Eqs.(4.3) and (4.4), we obtain

(4.5) $$(1 + R)(1 - t^2) + 2R' = 0,$$

(4.6) $$2i(R - 1)t + R'(t^2 - 1) = 0.$$

Eqs. (4.5) and (4.6) are solved to give

(4.7) $$R = \frac{-(t^2 - 1)^2 + 4it}{(t^2 - 1)^2 + 4it},$$

(4.8) $$R' = \frac{4it(t^2 - 1)}{(t^2 - 1)^2 + 4it},$$

and the notation

(4.9) $$t = \tan\theta$$

has been introduced. Note that t should be distinguished from the time variable t used earlier.

From (4.7) it follows that $|R| = 1$ for all angles on incidence and $|R|$ does not vanish for any angle of incidence. In addition both $|R|$ and $|R'|$ are independent of the material parameters and the stretch λ.

Graphical result showing the dependence of $|R'|$ on θ is described in Section 5.1.

4.2. **Case B:** $\beta = \sqrt{\alpha\gamma}$.   In this case, from (3.14), after using $\alpha/\gamma = \lambda^4$ and Snell's law $\cos\theta/c = 1/c'$,we have

(4.10) $$(1 + t^2)(\lambda^2 - m^2)^2 = (1 - m^2)(\lambda^2 + t^2)^2,$$

which can be organized as

(4.11) $$(m^2 + t^2)[m^2(1 + t^2) - t^2 + \lambda^2(\lambda^2 - 2)] = 0.$$

Note that $m = +it$ and $m = -it$ are the solutions of (4.11) corresponding to the incident and reflected waves respectively. The other solutions are

(4.12) $$m = \pm\sqrt{1 - (\lambda^2 - 1)^2/(1 + t^2)}.$$

If $\lambda \leq \sqrt{2}$ then m is real for all θ and the positive solution of (4.12) corresponds to a surface wave in $x_2 < 0$. If $\lambda > \sqrt{2}$ then there is a critical value of θ, θ_c say, for which $m = 0$ and this is given by

$$(4.13) \qquad t_c{}^2 = \lambda^2(\lambda^2 - 2),$$

where the notation $t_c = \tan \theta_c$ is used. It follows that m is real for $\theta_c \leq \theta \leq \pi/2$. For $\theta_c < \theta \leq \pi/2$ there is a reflected wave accompanied by a surface wave and for $\theta = \theta_c$ the surface wave becomes a plane shear (body) wave propagating parallel to the boundary in $x_2 < 0$ (grazing reflection). When $0 < \theta < \theta_c$ the surface wave is replaced by a second reflected wave with angle of reflection, θ' say, obtained from (4.12) by replacing m by $-i \tan \theta'$ to give

$$(4.14) \qquad t'^2 = \{\lambda^2(\lambda^2 - 2) - t^2\}/(1 + t^2),$$

where $t' = \tan \theta'$. The speed c' of the second reflected wave is obtained from Snell's law in the form $c' = c \cos \theta' / \cos \theta$ together with (4.14).

The two reflected waves coincide when the angle of incidence, θ_0 say, is given by

$$(4.15) \qquad t^2{}_0 = \lambda^2 - 2,$$

where $t_0 = \tan \theta_0$. Clearly, this gives a non-trivial real angle only if $\lambda > \sqrt{2}$.

As in Case A, the solution ψ may be written in the form (4.1), with m in (4.1) given by the positive solution of (4.12) when real and replaced by $-i \tan \theta'$ when imaginary.

The coefficients R, R' are determined by using the boundary conditions (4.3) and (4.4), with (4.4) taking the form

$$(4.16) \qquad (2\lambda^2 + 1)\psi_{,112} + \psi_{,222} - \bar{\rho}\ddot{\psi}_{,2} = 0$$

in this case. By substituting the value of ψ from (4.1) in (4.3), and (4.16) we obtain

$$(4.17) \qquad (R + 1)(1 - t^2) + R'(1 + m^2) = 0,$$

$$(4.18) \qquad (R - 1)it(m^2 + 1) + R'm(t^2 - 1) = 0,$$

respectively. In the latter equation use has been made of Eq. (4.12) in order to simplify the coefficients.

The solutions of (4.17) and (4.18) may be written in the form

$$(4.19) \qquad R = \frac{t(1 + m^2)^2 + im(t^2 - 1)^2}{t(1 + m^2)^2 - im(t^2 - 1)^2},$$

$$(4.20) \qquad R' = \frac{2(m^2 + 1)(t^2 - 1)t}{t(1 + m^2)^2 - im(t^2 - 1)^2}.$$

In these equations, for given t, m is obtained from Eq. (4.12) so as to have positive real part. Note that the Eqs. (4.19) and (4.20) reduce to (4.7) and (4.8) when m is set equal to 1.

From (4.19) it follows that $|R| = 1$ provided m is real for all angles on incidence. In addition $|R'|$ does depend upon the stretch λ.

FIGURE 3. Plot of $|R'|$ (surface wave amplitude) against $\theta\,(0 \le \theta \le \pi/2)$ for $2\beta = \alpha + \gamma$.

In Section (5.2) graphical results for R and R' (when real) and their absolute values (when complex) are given for illustration.

5. NUMERICAL RESULTS

In Sections 5.1 and 5.2 some graphical results are given for Cases A and B, respectively.

5.1. **Case A:** $2\beta = \alpha + \gamma$.   Recalling (4.8) it is convenient to display $|R'|$ as a function of θ for the range $0 \le \theta \le \pi/2$. In Fig. 3, $|R'|$ is plotted and it is easily seen that $|R'|$ vanishes where the numerator in (4.8) becomes zero i.e. at $\theta = 0$ and $\theta = \pi/4$. As well as in the case of normal incidence ($\theta = \pi/2$), the amplitude of the surface wave becomes zero.

There is no non-trivial result for grazing incidence since $|R| = 1$ and $R' = 0$ when $\theta = 0$. In this case there is only one reflected wave and one surface wave for each angle of incidence.

5.2. **Case B:** $\beta = \sqrt{\alpha\gamma}$.   For this case graphical results are given in Figs. 4-7. In Figs. 4(a-c), with reference to the slowness curves in Figs. 2(a-b), there is one reflected wave accompanied by a surface wave in $x_2 < 0$ for each angle of incidence. As in Case A, there are no non-trivial results for grazing incidence.

In Figs. 4(a-c), we have $|R| = 1$ since $\lambda \le \sqrt{2}$ as discussed in Section 4 and the character of surface wave amplitude $|R'|$ (against different stretches) for m to be real and positive is shown and is very similar to that shown in Fig. 3 which is independent of the stretch λ. Note that the value of $|R'|$ varies prominently with λ when $0 < \theta < \pi/4$.

With reference to the slowness curves in Fig. 2(c) we have two intervals, namely $0 \le \theta \le \theta_c$ with two reflected waves and $\theta_c \le \theta \le \pi/2$ with one reflected wave accompanied by a surface wave, where θ_c, approximately 1.1 here, is the critical value identified in Eq. (4.13). Note that for $\theta = \theta_0$, approximately 0.995, the two reflected waves coincide and then $|R'| = 1$ and $R = 0$.

As for the previous examples it is convenient to display results on the interval $(0, \pi/2)$ when there are two reflected waves, R and R' are real. Otherwise, they are complex. We therefore plot the results by showing R and R' for the interval $(0, \theta_c)$ (in Figs. 5 and 6(a)) and $|R'|$ only for $(\theta_c, \pi/2)$, (in Fig. 6(b)) by bearing in mind that $|R| = 1$ and further $|R|$ and $|R'|$ are continuous across the boundaries between intervals. Note that the horizontal scales in Fig. 6(a) and Fig. 6(b) are very different and, in particular, the interval corresponding to Fig. 6(b) is very short. The fine detail of the behavior exemplified would not show up clearly on a smaller scale.

(a) (b) (c)

FIGURE 4. Plots of $|R'|$ (surface wave amplitude) against θ $(0 \le \theta \le \pi/2)$ for $\beta = \sqrt{\alpha\gamma}$ and the following values of λ: (a) 1.4 (b) 0.9 (c) 0.05.

FIGURE 5. Plot of R (reflection coefficient) against θ $(0 \le \theta \le 1.1)$ for $\beta = \sqrt{\alpha\gamma}$ and $\lambda = 2$.

(a) (b)

FIGURE 6. Plot of R' (reflected wave) against θ in (a) and of $|R'|$ (surface wave amplitude) in (b) for $\beta = \sqrt{\alpha\gamma}$ with $\lambda = 2$.

FIGURE 7. Plot of $|R'|$ against θ $(0 \le \theta \le \pi/2)$ for $\beta = \sqrt{\alpha\gamma}$ with $\lambda = 2$.

The results in Fig. 6 are combined as plots of $|R'|$ on the single interval $(0, \pi/2)$ in Fig. 7 in order to facilitate comparison with the results shown in Fig. 4. In Figs. 4(a-c) $|R'|$ vanishes at three values of θ, we find, by contrast, that here it can vanish at 4 points. 4th point is the critical angle ≈ 1.1. The differences arise for $\lambda = 2$ as compared with $\lambda \le \sqrt{2}$ are due essentially to conversion of a surface wave into a second reflected wave.

In Fig. 7 note that the maximum value of $|R'|$ increases with λ, and the strength of the surface wave is focussed more and more in a narrow band of incident angles as λ increases. This latter effect is less marked than for the surface wave amplitudes in Figs. 4(a-c). The changes in the vertical scale of Fig. 7 with each of Figs. 4 should be noted.

The above results show the general character of the effect of pure shear on the reflection of plane waves at the boundary of an elastic half-space. All the figures have been produced using Mathematica [14].

References

[1] M. A. Biot, *Mechanics of Incremental Deformations,* New York: Wiley, 1965.

[2] P. Chadwick and D. A. Jarvis,   Surface waves in a pre-stressed elastic body, *Proc. R. Soc. Lon.* A **366** (1979), 517-536.

[3] P. Connor and R. W. Ogden,   The effect of shear on the propagation of elastic surfce waves, *Int. J. Engng Sci.,* **33** (1995), 973-982.

[4] M. Destrade and M. Hayes,   Inhomogeneous 'longitudinal' plane waves in a deformed elastic material, *J. Elasticity,* **75** no. 2 (2004), 147-165.

[5] M. Destrade and R. W. Ogden,   Surface waves in a stretched and sheared incompressible elastic material, *Int. J. Nonlinear Mech.,* **40** (2005), 241-253.

[6] M. A. Dowaikh and R. W. Ogden,   On surface waves and deformations in a pre-stressed incompressible elastic solid, *IMA J. Appl. Math.,* **44** (1990), 261-284.

[7] M. A. Dowaikh and R. W. Ogden,   On surface waves and deformations in a compressible elastic half-space, *Stability Appl. Anal. Continuous Media,* **1** (1991), 27-45.

[8] M. A. Hayes and R. S. Rivlin,   Surface waves in deformed elastic materials, *Arch. Rat. Mech. Anal.,* **8** (1961), 358-380.

[9] W. Hussain and R. W. Ogden, On the reflection of plane waves at the boundary of an elastic half-space subject to simple shear, *Int. J. Engng Sci.,* **37** (1999), 1549-1576.

[10] A. N. Norris, Propagation of plane waves in a pre-stressed elastic medium, *J. Acoust. Soc. Am.,* **74** (1983), 1642-1643.

[11] R. W. Ogden, *Non-linear Elastic Deformations,* Dover Publications, Inc. Mineola, New York, 1997.

[12] F. Santosa and W. W. Symes, A dispersive effective medium for wave propagtion in periodic composites, *Siam J. Appl. Math.,* **51** no. 4 (1991), 984-1005.

[13] R. S. Sidhu and S. J. Singh, Reflection of P and SV waves at the free surface of a pre-stressed elastic half-space, *J. Acoust. Soc. Am.,* **76** (1984), 594-598.

[14] S. Wolfram, *Mathematica*, version 5. Wolfram Research Inc., Champaign, Illinois, 2003.