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G-Injective Envelope of Separable G-C*-algebras

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Abstract. Argerami and Farenick have found conditions for the injective envelope of a separable C^*-algebra to be a von Neumann algebra. In this paper, we introduce an equivalent version of this result by finding conditions for the G-injective envelope of a separable G-C^*-algebra A to be a von Neumann algebra, when G is a discrete group acting on A.

Keywords: G-W^*-algebra, G-AW^*-algebra, G-Injective envelope, G-Regular monotone completion, Type I C^*-algebra, G-invariant Essential ideal, G-Local multiplier algebra, Discrete group.

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1. Introduction

1.1. **Notice.** In 1979, Hamana [7, theorem 4.1] used the Arveson extension theorem to prove that any C^*-algebra has an injective envelope which is unique up to *-isomorphism. Indeed, he showed that if A is a C^*-algebra, then the image of a unit-preserving idempotent contractive linear map φ of an Arveson injective extension B into itself, is the injective envelope of A. Later, in 1985, Hamana found an equivariant version of his result [9] by showing that there exists a unique G-injective envelope $(I_G(A), \kappa)$, for any G-operator system A, such that if $(B, \acute{\kappa})$ is any G-injective envelope of A, there exists a complete

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order isomorphism $\varphi : I_G(A) \longrightarrow B$, satisfying $\varphi \circ \kappa = \acute{\kappa}$, where G is a discrete group acting on A and B.

On the other hand, an injective operator system is unitally and completely order isomorphic to a unital, monotone complete AW^*-algebra [5, 12]. In the above cited result of Hamana, if $\varphi : B \longrightarrow B$ is a minimal A-projection, then the multiplication on $I_G(A) = \varphi(B)$ is given by the Choi-Effros product, that is, by

$$x \circ y = \varphi(xy), \quad x, y \in I_G(A)$$

and the involution and norm on $I_G(A)$ are inherited from B. Furthermore, if A is a unital G-C^*-algebra, then A embeds into its G-injective envelope as a G-invariant unital C^*-subalgebra. In the case when $G = \{1\}$, the above product yields a C^*-algebra injective structure on the injective envelope $I(A)$ of A.

In this paper, we extend a result of M. Argerami and D. R. Farenick [2, Theorem 1.2] to the setting of discrete C^*-dynamics. In the next section, we set up the terminology and notations for G-C^*-algebras and G-W^*-algebras. In the main result of the paper in section 3, we show that parts $(i), (ii)$ and (v) of Theorem 1.2 in [2] remain equivalent in separable G-C^*-algebras for discrete C^*-dynamics.

2. G-C^*-algebras

Let $B(H)$ and $K(H)$ be the set of bounded and compact operators on a complex Hilbert space H, respectively. A C^*-algebra A is a W^*-algebra if A, as a Banach space, is the dual space X^* of some (in fact, unique) Banach space X. It is a classical fact that a C^*-algebra A is a W^*-algebra iff A has a representation as a von Neumann algebra of operators acting on some complex Hilbert space. A C^*-algebra A is an AW^*-algebra if the left annihilator of each right ideal in A is of the form Ap, for some projection $p \in A$, or equivalently, if every maximal abelian C^*-subalgebra $D \subseteq A$ is monotone complete [3]. Any W^*-algebra is an AW^*-algebra, but the converse is not true, i.e., there exists AW^*-algebras that fail to have any faithful representation as a von Neumann algebra.

In the category of C^*-algebras and completely positive (c.p.) linear maps, the pair (B, κ) is an extension of a C^*-algebra A, if B is a C^*-algebra and $\kappa : A \longrightarrow B$ is a c.p. map. A C^*-algebra A is *injective* if we can extend any A-valued completely positive linear map of subspace S of a C^*-algebra C to an A-valued completely positive linear map of the C^*-algebra C. An extension (B, κ) of a C^*-algebra A is called the *injective envelope* of A if B is injective and the only completely positive linear map of B into itself that fixes each element of $\kappa(A)$, is the identity map id_B. In [7], Hamana proved that any C^*-algebra has a unique injective envelope. Following Choi and Effros [4], he considered a completely positive linear map ϕ of the C^*-algebra B into itself, and observed that $Im(\phi)$ with multiplication " $\circ$ ", $x \circ y = \phi(xy)$ for all $x, y \in Im(\phi)$, and involution and norm induced by those of B, is a unital C^*-algebra. The C^*-algebra $Im(\phi)$ is denoted by $C^*(\phi)$. Hamana proved that $C^*(\phi)$ is injective if B is injective in this category. Finally, if A is a C^*-algebra, there exists an injective C^*-algebra C containing A as a C^*-subalgebra, by the Arveson extension theorem (which asserts that the algebra of bounded operators on a complex Hilbert space is injective). By [7, Theorem 3.4], there exists a minimal A-projection ϕ on C. If $B = C^*(\phi)$ and κ is the canonical inclusion of A into B, then (B, κ) is an injective envelope of A.

In this section we generalize some of the results obtained in the category of C^*-algebras and completely positive linear maps to the category of G-C^*-algebras and completely positive G-linear maps. We assume throughout this paper that G is a discrete group.

A G-C^*-algebra is a C^*-algebra which equipped with an action of G by automorphisms. In other words, a G-C^*-algebra A is a C^*-algebra and a left G-module. Given two G-C^*-algebras A and B, the unital completely positive linear map $\varphi : A \longrightarrow B$ is G-*equivariant*, if $\varphi(g \cdot a) = g \cdot \varphi(a)$, for any $g \in G$ and $a \in A$. A G-C^*-algebra B can be viewed as a C^*-algebra over the discrete group algebra $L^1(G)$ with the module operation defined by

$$f \cdot x = \int f(g)\theta_g(x)dg \quad , \quad f \in L^1(G), x \in B$$

One could define the category of G-W^*-algebras and G-injective objects in this category in an analogous manner. A G-C^*-algebra B is a G-W^*-*algebra* if B is a W^*-algebra with the $L^1(G)$-module structure such that the map $x \mapsto f \cdot x$ in B is positive and normal, for each $f \in L^1(G)^+$.

A G-C^*-algebra A is said to be G-*injective* if for any G-C^*-algebras B and C, any G-equivariant complete isometry $\kappa : B \longrightarrow C$ and any G-equivariant u.c.p map $\varphi : B \longrightarrow A$, there exists a G-equivariant u.c.p map $\tilde{\varphi} : C \longrightarrow A$ satisfying $\tilde{\varphi} \circ \kappa = \varphi$, i.e., the following diagram commutes,

$$\begin{tikzcd} B \arrow[r, "\kappa"] \arrow[dr, "\varphi"'] & C \arrow[d, "\tilde{\varphi}"] \\ & A \end{tikzcd}$$

This simply means that G-equivariant u.c.p maps into A have G-equivariant u.c.p extensions.

Suppose that A and B are G-C^*-algebras. We say that;

(i) (B, κ) is a G-*extension* of A, if $\kappa : A \longrightarrow B$ be a G-equivariant and u.c.p $*$-monomorphism.

(ii) The G-extension (B, κ) is G-*essential* if for any G-C^*-algebra C and any G-equivariant u.c.p map $\varphi : B \longrightarrow C$, φ is completely isometric whenever $\varphi \circ \kappa$ is.

(iii) The G-extension (B, κ) is *G-rigid* if the only G-equivariant u.c.p map $\varphi : B \longrightarrow B$ satisfying $\varphi \circ \kappa = \kappa$ is the identity map id_B.

The pair (B, κ) is a *G-injective envelope* of A, if (B, κ) is G-essential , G-rigid and B is G-injective.

Throughout this paper, we denote the G-injective envelope of a G-C^*-algebra A by $I_G(A)$. When G is trivial we are back to the notations of injectivity for C^*-algebras, as well as plain essentiality and rigidity of extensions.

Let A be a unital G-C^*-algebra and let $\theta : G \longrightarrow Aut(A)$ be a G-action. Writing $\theta_g = \theta(g)$, for all $g \in G$, by injectivity each $\theta_g : A \longrightarrow A$ $(a \longrightarrow g \cdot a)$ extends to a $*$-isomorphism $I_G(A) \longrightarrow I_G(A)$, still denoted by θ_g. Due to rigidity, one can show that $\theta_g \circ \theta_h = \theta_{gh}$ on $I_G(A)$, for all $g, h \in G$, so that $I_G(A)$ becomes a unital G-C^*-algebra containing A as a G-invariant C^*-subalgebra. Further, the inclusion $A \hookrightarrow I_G(A)$ is a G-essential extension of A.

In [9], Hamana proved that there exist a unique G-injective envelope $(I_G(A), \kappa)$, for any G-operator system A, such that if $(B, \acute{\kappa})$ is any other G-injective envelope of A, there exists a complete order isomorphism $\varphi : I_G(A) \longrightarrow B$ satisfying $\varphi \circ \kappa = \acute{\kappa}$.

Let H be a complex Hilbert space and A be an operator system in $B(H)$, then $\ell^\infty(G, A)$ becomes a G-operator subsystem of $B(H \otimes \ell^2(G))$ with the action of G given by the left translation, i.e.,

$$(gf)(h) = f(g^{-1}h), \qquad g, h \in G, \qquad f \in \ell^\infty(G, A)$$

and each $f \in \ell^\infty(G, A)$ is acting on $H \otimes \ell^2(G)$ by $f(\xi \otimes \delta_g) = f(g)\xi \otimes \delta_g$, for $\xi \in H$ and $g \in G$.

Hamana showed that if A is an injective operator system, then $\ell^\infty(G, A)$ is G-injective, and that any G-injective G-operator system is injective.

If $A \subseteq B$ and B is a G-injective G-operator system, then an A-projection on B is a G-equivariant u.c.p map $\varphi : B \longrightarrow B$ satisfying $\varphi|_A = id_A$. A partial ordering on the set of A-projections on B can be defined by $\varphi \prec \psi$, for A-projections $\varphi, \psi : B \longrightarrow B$ if $\varphi \circ \psi = \psi \circ \varphi = \varphi$.

By the Zorn's lemma, there exists a minimal A-projection $\varphi : B \longrightarrow B$ on the set of seminorms induced by A-projection on B. In this argument, letting $\kappa : A \longrightarrow B$ be the inclusion map, then $(\varphi(B), \kappa)$ is a G-rigid and G-C^*-injective extension of A. Therefore, $(\varphi(B), \kappa)$ is the G-injective envelope of A.

A canonical G-injective G-operator system is $\ell^\infty(G, B)$, where B is an injective C^*-algebra. Let A be a unital G-C^*-algebra and B be a unital injective C^*-algebra containing A Let $\kappa : A \longrightarrow \mathcal{M} = \ell^\infty(G, B)$ be the G-equivariant injective $*$-homomorphism given by

$$\kappa(x)(g) = g^{-1}x, \qquad x \in A, \quad g \in G.$$

Then there is a $\kappa(A)$-projection $\varphi : \mathcal{M} \longrightarrow \mathcal{M}$ such that $(\varphi(\mathcal{M}), \kappa)$ is the G-injective envelope of A. Thus, for any injective extension B of a unital G-C^*-algebra A, the map $\kappa : A \longrightarrow \ell^\infty(G, B)$ is the canonical inclusion map.

Any injective operator system is unitally and completely order isomorphic to a unital, monotone complete AW^*-algebra [5, 12]. In our setting, if $A \subseteq B$ are as above and $\varphi : B \longrightarrow B$ is a minimal A-projection, then the multiplication on $I_G(A) = \varphi(B)$ is given by the Choi-Effros product, i.e., by

$$x \circ y = \varphi(xy), \quad x, y \in I_G(A)$$

and the involution and norm on $I_G(A)$ are inherited from B [7]. Further, if A is a unital G-C^*-algebra, then A embeds into its G-injective envelope as a G-invariant unital C^*-subalgebra. In the case when $G = \{1\}$, the above product yields a C^*-algebra injective structure on the injective envelope $I(A)$ of A.

A G-C^*-algebra A is a *G-monotone complete* if underlying C^*-algebra A is a monotone complete. A G-W^*-algebra is G-monotone complete if the underlying W^*-algebra is so as a C^*-algebra. A linear subspace A of a G-C^*-algebra B is called G-C^*-subalgebra of B, written $A \preceq B$, if A is a G-C^*-algebra in the restricted action of G.

Given two G-C^*-algebra $A \preceq B$, A is said to be *G-closed* in B if $y \in B$ and $g \cdot y \in A$, for all $g \in L^1(G)$, imply $y \in A$. For any G-C^*-algebras $A \preceq B$ the smallest G-closed G-C^*-subalgebra of B containing A is called the *G-closure* of A in B, written G-$cl_B A$, i.e., $G\text{-}cl_B A = \{y \in B : f \cdot y \in A \text{ for all } f \in L^1(G)\}$. A G-C^*-algebra A is *G-complete* if for any G-C^*-algebra B with $A \preceq B$, A is a G-closed in B.

A *G-regular completion* of a G-C^*-algebra A is a G-C^*-algebra, written $\overline{A}_G$, such that;
(1) $\overline{A}_G$ is G-complete,
(2) $A \preceq \overline{A}_G$,
(3) If $A \preceq B$ and B is G-C^*-complete, there are a G-C^*-algebra B' with $A \preceq B' \preceq B$ and a G-isomorphism $\psi : \overline{A}_G \longrightarrow B'$ with $\psi|_A = id_A$.

In fact, the $\overline{A}_G$ is the smallest G-complete containing A. Hence, $\overline{A}_G$ exists and is unique. Now the Hamana's construction [9] of $\overline{A}_G$ is via the G-injective envelope of A. Namely, $\overline{A}_G$ is the G-closure of A in $I_G(A)$.
For each G-C^*-algebra A, there is a representation in which

$$A \preceq \overline{A}_G \preceq I_G(A),$$

where each containment is as a G-C^*-subalgebra. An important feature of this sequence of containments is that $\overline{A}_G$ is G-monotone closed in $I_G(A)$

An ideal I of A is *essential* if $K \cap I \neq \{0\}$, for any non-zero ideal $K \subseteq A$. Equivalently, if $aI = 0$, for all $a \in A$, then $a = 0$. Any essential ideal is necessarily non-zero. The multiplier algebra $M(A)$ of a C^*-algebra A is a C^*-subalgebra of the enveloping von Neumann algebra A^{**} that consists of all $x \in A^{**}$ for which $xa \in A$ and $ax \in A$, for all $a \in A$.

An essential ideal I of a G-C^*-algebra A is *G-essential ideal* if I is G-invariant. For a G-invariant ideal I of A, the *G-multiplier algebra* $M_G(I)$ of I is the G-regular completion of the multiplier algebra $M(I)$, endowed with the canonical strictly continuous action of G,that is, $M_G(I) = \overline{M(I)}_G$.

If $J \subseteq A$ is a G-invariant ideal, then J^{**} is identified with the closure of J in A^{**} with respect to the strong operator topology. Thus, if J and K are G-invariant ideals of A, and if $J \subseteq K$, then $M_G(J) \succeq M_G(K) \succeq M_G(A)$.

Consider the G-multiplier algebra $M_G(J)$ of any G-essential ideal J of A. If $\varepsilon_G(A)$ is the set of G-essential ideals of A, partially ordered by reverse inclusion, then the set $\xi(A)$ of G-multiplier algebras $M_G(K)$ of $K \in \varepsilon_G(A)$ is a directed system of G-C^*-algebras. We define a *G-local multiplier algebra*, denoted by $M_G^{loc}(A)$, as follows

$$M_G^{loc}(A) = \varinjlim\{M_G(K); K \in \varepsilon_G(A)\}.$$

In fact, the $M_G^{loc}(A)$ is defined to be the C^*-direct limit over the downward directed system $K \in \varepsilon_G(A)$, and $M_G^{loc}(A)$ is realized by idealizers in $I_G(A)$ of G-essential ideals of A. By an argument similar to [6, Corollary 4.3]

$$M_G^{loc}(A) = cl\left(\bigcup_{K\in\varepsilon_G(A)}\{x \in I_G(A); xK + Kx \subseteq K\}\right)$$

where the closure is with respect to the norm topology of $I_G(A)$. Thus,

$$A \preceq M_G^{loc}(A) \preceq I_G(A)$$

is an inclusion of G-C^*-subalgebras.

Lemma 2.1. *If A is a G-C^*-algebra for which $I_G(A)$ is a G-W^*-algebra, then $\overline{A}_G$ is a G-W^*-algebra.*

Proof. Suppose that $I_G(A)$ is a G-W^*-algebra. Then $I_G(A)$ is represented as a von Neumann algebra acting on a Hilbert space. We assume that $\{h_\alpha\}_\alpha$ be any bounded increasing net in $(\overline{A}_G)_{sa}$. Because $I_G(A)$ is G-monotone complete, $\{h_\alpha\}_\alpha$ has a least upper h such that $h = \lim_\alpha h_\alpha = \sup_\alpha h_\alpha$ in the strong operator topology. Since, $\overline{A}_G$ is G-monotone closed in $I_G(A)$, $h \in \overline{A}_G$. Thus $\overline{A}_G$ is a G-C^*-algebra of operators in which the limit of every bounded increasing net of hermitian elements again belongs to $\overline{A}_G$. Therefore, $\overline{A}_G$ is a G-W^*-algebra by [10, lemma 1]. □

Proposition 2.2. *For any G-C^*-algebra A the G-closure of A in its G-injective envelope $I_G(A)$ is the G-regular completion $\overline{A}_G$ of A.*

Proof. Let A_1 be the G-closure of A in $I_G(A)$ and $A \preceq B$, then $A \preceq B \preceq B_1$ for some G-injective B_1, and there are an idempotent G-morphism $\phi : B_1 \longrightarrow B_1$ and a G-isomorphism $\psi : I_G(A) \longrightarrow \phi(B_1)$ such that $\phi|_A = id_A = \psi|_A$. We have G-$cl_{B_1}A \preceq \phi(B_1)$. Indeed, if $b \in G$-$cl_{B_1}A$, then $f \cdot b \in A$ for all $f \in L^1(G)$ and $f \cdot b = \phi(f \cdot b) = f \cdot \phi(b)$ in B_1 for all $f \in L^1(G)$; hence $b = \phi(b) \in \phi(B_1)$. Thus

$$G\text{-}cl_{\phi(B_1)}A = (G\text{-}cl_{B_1}A) \cap \phi(B_1) = G\text{-}cl_{B_1}A.$$

Further, since ψ is a G-isomorphism and $\psi|_A = id_A$, we have $\psi(A_1) = G - cl_{\phi(B_1)}A$, and so $\psi(A_1) = G - cl_{B_1}A$. First we assume that $y \in \psi(A_1)$, then there is a $a_1 \in A_1$ such that $y = \psi(a_1) \in \phi(B_1)$. On the other hand, since A_1 is a G-closure of A, $f \cdot a_1 \in A$ for all $f \in L^1(G)$, and since $\psi|_A = id_A$, we have

$$f \cdot y = f \cdot \psi(a_1) = \psi(f \cdot a_1) = f \cdot a_1 \in A.$$

Hence, $y \in G\text{-}cl_{\phi(B_1)}A$.
Now, let $y \in G - cl_{\phi(B_1)}A$. By definition, we have $f \cdot y \in A$ and $y \in \phi(B_1)$. Suppose that $b_1 \in B_1$, with $y = \phi(b_1)$. Since ψ is a G-isomorphism, there exists $a_1 \in I_G(A)$ such that $y = \phi(b_1) = \psi(a_1)$. On the other hand, since A_1 is a G-closure of A in $I_G(A)$,

$$\psi(f \cdot a_1) = f \cdot \psi(a_1) = f \cdot y \in A \Rightarrow f \cdot a_1 \in A \Rightarrow a_1 \in A_1 \Rightarrow y = \psi(a_1) \in \psi(A_1).$$

If $A_1 = A$, namely, A is G-closed in $I_G(A)$. Then so is A in $\phi(B_1)$, and $A = G\text{-}cl_{B_1}A$. Hence, $A = G\text{-}cl_B A$, that is, A is G-closed in B. Since $A \preceq B \preceq B_1$, $G\text{-}cl_B A \preceq G\text{-}cl_{B_1}A$. As B is arbitrary, this means that A is G-complete.

Next, suppose that A is arbitrary, but B is G-complete. Since $I_G(A_1) = I_G(A)$ and A_1 is G-closed in $I_G(A)$, it follows from the foregoing argument that A_1 is G-complete. As B is G-complete, $G\text{-}cl_{B_1}A \preceq G\text{-}cl_{B_1}B = B$, and $\psi(A_1) = G\text{-}cl_{B_1}A \preceq B$ with $\psi(A_1) \cong A_1$. Therefore, A_1 is the G-regular completion of A.

Finally, let only that $A \preceq B$. By the above argument to $A \preceq B \preceq \overline{B}_G$, there is a G-isomorphism ψ of A_1 onto $G\text{-}cl_{\overline{B}_G}A$ with $\psi|_A = id_A$. Hence, since $A \preceq G\text{-}cl_B A \preceq G\text{-}cl_{\overline{B}_G}A$, $G\text{-}cl_B A$ is isomorphic to the G-C^*-subalgebra $\psi^{-1}(G\text{-}cl_B A)$ of A_1. □

3. Separable C^*-algebra of a discrete group

The main result of this paper is Theorem (3.4) on separable discrete C^*-dynamics. Before turning to the proof of Theorem (3.4), we prove some preliminary results. We need the notion of *covariant representation* and the relation between G-local multiplier algebra and G-regular completion of G-C^*-algebras.

Definition 3.1. A C^*-algebra A is called *elementary* if $A \cong K(H)$ for some Hilbert space H.

The separable elementary C^*-algebras are the finite-dimensional matrix algebras and the C^*-algebras of compact operators of separable infinite-dimensional Hilbert space. Every elementary C^*-algebra is simple and the converse is true when the C^*-algebra is of type I. If A is a C^*-subalgebra of $K(H)$ acting irreducibly on Hilbert space H, then A is elementary.

Definition 3.2. A *covariant representation* of a G-C^*-algebra A is a pair (π, σ) where (π, H) is a representation of A , (σ, H) is a unitary representations of G, such that

$$\sigma(g)\pi(a)\sigma(g)^{-1} = \pi(\theta_g(a)) = \pi(g \cdot a)$$

for every $a \in A$, $g \in G$.
A covariant representation (π, σ) of a G-C^*-algebra A on a Hilbert space H is *normal* if (π, H) is normal.

Proposition 3.3. *$\overline{M_G^{loc}(A)} = \overline{A}_G$ for every G-C^*-algebra A.*

Proof. Since $M_G^{loc}(A)$ is G-equivariant *-isomorphically embedded into $I_G(A)$, extending the canonical G-equivariant *-monomorphism of A into $I_G(A)$, the G-C^*-algebra $I_G(A)$ serves as an injective G-extension of the G-C^*-algebra $M_G^{loc}(A)$. Therefore, the identity map on $M_G^{loc}(A)$ admits a unique G-extension to a G-equivariant completely positive map of $I_G(A)$ into itself with the same completely bounded norm one. Since $A_G \preceq M_G^{loc}(A) \preceq I_G(A)$ by construction and $I_G(A)$ is the G-injective envelope of A, $I_G(A)$ has to be the G-injective envelope of $M_G^{loc}(A)$. Since the G-regular completion of a G-C^*-algebra B is the G-monotone closure of B in the G-injective envelope $I_G(A)$,

$$A_G \preceq M_G^{loc}(A) \preceq \overline{A}_G \preceq I_G(A) = I(M_G^{loc}(A))$$

implies that $\overline{A}_G \preceq \overline{M_G^{loc}(A)} \preceq \overline{\overline{A}}_G$. Thus, $\overline{M_G^{loc}(A)} = \overline{A}_G$. $\square$

Theorem 3.4. *The following statements are equivalent for a separable G-C^*-algebra A:*
(i) $\overline{A}_G$ is a G-W^*-algebra.
(ii) $I_G(A)$ is a G-W^*-algebra.
(iii) A contains a G-invariant minimal essential ideal that is G-isomorphic to a direct sum of elementary G-C^*-algebras.

Proof. By Lemma (2.1), the proof of (ii)⇒(i) is clear.
(ii)⇒(iii): We have divided the proof into two stages. In the first stage, let us first show that there exists a faithful representation $\pi : A_G \longrightarrow B(H)$ such that the von Neumann algebra $\pi(A_G)''$ is generated by its minimal projections, each of which is contained in $\pi(A_G)$. For this, let $I_G(A)$ be a G-W^*-algebra. By [11, lemma 7.4.9], there is a faithful G-equivariant representation $\widetilde{\pi} : I_G(A) \longrightarrow B(H)$ such that $\pi(A_G)$ is a G-C^*-subalgebra of $\widetilde{\pi}(I_G(A))$, with $\pi = \widetilde{\pi}|_{(A_G)}$. Without loss of generality, suppose that $I_G(A)$ is a von Neumann algebra acting on a Hilbert space. Since the G-regular completion $\overline{A}_G$ of A_G is G-monotone closed in $I_G(A)$ and because $I_G(A)$ is a von Neumann algebra, $\overline{A}_G$ is a von Neumann algebra by Lemma (2.1). Thus, $A_G'' \subseteq \overline{A}_G'' = \overline{A}_G$, A_G'' being the double commutant of A_G.

Now, let ω be a normal state on von Neumann algebra A_G'' that is faithful on A_G. Assume that $\omega(h) = 0$, where $h \in A_G''^{+}$. Because $h = \sup\{k \in A^+; k \leq h\}$, we have $0 \leq \omega(k) \leq \omega(h) = 0$, for each $k \in A_G^+$ with $k \leq h$. Thus $\omega(k) = 0$, which implies that $k = 0$ because ω is faithful on A. Hence, $h = 0$ and so

ω is faithful on A''_G. Namely, any normal state $\omega \in A''_G$ is faithful precisely when its restriction $\omega|_{A_G}$ to A_G is faithful. By [13, P. 139], because A_G is separable and order dense in A''_G, A''_G is generated by its minimal projections, each of which is contained in A_G. Furthermore, since A''_G is a direct product of type I factors by [3, lemma 2.2], A''_G is injective by [3, corollory 2.3]. Because $A_G \subseteq A''_G \subseteq I_G(A)$, we conclude that $A''_G = \overline{A}_G = I_G(A)$, by minimality of the injective envelope.

The second stage, without loss of generality, assumes that A_G is already represented as a subalgebra of $B(H)$ and that $M = A''_G$ is generated by its minimal projections, each of with lie in A_G. Let $K \subseteq A_G$ be the ideal of A_G generated by the minimal projections of M. We claim that K is an essential ideal, minimal among all essential ideals of A_G. Suppose that $J \subseteq A_G$ is a nonzero ideal. Choose any nonzero $h \in J^+$. There is a strictly positive λ in the spectrum $\sigma(h)$ of h. Let $e \in M$ be the spectral projection $e = e^h([\lambda, +\infty))$, where e^h denotes the spectral resolution of h. Thus, $0 \neq \lambda e \leq he$, and there is a minimal projection p of M such that $ep = pe = p$ and $0 \neq \lambda p = \lambda p^2 = p\lambda p \leq php \in J \cap K$. Then $J \cap K \neq \{0\}$.

By [3, lemma 2.2], since $M = A''_G$ is generated by its minimal projections, M is a discrete type I von Neumann algebra. Therefore, there is a faithful normal covariant *-representation γ of M on a Hilbert space H of the form $H = \bigoplus_n H_n$ by [11, lemma 7.4.9], such that

$$\gamma(K) \subseteq \gamma(A_G) \subseteq \gamma(M) = \prod_n B(H_n)$$

It fact, the minimal projections of any $B(H_n)$ are minimal projection of $\gamma(M)$. Hence, elements of $\gamma(K)$. Moreover, if e is a minimal projection of $\prod_n B(H_n)$, $e \in B(H_n)$, for some $n \in N$. Therefore, $\bigoplus_n K(H_n) \subseteq \gamma(K)$. Since $\gamma(K)$ is the smallest G-C^*-algebra that contains the minimal projections of $\gamma(M)$, it follows that $\gamma(K) = \bigoplus_n K(H_n)$. Since, $K \cong \bigoplus_n K(H_n)$, K is G-invariant minimal essential ideal of A_G.

(iii)$\Rightarrow$(ii): Suppose that A_G has a G-invariant minimal essential ideal K such that $K \cong \bigoplus_n K(H_n)$. Thus, by [1, Lemma 1.2.21],

$$M(K) = M(\bigoplus_n K(H_n)) = \prod_n M(K(H_n)) = \prod_n B(H_n),$$

and this shows that $M(K)$ is a type I W^*-algebra. Since K is a G-invariant minimal essential ideal of A_G, by [1, Remark 2.3.7] $M(K) = M_G^{loc}(A)$. Hence, $M_G^{loc}(A)$ is an injective G-W^*-algebra. We know that $A_G \subseteq M_G^{loc}(A) \subseteq I_G(A)$ as G-C^*-subalgebras, it must be that $M_G^{loc}(A) = I_G(A)$ by definition of injective envelope, and this is precisely the proof of the G-W^*-algebra of $I_G(A)$.

(i)$\Rightarrow$(ii): For the G-W^*-algebra $\overline{A}_G$, $\overline{A}_G = A''_G$ by the proof of (ii)$\Rightarrow$(iii). Since A''_G is a direct product of type I factors, so A''_G is injective. Therefore,

$\overline{A}_G$ is injective. Hence, $\overline{A}_G = I_G(A)$, which yields that $I_G(A)$ is a G-W^*-algebra. □

EXAMPLE 3.5. by [8, lemma 2.2], $A = \ell^\infty(G, B(H))$ is G-injective, where G acts trivially on $B(H)$. Thus $I_G(A) = A$ which is a G-W^*-algebra. Now the minimal essential ideal of A is $c_0(G) \otimes K(H)$ which is essential ideal and dense and is direct sum of $|G|$-copies of elementary C^*-algebras $\mathbb{C} \otimes K(H)$ [This is an infinite direct sum if the cardinal $|G|$ is not finite]. Also A is already G-complete, so the G-closure of A is A itself, which is a G-W^*-algebra.

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