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On the Volume of μ-way G-trade

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Abstract. A μ-way G-trade ($\mu \geq 2$) consists of μ disjoint decompositions of some simple (underlying) graph H into copies of a graph G. The number of copies of the graph G in each of the decompositions is the volume of the G-trade and denoted by s. In this paper, we determine all values s for which there exists a μ-way $K_{1,m}$-trade of volume s for underlying graph $H = K_{2m,2m}$ and $H = K_{2m}$.

Keywords: Trade, G-trade, μ-way G-trade ,Trade spectrum.

2000 Mathematics subject classification: 05B99.

1. Introduction

Trades are useful combinatorial objects with many applications in various areas of combinatorial configurations. In other words, a combinatorial trade is a subset of a combinatorial configuration which may be "exchanged" without changing overall parameters in the configuration. The combinatorial configuration may be a block design, a latin square, or in the case of this literature, a graph.

The concept of trade was first introduced in the block designs by Hedayat (see [16]), however trades were used back in 1917 by White, Cole and Cummings [12]. Many papers on trades in block designs are concentrated on the existence

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and non-existence of trades (see [4, 17, 21, 26]). Trades were used in the latin squares, named as latin trades (see [5, 18, 19]). Afterwords trades have been introduced in graph theory, named as graphical trades (see [8, 27]).

A decomposition of a graph H is a collection of edge-disjoint subgraphs of H, such that partitions the edges of H. If each of the subgraphs is isomorphic to some graph G, then the decomposition is called a G-decomposition of H.

Suppose G is a simple graph. A G-trade of volume s is a pair $\{T_1, T_2\}$ where each $T_i (i = 1, 2)$ consists of s pairwise edge-disjoint graphs isomorphic to G. The s copies in T_1 are distinct from the s copies in T_2 and the edge-set of the graphs in T_1 and T_2 are identical and forming a simple graph say underlying graph H. Therefore T_1 and T_2 are two disjoint G-decompositions of underlying graph H. The number of vertices in the underlying graph H is the foundation of G-trade and denoted by v.

In some papers on trades in graph theory, the underlying graph is unrestricted and it is obtained from the union of blocks in each T_i (see [8, 9, 24, 25]). In some other works, the underlying graph is a fixed graph with definite vertices and edges (see [7, 10]).

The concept of trades have been generalized in latin trades and block designs before, see [5, 28], also see([2, 14, 15, 29, 31]). Recently, the idea of generalization of G-trade is investigated in [20], which we explain as follows.

Definition 1.1. A μ-way G-trade of volume s with underlying graph H consists of μ disjoint decompositions of graph H into s edge-disjoint copies of G. In other words:

$$
\begin{aligned}
&T_1 = \{G_1^{(1)}, G_2^{(1)}, ..., G_s^{(1)}\},\\
&T_2 = \{G_1^{(2)}, G_2^{(2)}, ..., G_s^{(2)}\},\\
&\vdots\\
&T_\mu = \{G_1^{(\mu)}, G_2^{(\mu)}, ..., G_s^{(\mu)}\},\\
&T_i \cap T_j = \emptyset \qquad \forall 1 \le i, j \le \mu,\\
&G_i^j \simeq G \quad \forall 1 \le i \le s,\ 1 \le j \le \mu,\\
&\bigcup_{i=1}^{s} G_i^{(1)} = \bigcup_{i=1}^{s} G_i^{(2)} = ... = \bigcup_{i=1}^{s} G_i^{(\mu)} = H,\\
&\bigcup_{i=1}^{s} E(G_i^{(1)}) = \bigcup_{i=1}^{s} E(G_i^{(2)}) = ... \bigcup_{i=1}^{s} E(G_i^{(\mu)}) = E(H),\\
&G_i^{(j)} \neq G_k^{(l)} \qquad 1 \le i, k \le s, \qquad 1 \le j, l \le \mu, \qquad i \neq k, j \neq l.
\end{aligned}
$$

The number of vertices in the underlying graph H is the foundation of μ-way G-trade and denoted by v. Each G_i is called a block.

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Moreover, this generalization is mentioned in [10] for a particular state when G is a cycle.

As usual K_n denotes the complete graph on n vertices and $K_{m,n}$, denotes the complete bipartite graph with parts of sizes m, n. A star is a complete bipartite graph $K_{1,x}$ that is called x-star. A $K_{1,x}$ with vertex set $\{a_0, b_1, b_2, ..., b_x\}$ and edge set $\{a_0 b_i \mid 1 \leq i \leq x\}$ is denoted by $[a_0 : b_1, b_2, ..., b_x]$.

EXAMPLE 1.2. Figure (1) is a 3-way $K_{1,2}$-trade of volume 2, with foundation 5 on underlying graph $H_1 = K_{1,4}$, where $V(H_1) = \{1, 2, 3, 4, 5\}$. In other words $H_1 = K_{1,4}$ is decomposed into three disjoint partition.

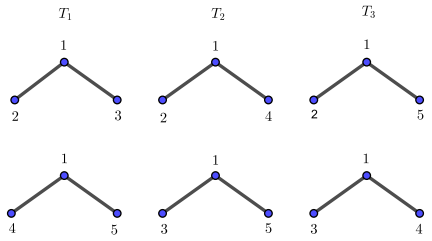

FIGURE 1. 3-way $K_{1.2}$-trade of volume 2

EXAMPLE 1.3. Figure (2) is a 3-way $K_{1,2}$-trade of volume 3, with foundation 7 on underlying graph $H_2 = K_{1,6}$, where $V(H_2) = \{1, a, b, c, d, e, f\}$. In other words $H_2 = K_{1,6}$ is decomposed into three disjoint partition.

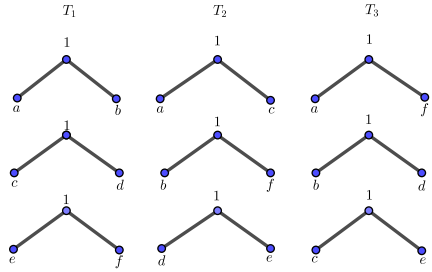

FIGURE 2. 3-way $K_{1,2}$-trade of volume 3

Trades are also intimately connected with the so-called *intersection problem* for combinatorial structures. This basically asks, given two combinatorial structures with the same parameters, and based on the same underlying set, such as a pair of block designs, a pair of latin squares, or a pair of graphs, in how

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many ways may they intersect? So for two block designs, how many common blocks may there be?

Let D_1 and D_2 be two block designs with the same parameters and the same set of points. Clearly if the blocks common to D_1 and D_2 are deleted, the remaining blocks T_1 in D_1 and T_2 in D_2 will form a trade $T = \{T_1, T_2\}$. So the possible volume T of the trade is intimately connected with the intersection problem for block designs.

The intersection problem has also been considered for more than just *pairs* of combinatorial structures; the intersection of μ combinatorial structures with $\mu > 2$ was dealt with in, for example, [1, 3, 30] for three block desgins and [6, 11, 13] for three latin squares. These correspond in the same manner to μ-way G- trades in the corresponding combinatorial structure.

This gives one of the motivations to study the spectrum of volumes of trades, besides that the problem of determining the possible size of combinatorial objects from some studied class is very natural, itself.

So the problem of determining the conditions for the existence, non-existence and the spectrum of volumes of μ-way G- trades is an important problem in combinatorial subjects. Not much is known for the mentioned questions on μ-way G- trades for $\mu \geq 3$ and most of the literature focused mainly on the case $\mu = 2$, see [22, 23].

Here, we determine all values s for which there exists a μ-way $K_{1,m}$-trade of volume s for underlying graph $H = K_{2m,2m}$ and $H = K_{2m}$.

Definition 1.4. The trade spectrum $TS_\mu(G)$ of G, is the set of values s for which there exists a μ-way G-trade of volume s. Let $X_\mu(G)$ denote the set of non-negative integers s for which no μ -way G-trade of volume s exists.

The trade spectrum of G is additive: if $x, y \in TS_\mu(G)$, then certainly $x+y \in TS_\mu(G)$, just take the union of two trades, of volume x and the other of volume y. So according to the above examples, we have a 3-way $K_{1,2}$-trade of volume $2 + 3 = 5$ on underlying graph $H = K_{1,10}$, where $V(H) = V(H_1) \cup V(H_2) = \{1, 2, 3, 4, 5, a, b, c, d, e, f\}$.

Note that $TS_\mu(K_2) = \{0\}$ because $TS_\mu(K_2) \subseteq TS(K_2)$ and $TS(K_2) = \{0\}$, where $TS(K_2)$ is the set of values for which there exists a K_2-trade of volume s, for $\mu = 2$.

Obviously $0 \in TS_\mu(G)$, moreover, $X_\mu(G)$ and $TS_\mu(G)$ partition the set of non-negative integers. Certainly $1 \in X_\mu(G)$, unless G contains an isolated vertex(see Lemma (2.1)). If $1 \in TS_\mu(G)$, then $TS_\mu(G)$ contains all the set of non-negative integers (because $TS_\mu(G)$ of G is additive).

2. Some general results

We start this section with general results for all graphs.

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lemma 2.1. *There exists a μ-way G-trade of each volume ($X_\mu(G) = \emptyset$) if and only if G has isolated vertices.*

Proof. Certainly $X_\mu(G) = \emptyset$ if and only if $1 \in TS_\mu(G)$ and μ different copies of G can have the same edge set if and only if G has isolated vertices. □

Example 2.2. Suppose G is a graph with one edge and one isolated vertex. In Figure (3) , there exists a 4-way G-trade of volume 1, with foundation 3 on underlying graph H, where H is a graph with 1 edge and 3 vertices, and here $H = G \simeq G^{(1)} \simeq G^{(2)} \simeq G^{(3)} \simeq G^{(4)}$.

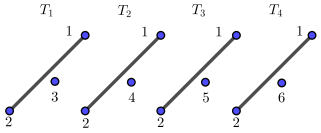

Figure 3. 4-way G-trade of volume 1

Theorem 2.3. *Let I be an independent subset of $V(G)$, such that $G \setminus I$ is not connected graph, then $X_\mu(G) \subseteq \{1, 2, \ldots, \mu - 1\}$ (for $t \geq \mu, t \in T_\mu(G)$).*

Proof. We may assume that G is non-empty. Let $t \geq \mu$, we need only show that $t \in TS_\mu(G)$. Let C be the vertex set of a component of $G \setminus I$, and D be the rest of the vertices of $G \setminus I$. Let $\mathbb{Z}_t$ denote the ring of integers modulo t. For each $i \in \mathbb{Z}_t$ place copy G_i of G on the vertex set $C \times \{i\} \bigcup I \bigcup D \times \{i\}$ respectively and we define:

$$
\begin{aligned}
&T_1 : \{(C \times \{i\}) \bigcup I \bigcup (D \times \{i\});\ i \in \mathbb{Z}_t\} \\
&T_2 : \{(C \times \{i\}) \bigcup I \bigcup (D \times \{i+1\});\ i \in \mathbb{Z}_t\} \\
&\vdots \\
&T_\mu : \{(C \times \{i\}) \bigcup I \bigcup (D \times \{i+\mu-1\})\ ; i \in \mathbb{Z}_t\}
\end{aligned}
$$

Obviously $T = \{T_1, T_2, T_3 \ldots T_\mu\}$ is a μ-way G-trade of volume t. □

Example 2.4. The trade in Figure (4) is constructed according to Theorem (2.3). Let G be a triangle graph with one hanging edge ($G = K_4 \setminus K_{1,2}$), with vertex set $V(G) = \{1, 2, 3, 4\}$ and with edge set $E(G) = \{12, 23, 13, 34\}$. Take $I = \{3\}, C = \{1, 2\}$ and $D = \{4\}$. It is clear that $G \setminus I$ is not connected graph and $3 \in T_3(G)$. The following trade is a 3-way G-trade of volume 3.

$$
\begin{aligned}
&T_1 : \{G_i;\ i \in \mathbb{Z}_3\} = \{(\{1,2\} \times \{i\}) \bigcup 3 \bigcup (4 \times \{i\});\ i \in \mathbb{Z}_3\} \\
&T_2 : \{G_i^{'};\ i \in \mathbb{Z}_3\} = \{(\{1,2\} \times \{i\}) \bigcup 3 \bigcup (4 \times \{i+1\});\ i \in \mathbb{Z}_3\}
\end{aligned}
$$

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$$T_3 : \{G_i^{''};\ i \in \mathbb{Z}_3\} = \{(\{1,2\} \times \{i\}) \bigcup 3 \bigcup (4 \times \{i+2\});\ i \in \mathbb{Z}_3\}$$

The underlying graph H has 10 vertices and 12 edges as follows:

$$\begin{aligned} V(H) =& \{(1,0),(1,1),(1,2),(2,0),(2,1),(2,2),3,(4,0),(4,1),(4,2)\},\\ E(H) =& \{3(1,0),3(1,1),3(1,2),3(2,0),3(2,1),3(2,2),(1,0)(2,0),\\ & (1,1)(2,1),(1,2)(2,2),3(4,0),3(4,1),3(4,2)\} \end{aligned}$$

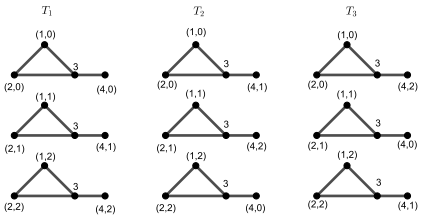

FIGURE 4. 3-way G-trade of volume 3

Theorem 2.5. *If for some independent subset $I \subseteq V(G)$, the graph $G \setminus I$ has k components $C_1, C_2, \ldots, C_k$, $(k \geq \mu)$, then $X_\mu(G) \subseteq \{1\}$.*

Proof. Let $t \geq 2$, we need only show that $t \in TS_\mu(G)$. Let $\mathbb{Z}_t$ denote the ring of integers modulo t, then we define:

$$\begin{aligned} &T_1 : \{C_1 \times \{i\} \bigcup C_2 \times \{i\} \bigcup \ldots \bigcup C_k \times \{i\} \bigcup I\ ;\ i \in \mathbb{Z}_t\}\\ &T_2 : \{C_1 \times \{i\} \bigcup C_2 \times \{i+1\} \bigcup \ldots \bigcup C_k \times \{i\} \bigcup I\ ;\ i \in \mathbb{Z}_t\}\\ &\vdots\\ &T_k : \{C_1 \times \{i\} \bigcup C_2 \times \{i\} \bigcup \cdots \bigcup C_k \times \{i+1\} \bigcup I\ ;\ i \in \mathbb{Z}_t\} \end{aligned}$$

Obviously $T = \{T_1, T_2, T_3 \ldots T_k\}$ is a k-way G-trade of volume t and $k \geq \mu$, so there is a μ-way G-trade of volume t.

□

EXAMPLE 2.6. The trade in Figure (5) is constructed according to Theorem (2.5). Let G be a tree graph with vertex set $V(G) = \{1,2,3,4,5\}$ and with edge set $E(G) = \{12, 23, 34, 35\}$. Take $I = \{3\}, C_1 = \{1,2\}, C_2 = \{4\}$ and $C_3 = \{5\}$. It is clear that $G \setminus I$ is not connected and $2 \in T_3(G)$. The following trade is a

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3-way G-trade of volume 2.

$$T_1 :\{G_i;\ i \in \mathbb{Z}_2\} = \{\{1,2\} \times \{i\} \bigcup \{4\} \times \{i\} \bigcup \bigcup \{5\} \times \{i\} \bigcup \{3\}\ ;\ i \in \mathbb{Z}_2\}$$
$$T_2 :\{G_i^{'};\ i \in \mathbb{Z}_2\} = \{\{1,2\} \times \{i\} \bigcup \{4\} \times \{i+1\} \bigcup \bigcup \{5\} \times \{i\} \bigcup \{3\}\ ;\ i \in \mathbb{Z}_2\}$$
$$T_3 :\{G_i^{''};\ i \in \mathbb{Z}_2\} = \{\{1,2\} \times \{i\} \bigcup \{4\} \times \{i\} \bigcup \bigcup \{5\} \times \{i+1\} \bigcup \{3\}\ ;\ i \in \mathbb{Z}_2\}$$

The underlying graph H has 9 vertices and 8 edges as follows:

$$V(H) = \{(1,0),(1,1),(2,0),(2,1),3,(4,0),(4,1),(5,0),(5,1)\},$$
$$E(H) = \{(1,0)(2,0),(1,1)(2,1),3(2,0),3(2,1),3(4,0),3(4,1),3(5,0),3(5,1)\}$$

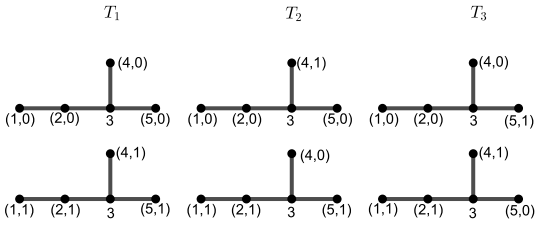

FIGURE 5. 3-way G-trade of volume 2

3. μ-WAY $K_{1,m}$-TRADE FOR $H = K_{2m,2m}$

In this section we determine the μ-way $K_{1,m}$-trade spectrum for underlying graph $H = K_{2m,2m}$.

lemma 3.1. *Let m and s be integers. If there exists a μ-way $K_{1,m}$-trade of volume s on $H = K_{1,2m}$, then $s = 2$ and $\mu \leq \dfrac{\binom{2m}{m}}{2}$.*

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Proof. There exist $\binom{2m}{m}$ graphs $K_{1,m}$ in $K_{1,2m}$. Since each T_i has s blocks, then $m \times s = 2m$, therefore $s = \frac{2m}{m} = 2$ and $\mu \leq \frac{\binom{2m}{m}}{2}$. □

Theorem 3.2. *Let $H = K_{1,2m}$ and $\mu \leq \frac{\binom{2m}{m}}{2}$, then there exists a μ-way $K_{1,m}$-trade of volume 2.*

Proof. Let $H = [0 : 1, 2, ..., 2m]$. Without loss of generality let the edge 01 is contained in the first block of each T_i. The number of $K_{1,m}$ which include the edge 01 is equal to $\binom{2m-1}{m-1}$. The second block in each T_i is determined uniquely. It is clear $\frac{1}{2}\binom{2m}{m} = \binom{2m-1}{m-1}$, then $\mu \leq \frac{\binom{2m}{m}}{2}$. □

lemma 3.3. *Let m and s be integers. If there exists a μ-way $K_{1,m}$-trade of volume s on $H = K_{2m,2m}$, then $s = 4m$ and $\mu \leq \binom{2m}{m}$.*

Proof. Since μ-way $K_{1,m}$-trade is of volume s, then $m \times s = 4m^2$, therefore $s = \frac{4m^2}{m} = 4m$. Let $V(H) = A \cup B$, where A and B are two disjoint sets and $|A| = |B| = 2m$, and all edges are between A and B. There exist $2m$ graphs $K_{1,2m}$ with centres in A and $2m$, $K_{1,2m}$ with centres in B. By Lemma (3.1) each $K_{1,2m}$ has $\binom{2m}{m}$, $K_{1,m}$. So we have $\frac{4m \times \binom{2m}{m}}{4m} = \binom{2m}{m}$ and $\mu \leq \binom{2m}{m}$. □

Theorem 3.4. *Let $H = K_{2m,2m}$ and $\mu \leq \binom{2m}{m}$, then there exists a μ-way $K_{1,m}$-trade of volume $4m$.*

Proof. The result follows from Theorem (3.2). □

4. μ-WAY $K_{1,m}$-TRADE FOR $H = K_{2m}$

In this section we determine the μ-way $K_{1,m}$ trade spectrum on underlying graph $H = K_{2m}$.

lemma 4.1. *Let m and s be integers. If there exists a μ-way $K_{1,m} - trade$ of volume s on $H = K_{2m}$, then $s = 2m - 1$ and $\mu \leq \frac{2m \times \binom{2m-1}{m}}{2m-1}$.*

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Proof. There exist $\sum_{v_i \in v(H)} \binom{degv_i}{m} = 2m \times \binom{2m-1}{m}$ graphs $K_{1,m}$ in K_{2m}. Since μ-way $K_{1,m}$-trade is of volume s, then $s \times m = \binom{2m}{2}$, therefore $s = \frac{\binom{2m}{2}}{m} = 2m-1$ and $\mu \leq \frac{2m \times \binom{2m-1}{m}}{2m-1}$.

□

Billington and Hoffman in [7] obtained a 2-way $K_{1,m}$-trade of volume $2m-1$ for underlying graph $H = K_{2m}$. Now we introduce two other disjoint $K_{1,m}$-decomposition for K_{2m}.

Theorem 4.2. *Let $H = K_{2m}(m > 2)$. Then there exists a 4-way $K_{1,m} - trade$ of volume $s = 2m-1$.*

Proof. Let $V(H) = \{0, 1, 2, ...2m-1\}$ be vertex set of underlying graph H. Suppose $K_{1,m}$ in T_1, T_2 be $[i : j_1, j_2, ..., j_m], [i : j'_1, j'_2, ..., j'_m]$ for $i, i', j_k, j'_k \in 0, 1, ..., 2m-1$, as constructed in [7].

Now we construct T_3, T_4 as follows:

$$T_3 : [i+m : j_1+m, j_2+m, ..., j_m+m]$$
$$T_4 : [i+m : j'_1+m, j'_2+m, ..., j'_m+m]$$

with modulo $2m$. We show that T_2 and T_3 are disjoint. Otherwise, assume that $B'_i = B''_j = [\alpha+m : \beta_1+m, \beta_2+m, ..., \beta_m+m]$, where $B'_i \in T_2$ and $B''_j \in T_3$. Therefore there exists a $B_k \in T_1$ that $B_k = [\alpha : \beta_1, \beta_2, ..., \beta_m]$. There exists a $\beta_i + m = \alpha + m - 1$. Hence the B_K in T_1 is following:

$B_K = [\alpha : ..., \alpha-1, ..., \alpha+1, ...]$. Therefore exists a block in T_2 that is $[\alpha : ..., \alpha-1, ...\alpha-1, ...]$. This gives a contradiction. □

Example 4.3. The trade in Figure (6) is constructed according to Theorem (4.2). Let G be a $K_{1,3}$ and H be a K_6.

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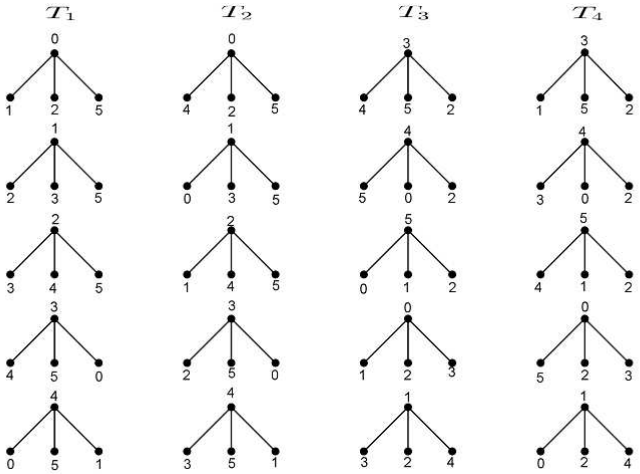

FIGURE 6. 4-way $K_{1,3}$-trade of volume 5

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