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# On A Class of Soc-Injective Modules

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Abstract. Let $R$ be a ring. The class of $SA$-injective right $R$-modules ($SAI_R$) is introduced as a class of soc-injective right $R$-modules. Let $N$ be a right $R$-module. A right $R$-module $M$ is said to be $SA$-$N$-injective if every $R$-homomorphism from a semi-artinian submodule of $N$ into $M$ extends to $N$. A module $M$ is called $SA$-injective, if $M$ is $SA$-$R$-injective. We characterize rings over which every right module is $SA$-injective. Conditions under which the class $SAI_R$ is closed under quotient (resp. direct sums, pure homomorphic images) are given. The definability of the class $SAI_R$ is studied. Finally, relations between SA-injectivity and certain generalizations of injectivity are given.

**Keywords:** Semi-artinian submodule, Definable class, Injective module, Noetherian module, Flat module.

**2020 Mathematics subject classification:** 16D50, 16D40, 16D10.

## 1. Introduction

Throughout $R$ is an associative ring with identity and all modules are unitary $R$-modules. If not otherwise specified, by a module (resp. homomorphism) we will mean a right $R$-module (resp. right $R$-homomorphism). We use $R$-Mod (resp. Mod-$R$) to denote the class of left (resp. right) $R$-modules. We will use $M^*$ to denote the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of a right module

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$M$. Let $\mathcal{G}$ (resp. $\mathcal{F}$) be a class of right (resp. left) $R$-modules. A pair $(\mathcal{F}, \mathcal{G})$ is called almost dual pair if $\mathcal{G}$ is closed under summands and direct products, and for any left $R$-module $M$, $M \in \mathcal{F}$ if and only if $M^* \in \mathcal{G}$ [12, p. 66]. An exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of right $R$-modules is said to be pure if the sequence $0 \to \mathrm{Hom}_R(N, A) \to \mathrm{Hom}_R(N, B) \to \mathrm{Hom}_R(N, C) \to 0$ is exact, for every finitely presented right $R$-module $N$ and we called that $\alpha(A)$ is a pure submodule of $B$ [21]. A right $R$-module $M$ is called $FP$-injective if every monomorphism $\alpha : M \to N$ is pure. A right $R$-module $M$ is called pure injective if $M$ is injective with respect to all pure short exact sequences [21]. If a subclass $\mathcal{G}$ of Mod-$R$ is closed under pure submodules, direct limits and direct products, then it is called a definable class[16]. We denote by $\mathrm{Soc}(M)$ to the socle of a module $M$. A right $R$-module $M$ is called semi-artinian if for any proper submodule $N$ of $M$ we have $\mathrm{Soc}(M/N) \neq 0$ [9, p. 238]. We will denote to the sum of all semi-artinian submodules of a right $R$-module $M$ by $\mathrm{Sa}(M)$. If $N$ is a submodule of a right $R$-module $M$, the notation $N \subseteq^{sa} M$ means that $N$ is a semi-artinian submodule of $M$.
We refer the reader to [2], [9], [16], [18] and [21], for general background materials.

Injective modules have been studied extensively, and several generalizations for these modules are given, for example, soc-injective modules [1], $\mathcal{L}$-injective Modules [13], and $n$-$FP$-injective modules [5]. If $\mathrm{Ext}^1(R/K, M) = 0$, for any semisimple right ideal $K$ of $R$, then a right $R$-module $M$ is called soc-injective [1], where $\mathrm{Ext}^1(A, B)$ is defined as the first right derived functor of $\mathrm{Hom}_R(A, B)$, for any two right $R$-modules $A, B$ (see [4, Ch. VI] for more details).

In section 2 of this paper, we introduce the class of $SA$-injective modules. This class of modules lies between injective modules and soc-injective modules. We first give examples to show that the notion of $SA$-injectivity is distinct from that of injectivity and soc-injectivity. We characterize rings over which every module is $SA$-injective. We prove the equivalence of the following statements: (1) Every right $R$-module is $SA$-injective; (2) Every semi-artinian module is $SA$-injective; (3) Every semi-artinian right ideal of $R$ is $SA$-injective; (4) Every semi-artinian right ideal of $R$ is a direct summand of $R$. Conditions under which the class of $SA$-injective right $R$-modules ($SAI_R$) is closed under quotient are given. For instance, we prove that the equivalence of the following: (1) The class $SAI_R$ is closed under quotient; (2) Sums of any two $SA$-injective submodules of any right $R$-module is $SA$-injective; (3) All semi-artinian right ideals of $R$ are projective. Finally, we give conditions such that the class $SAI_R$ is closed under direct sums. For instance, we prove that the following are equivalent. (1) $Sa(R_R)$ is noetherian; (2) Any direct sum of $SA$-injective right $R$-modules is $SA$-injective; (3) The class $SAI_R$ is closed under pure submodules; (4) All $FP$-injective modules are $SA$-injective.

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In section 3, we study the definability of the class $SAI_R$. It is shown that the following assertions are equivalent: (1) $SAI_R$ is definable; (2) The class $SAI_R$ is closed under pure submodules and pure homomorphic images; (3) Every semi-artinian right ideal in $R$ is finitely presented; (4) A module $M \in SAI_R$ iff $M^* \in (SAI_R)^{\ominus}$; (5) A module $M \in SAI_R$ iff $M^{**} \in SAI_R$. Finally, we prove that if the class $SAI_R$ is a definable, then the class of flat left $R$-modules and the class $(SAI_R)^{\ominus}$ are coincide iff all modules in $SAI_R$ are $FP$-injective iff all pure-injective modules in $SAI_R$ are injective.

In section 4, we give relations between SA-injectivity and certain generalizations of injectivity (in particular, quasi-injectivity and F-injectivity). Firstly, we prove that a ring $R$ is a right semi-artinian ring iff every $SA$-injective right $R$-module is quasi-injective iff every cyclic $SA$-injective right $R$-module is quasi-injective. Then, we prove that a commutative ring $R$ is semisimple if and only if $R$ is a semi-artinian ring and every quasi-injective $R$-module is $SA$-injective. Also, we prove that $\mathrm{Sa}(R_R)$ is a noetherian right $R$-module if and only if every F-injective right $R$-module is $SA$-injective. Finally, we prove that a ring $R$ is a (von Neumann) regular and every P-injective right $R$-module is $SA$-injective if and only if every $SA$-injective right $R$-module is P-injective and every semi-artinian right ideal of $R$ is a direct summand of $R_R$.

## 2. $SA$-Injective Modules

**Definition 2.1.** Let $N$ be a module. A module $M$ is called $SA$-$N$-injective, if for any semi-artinian submodule $K$ of $N$, any homomorphism $f : K \to M$ extends to $N$. $M$ is called $SA$-injective if $M$ is $SA$-$R$-injective. A ring $R$ is called $SA$-injective if the module $R_R$ is $SA$-injective.

We will use $SAI_R$ to denote the class of $SA$-injective right $R$-modules.

Examples 2.2. **(1)** All injective modules are $SA$-injective. Since 0 is the only semi-artinian right ideal in $\mathbb{Z}$, we have the right $\mathbb{Z}$-module $\mathbb{Z}$ is a $SA$-injective but it is not injective. Hence $SA$-injectivity is a proper generalization of injectivity.
**(2)** Since every semisimple module is semi-artinian, we have every $SA$-injective module is soc-injective. The converse is not true in general, for example: let $R = \mathbb{Z}_2[x_1, x_2, ...]$ where $x_i^3 = 0$ for all $i$, $x_i^2 = x_j^2 \neq 0$ for all $i$ and $j$ and $x_i x_j = 0$ for all $i \neq j$. By [1, Example 5.7], $R$ is a semiprimary commutative and soc-injective ring but it is not self injective. By [18, Example 1, p. 184], $R$ is a right semi-artinian ring, so that Proposition 2.5 in [18, p. 183] implies that $I \subseteq^{sa} R_R$ for any right ideal $I$ in $R$ and hence $R$ is not $SA$-injective ring.
**(3)** Clearly, if $\mathrm{Soc}(N_R) = 0$, then 0 is the only semi-artinian submodule of $N$ and hence every module is $SA$-$N$-injective. Particularly, all $\mathbb{Z}$-modules are $SA$-injective.

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**(4)** All modules with zero socles are $SA$-injective, this follows from the fact that $\mathrm{Soc}(M) = 0$ if and only if $\mathrm{Sa}(M) = 0$, for any module $M$.

**Proposition 2.3.** *Let $N$ be a module. Then following statements hold:*
(1) *The class of $SA$-$N$-injective modules is closed under isomorphic copies, direct products, direct summands and finite direct sums.*
(2) *For any submodule $K$ of $N$, if $M$ is $SA$-$N$-injective module, then $M$ is $SA$-$K$-injective.*
(3) *If $M$ is $SA$-$N$-injective module, then $M$ is $SA$-$K$-injective, for any module $K$ isomorphic to $N$.*

*Proof.* Clear. □

**Corollary 2.4.** *The class of $SA$-injective right $R$-modules ($SAI_R$) is closed under isomorphic copies, direct products, direct summands and finite direct sums.*

**Proposition 2.5.** *Let $M$ be a module and $\{N_i : i \in I\}$ be a family of modules. If $\bigoplus_{i\in I} N_i$ is a multiplication module, then $M$ is $SA$-$\bigoplus_{i\in I} N_i$-injective iff $M$ is $SA$-$N_i$-injective, for all $i \in I$.*

*Proof.* ($\Rightarrow$) By Proposition 2.3((2),(3)).

($\Leftarrow$) Let $K \subseteq^{s.a} \bigoplus_{i\in I} N_i$. Since $\bigoplus_{i\in I} N_i$ is a multiplication module (by hypothesis), we have from [20, Theorem 2.2, p. 3844] that $K = \bigoplus_{i\in I} K_i$ with $K_i$ is a submodule of $N_i$, for all $i \in I$. By [9, p. 238], $K_i \subseteq^{s.a} N_i$. For $i \in I$, consider the following diagram:

$$\begin{array}{ccc} K_i & \xrightarrow{\ i_2\ } & N_i \\ {\scriptstyle i_{K_i}}\downarrow & & \downarrow{\scriptstyle i_{N_i}} \\ K = \bigoplus_{i\in I} K_i & \xrightarrow{\ i_1\ } & \bigoplus_{i\in I} N_i \\ {\scriptstyle f}\downarrow & & \\ M & & \end{array}$$

where $i_{K_i}$, $i_{N_i}$ are injection maps and $i_1$, $i_2$ are inclusion maps. The hypothesis implies that there exists homomorphism $h_i : N_i \longrightarrow M$ such that $h_i \circ i_2 = f \circ i_{K_i}$. By [9, Theorem 4.1.6(2)], there exists exactly one homomorphism $h : \bigoplus_{i\in I} N_i \longrightarrow M$ satisfying $h_i = h \circ i_{N_i}$ . Thus $f \circ i_{K_i} = h_i \circ i_2 = h \circ i_{N_i} \circ i_2 = h \circ i_1 \circ i_{K_i}$ for all $i \in I$. Let $(a_i)_{i\in I} \in \bigoplus_{i\in I} K_i$, thus $a_i \in K_i$, for all $i \in I$ and $f((a_i)_{i\in I}) = f(\sum_{i\in I} i_{K_i}((a_i)_{i\in I})) = (h \circ i_1)((a_i)_{i\in I})$. Thus $f = h \circ i_1$ and the proof is complete. □

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Recall that a ring $R$ is called a right invariant if each of its right ideals is an ideal of $R$ [20, p. 3839].

**Corollary 2.6.** (1) *Let $M$ be a module over a right invariant ring $R$ and $1 = \lambda_1 + \lambda_2 + ... + \lambda_m$ in $R$ such that $\lambda_j$ are orthogonal idempotent. Then $M$ is $SA$-injective iff $M$ is $SA$-$\lambda_j R$-injective for every $j = 1, 2, ..., m$.*
(2) *If $M$ is $SA$-$aR$-injective module and $aR \cong bR$, where $a$ and $b$ are idempotents of $R$, then $M$ is $SA$-$bR$-injective.*

*Proof.* (1) By [2, Corollary 7.3], $R = \bigoplus_{j=1}^{m} \lambda_j R$. Since $R$ is a right invariant ring, we get from [20, Proposition 3.1, p. 3855] that $R$ is a multiplication module and hence Proposition 2.5 implies that $M$ is $SA$-injective iff $M$ is $SA$-$\lambda_j R$-injective for all $1 \leq j \leq m$.

(2) By Proposition 2.3(3). □

**Proposition 2.7.** *The following statements are equivalent for a module $M$.*
(1) *All modules are $SA$-$M$-injective.*
(2) *All semi-artinian modules are $SA$-$M$-injective.*
(3) *All semi-artinian submodules of $M$ are $SA$-$M$-injective.*
(4) *All semi-artinian submodules of $M$ are direct summands of $M$.*

*Proof.* Straightforward. □

Proposition 2.7 implies the next result.

**Corollary 2.8.** *For a ring $R$, the following conditions are equivalent.*
(1) *Mod-$R = SAI_R$.*
(2) *All semi-artinian modules are $SA$-injective.*
(3) *All semi-artinian right ideals of $R$ are $SA$-injective.*
(4) *If $I \subseteq^{sa} R_R$, then $I$ is a direct summand of $R_R$.*

**Corollary 2.9.** *A module $M$ is semisimple if and only if $M$ is semi-artinian and all modules are $SA$-$M$-injective.*

*Proof.* ($\Rightarrow$) It is obvious.

($\Leftarrow$) If $K$ is a submodule of $M$, then $K$ is semi-artinian by [9, p. 238] and hence Proposition 2.7 implies that $K$ is a direct summand of $M$. Thus $M$ is a semisimple module. □

As a special case of Corollary 2.9, we have the following corollary.

**Corollary 2.10.** *A ring $R$ is a right semisimple ring if and only if it is a right semi-artinian ring and Mod-$R = SAI_R$ .*

In general, not every semi-artinian submodule of a projective module is projective, for example, if $M = \mathbb{Z}_4$ as $\mathbb{Z}_4$-module and $K = \overline{2}\mathbb{Z}_4$, then $K \subseteq^{sa} M$ but $K$ is not a projective $\mathbb{Z}_4$-module.

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**Theorem 2.11.** *The following conditions are equivalent for a projective module $M$.*
(1) *The class of $SA$-$M$-injective modules is closed under quotient.*
(2) *Every quotient of an injective module is $SA$-$M$-injective.*
(3) *If $K_1$ and $K_2$ are two $SA$-$M$-injective submodules of a module $N$, then $K_1 + K_2$ is $SA$-$M$-injective.*
(4) *If $K_1$ and $K_2$ are two injective submodules of a module $N$, then $K_1 + K_2$ is $SA$-$M$-injective.*
(5) *If $K \subseteq^{sa} M$, then $K$ is projective.*

*Proof.* (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are obvious.

(2) $\Rightarrow$ (5) Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \overset{i}{\hookrightarrow} & M \\ & & \downarrow{\scriptstyle f} & & \\ E & \xrightarrow{h} & N & \longrightarrow & 0 \end{array}$$

where $N$ and $E$ are modules, $K$ is a semi-artinian submodule of $M$, $h$ is an epimorphism and $f$ is a homomorphism. We can assume that $E$ is injective (see, e.g. [3, Proposition 5.2.10]). By $SA$-$M$-injectivity of $N$, $f$ can be extended to a homomorphism $g : M \longrightarrow N$. By projectivity of $M$, there is a homomorphism $\tilde{g} : M \longrightarrow E$ such that $h \circ \tilde{g} = g$. Let $\tilde{f} : K \longrightarrow E$ be the restriction of $\tilde{g}$ over $K$. It is clear that $h \circ \tilde{f} = f$. Then $K$ is projective.

(5) $\Rightarrow$ (1) Let $L$ and $N$ be modules such that $N$ is $SA$-$M$-injective and $h : N \longrightarrow L$ is an epimorphism. If $K \subseteq^{sa} M$ and $f : K \longrightarrow L$ is any homomorphism, then the hypothesis implies that $K$ is projective and hence there is a homomorphism $g : K \longrightarrow N$ with $h \circ g = f$. By $SA$-$M$-injectivity of $N$, there is a homomorphism $\tilde{g} : M \longrightarrow N$ with $\tilde{g} \circ i = g$. Let $\beta = h \circ \tilde{g} : M \longrightarrow L$. Then $\beta \circ i = h \circ \tilde{g} \circ i = h \circ g = f$. and hence $L$ is an $SA$-$M$-injective module.

(1) $\Rightarrow$ (3) Let $K_1$ and $K_2$ be two $SA$-$M$-injective submodules of a module $K$. Thus $K_1 + K_2$ is a homomorphic image of the direct sum $K_1 \oplus K_2$. $SA$-$M$-injectivity of $K_1 \oplus K_2$ and the hypothesis imply that $K_1 + K_2$ is $SA$-$M$-injective.

(4) $\Rightarrow$ (2) Let $F$ be an injective module with submodule $D$. Let $B = F \oplus F$, $L = \{(x, x) \mid x \in D\}$, $\bar{B} = B/L$, $K_1 = \{b + L \in \bar{B} \mid b \in F \oplus 0\}$, $K_2 = \{b + L \in \bar{B} \mid b \in 0 \oplus F\}$. Then $\bar{B} = K_1 + K_2$. Since $(F \oplus 0) \cap L = 0$ and $(0 \oplus F) \cap L = 0$, $F \cong K_i$, $i = 1, 2$. Since $K_1 \cap K_2 = \{b + L \in \bar{B} \mid b \in D \oplus 0\} = \{b + L \in \bar{B} \mid b \in 0 \oplus D\}$, $K_1 \cap K_2 \cong D$ under $b \mapsto b + L$ for all $b \in D \oplus 0$. By hypothesis, $\bar{B}$ is $SA$-$M$-injective. Injectivity of $K_1$ implies that $\bar{B} = K_1 \oplus A$ for some submodule $A$ of $\bar{B}$, so $A \cong (K_1 + K_2)/K_1 \cong K_2/K_1 \cap K_2 \cong F/D$. By Proposition 2.3(5), $F/D$ is $SA$-$M$-injective. □

Theorem 2.11 implies the following result.

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**Corollary 2.12.** *The following statements are equivalent.*
(1) *The class $SAI_R$ is closed under quotient.*
(2) *Every quotient of an injective module is SA-injective.*
(3) *For any module $N$, if $N_1$ and $N_2$ are submodules of $N$ with $N_1$, $N_2 \in SAI_R$, then $N_1 + N_2 \in SAI_R$.*
(4) *For any module $N$, if $N_1$ and $N_2$ are injective submodules of $N$, then $N_1 + N_2 \in SAI_R$.*
(5) *If $I \subseteq^{sa} R_R$, then $I$ is projective.*

**Theorem 2.13.** *If $M$ is a finitely generated module, then the following statements are equivalent.*
(1) $\mathrm{Sa}(M)$ *is noetherian.*
(2) *The class of SA-M-injective modules is closed under direct sums.*
(3) *Direct sums of injective modules are SA-M-injective.*
(4) *If $K$ is injective module, then $K^{(S)}$ is SA-M-injective for any index set $S$,*
.
(5) *If $K$ is injective module, then $K^{(\mathbb{N})}$ is SA-M-injective.*

*Proof.* (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) Clear.

(1) $\Rightarrow$ (2) Let $E = \bigoplus_{i\in I} M_i$, where $M_i$ are $SA$-$M$-injective modules and $f : K \to E$ be a homomorphism with $K \subseteq^{sa} M$. Since $\mathrm{Sa}(M)$ is a noetherian module, we have $K$ is finitely generated and hence $f(K) \subseteq \bigoplus_{j\in I_1} M_j$, for some finite subset $I_1$ of $I$ and hence $\bigoplus_{j\in I_1} M_j$ is $SA$-injective. Then $f$ can be extended to a homomorphism $g : M \to E$ and so $E$ is $SA$-injective.

(5) $\Rightarrow$ (1) Let $K_1 \subseteq K_2 \subseteq ...$ be a chain of submodules of $\mathrm{Sa}(M)$. For each $i \geq 1$, let $F_i = E(M/K_i)$, $F = \bigoplus_{i=1}^{\infty} F_i$ and $M_i = \prod_{j=1}^{\infty} F_j = F_i \oplus (\prod_{\substack{j=1\\ j\neq i}}^{\infty} F_j)$, then $M_i$ is injective. By hypothesis, $\bigoplus_{i=I}^{\infty} M_i = (\bigoplus_{i=1}^{\infty} F_i) \oplus (\bigoplus_{i=1}^{\infty} \prod_{\substack{j=1\\ j\neq i}}^{\infty} F_j)$ is $SA$-$M$-injective and hence Proposition 2.3(1) implies that $F$ it self is $SA$-$M$-injective.
Define $f : H = \bigcup_{i=1}^{\infty} K_i \longrightarrow F$ by $f(a) = (a + K_i)_i$. Clearly, $f$ is a well defined homomorphism. Since $\mathrm{Sa}(M) \subseteq^{sa} M$ (by [9, p. 238]), we have $\bigcup_{i=1}^{\infty} K_i \subseteq^{sa} M$ and hence $f$ can be extended to a homomorphism $g : M \longrightarrow F$. Since $M$ is finitely generated, we have $g(M) \subseteq \bigoplus_{i=1}^{n} E(M/K_i)$ for some $n$ and hence $f(\bigcup_{i=1}^{\infty} K_i) \subseteq \bigoplus_{i=1}^{n} E(M/K_i)$. Since $\pi_i f(x) = \pi_i(x + K_j)_{j\geq 1} = x + K_i$, for all $x \in H$ and $i \geq 1$, where $\pi_i : \bigoplus_{j\geq 1} E(M/K_j) \longrightarrow E(M/K_i)$ is the projection map, $\pi_i f(H) = H/K_i$ for all $i \geq 1$. Since $f(H) \subseteq \bigoplus_{i=1}^{n} E(M/K_i)$, $H/K_i = \pi_i f(H) = 0$,

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for all $i \geq n+1$, so $H = K_i$ for all $i \geq n+1$ and hence the chain $K_1 \subseteq K_2 \subseteq ...$ terminates at $K_{n+1}$. Thus $\mathrm{Sa}(M)$ is a noetherian module. □

**Proposition 2.14.** *The following statements are equivalent.*
(1) $\mathrm{Sa}(R_R)$ *is noetherian.*
(2) *The class* $SAI_R$ *is closed under direct sums.*
(3) *Any direct sum of injective modules is* $SA$*-injective.*
(4) *If* $K$ *is injective module, then* $K^{(S)}$ *is* $SA$*-injective for any index set* $S$*.*
(5) *If* $K$ *is injective module, then* $K^{(\mathbb{N})}$ *is* $SA$*-injective.*
(6) *The class* $SAI_R$ *is closed under pure submodules.*
(7) *All* $FP$*-injective modules are* $SA$*-injective.*

*Proof.* By applying Theorem 2.13, we have the equivalent of $(1), (2), (3), (4)$ and $(5)$.

$(1) \Rightarrow (6)$. Let $N \in SAI_R$ and $K$ a pure submodule of $N$. Let $C \subseteq^{sa} R_R$, thus the hypothesis implies that $C$ is finitely generated and so $R/C$ is a finitely presented. Hence the sequence $\mathrm{Hom}_R(R/C, N) \to \mathrm{Hom}_R(R/C, N/K) \to 0$ is exact. By [8, Theorem XII.4.4 (4), p. 491], the sequence $\mathrm{Hom}_R(R/C, N) \to \mathrm{Hom}_R(R/C, N/K) \to \mathrm{Ext}^1(R/C, K) \to \mathrm{Ext}^1(R/C, N)$ is exact. Thus $\mathrm{Ext}^1(R/C, K) = 0$ and hence $K \in SAI_R$. Therefore, the class $SAI_R$ is closed under pure submodules.

$(6) \Rightarrow (7)$. If $M$ is any $FP$-injective module, then $M$ is a pure submodule of a $SA$-injective module. By hypothesis, $M \in SAI_R$.

$(7) \Rightarrow (1)$. Let $I$ be a submodule of $\mathrm{Sa}(R_R)$, thus $I \subseteq^{sa} R_R$. Let $\alpha : I \to M$ be a homomorphism, where $M$ is a $FP$-injective module. By hypothesis, $M$ is $SA$-injective and hence $\alpha$ extends to $R_R$. By [6], $I$ is finitely generated and hence $\mathrm{Sa}(R_R)$ is a noetherian module. □

## 3. Definability of the class $SAI_R$

If $\mathcal{X} \subseteq$ Mod-$R$, then we write $\mathcal{X}^{\ominus} = \{M \in R\text{-Mod} \mid M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \in \mathcal{X}\}$ and $\mathcal{X}^{+} = \{M \in \text{Mod-}R \mid M \text{ is a pure submodule of a module in } \mathcal{X}\}$ .

**Lemma 3.1.** *The pair* $((SAI_R)^{\ominus}, SAI_R)$ *is an almost dual pair over a ring* $R$*.*

*Proof.* By Corollary 2.4 and [12, Proposition 4.2.11, p. 72]. □

**Corollary 3.2.** *Consider the following conditions for the class* $SAI_R$ *over a ring* $R$*.*
(1) *The class* $SAI_R$ *is definable.*
(2) $(SAI_R, (SAI_R)^{\ominus})$ *is an almost dual pair over a ring* $R$*.*
(3) $(SAI_R)^* \subseteq (SAI_R)^{\ominus}$*.*
(4) $(SAI_R)^{**} \subseteq SAI_R$*.*
(5) *The class* $SAI_R$ *is closed under pure homomorphic images.*

*Then* $(1) \Leftrightarrow (2), (1) \Rightarrow (3), (1) \Rightarrow (5)$ *and* $(3) \Leftrightarrow (4)$*. Moreover, if* $Sa(R_R)$ *is noetherian, then all five conditions are equivalent.*

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*Proof.* (1) $\Leftrightarrow$ (2). By Lemma 3.1 and [12, Proposition 4.3.8, p. 89].

(1) $\Rightarrow$ (3). Since $SAI_R$ is a definable class, it is closed under pure submodules and hence $(SAI_R)^+ = SAI_R$. Since $((SAI_R)^{\ominus}, SAI_R)$ is an almost dual (by Lemma 3.1), it follows from [12, Theorem 4.3.2, p. 85], that $(SAI_R)^* \subseteq (SAI_R)^{\ominus}$.

(1) $\Rightarrow$ (5). By [16, 3.4.8, p. 109].

(3) $\Rightarrow$ (4). By Lemma 3.1 and [12, Theorem 4.3.2, p. 85].

(4) $\Rightarrow$ (1) and (5) $\Rightarrow$ (1). Suppose that $\mathrm{Sa}(R_R)$ is a noetherian module. By Proposition 2.14, the class $SAI_R$ is closed under pure submodules and hence $(SAI_R)^+ = SAI_R$. Thus the results follow from [12, Theorem 4.3.2, p. 85]. □

**Corollary 3.3.** *If every SA-injective modules is pure-injective, then the following statements are equivalent for a class $SAI_R$ over a ring $R$.*
(1) *$SAI_R$ is definable.*
(2) *The class $SAI_R$ is closed under direct sums.*
(3) $(SAI_R)^+ = SAI_R$
(4) $\mathrm{Sa}(R_R)$ *is a noetherian module.*

*Proof.* By Proposition 2.14, Lemma 3.1 and [12, Theorem 4.5.1, p. 103]. □

If $A$ is a a right $R$-module and $B$ is a left $R$-module, then $\mathrm{Tor}_1(A, B)$ is defined as the first left derived functor of the tensor product $A \otimes_R B$ (see [4, Ch. VI] for more details).

**Lemma 3.4.** *A left $R$-module $M \in (SAI_R)^{\ominus}$ iff $Tor_1(R/I, M) = 0$, for any semi-artinian right ideal $I$ of a ring $R$.*

*Proof.* Let $M$ be a left $R$-module and $I \subseteq^{sa} R_R$. By [7, Theorem 3.2.1, p. 75], $\mathrm{Ext}^1(R/I, M^*) \cong (\mathrm{Tor}_1(R/I, M))^*$, so that $\mathrm{Tor}_1(R/I, M) = 0$ if and only if $M^* \in SAI_R$. Hence $({}_RSAF, SAI_R)$ is an almost dual, where ${}_RSAF = \{M \in R\text{-Mod} |\ \mathrm{Tor}_1(R/I, M) = 0$, for any semi-artinian right ideal $I$ of a ring $R\}$. By [12, Proposition 4.2.11, p. 72], $(SAI_R)^{\ominus} =_R SAF$. □

A module $M$ is called $n$-presented if there is an exact sequence $F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$, with each $F_i$ is a finitely generated free modules [5].

**Theorem 3.5.** *The following statements are equivalent for a class $SAI_R$ over a ring $R$.*
(1) *$SAI_R$ is definable.*
(2) *The class $SAI_R$ is closed under pure submodules and pure homomorphic images.*
(3) *Every semi-artinian right ideal in $R$ is finitely presented.*
(4) *A module $M \in SAI_R$ iff $M^* \in (SAI_R)^{\ominus}$.*
(5) *A module $M \in SAI_R$ iff $M^{**} \in SAI_R$.*

*Proof.* (1) $\Rightarrow$ (2). By [16, 3.4.8, p. 109].

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(2) $\Rightarrow$ (3). Let $N$ be any $FP$-injective module, thus there is an injective module $H$ with pure exact sequence $0 \to N \xrightarrow{i} H \xrightarrow{\pi} H/N \to 0$. By hypothesis, $H/N \in SAI_R$. Let $K \subseteq^{sa} R_R$, thus $\mathrm{Ext}^1(R/K, H/N)$
$= 0$. By [8, Theorem 4.4 (4), p. 491], the sequence $0 = \mathrm{Ext}^1(R/K, H/N) \to$ $\mathrm{Ext}^2(R/K, N) \to$
$\mathrm{Ext}^2(R/K, H) = 0$ is exact and hence $\mathrm{Ext}^2(R/K, N) = 0$. By [8, Theorem 4.4 (3), p. 491], the sequence $0 = \mathrm{Ext}^1(R, N) \to \mathrm{Ext}^1(K, N) \to \mathrm{Ext}^2(R/K, N) = 0$ is exact, so that $\mathrm{Ext}^1(K, N) = 0$. By hypothesis, $SAI_R$ is closed under pure submodules, so that $K$ is finitely generated by Proposition 2.14 and hence [6, Proposition, p. 361] implies that $K$ is finitely presented.

(3) $\Rightarrow$ (1). Let $M \in SAI_R$. Let $K \subseteq^{sa} R_R$, thus $K$ is finitely presented (by hypothesis) and hence there is an exact sequence $F_2 \xrightarrow{\alpha_2} F_1 \xrightarrow{\alpha 1} K \to 0$, where $F_1, F_2$ are finitely generated free modules. Let $\beta = i\alpha_1$, where $i : K \to R$ is the inclusion mapping, thus the sequence $F_2 \xrightarrow{\alpha_2} F_1 \xrightarrow{\beta} R \xrightarrow{\pi} R/K \to 0$ is exact, where $\pi : R \to R/K$ is the natural epimorphism. Hence $R/K$ is a 2-presented module, so that from [5, Lemma 2.7 (2)] we have $\mathrm{Tor}_1(R/K, M^*) \cong (\mathrm{Ext}^1(R/K, M))^* = 0$. By Lemma 3.4, $M^* \in (SAI_R)^{\ominus}$ and hence $(SAI_R)^* \subseteq (SAI_R)^{\ominus}$. By hypothesis, every semi-artinian right ideal in $R$ is finitely generated, so that $\mathrm{Sa}(R_R)$ is noetherian. By Corollary 3.2, $SAI_R$ is a definable class.

(1) $\Rightarrow$ (4). By Corollary 3.2, $(SAI_R, (SAI_R)^{\ominus})$ is an almost dual pair and hence a module $M \in SAI_R$ iff $M^* \in (SAI_R)^{\ominus}$.

(4) $\Rightarrow$ (5). By hypothesis, $(SAI_R)^* \subseteq (SAI_R)^{\ominus}$. By Corollary 3.2, $(SAI_R)^{**} \subseteq SAI_R$. Hence for any module $M$, if $M \in SAI_R$, then $M^{**} \in SAI_R$.
Conversely, if $M^{**} \in SAI_R$, then $M^* \in (SAI_R)^{\ominus}$. By hypothesis, $M \in SAI_R$.

(5) $\Rightarrow$ (1). Let $N$ be a $FP$-injective module, thus there is a pure exact sequence $0 \to N \to E \to E/N \to 0$, where $E$ is an injective module. By [21, 34.5, p. 286], the sequence $0 \to N^{**} \to E^{**} \to (E/N)^{**} \to 0$ is split. By hypothesis, $E^{**} \in SAI_R$ and hence $N^{**} \in SAI_R$. By hypothesis, $N \in SAI_R$ so that $\mathrm{Sa}(R_R)$ is noetherian by Proposition 2.14. Thus $SAI_R$ is definable class by Corollary 3.2. □

Note that if the class $SAI_R$ is closed under pure submodules, then $(SAI_R)^+ = SAI_R$. Thus we have the following corollary.

**Corollary 3.6.** *The class $SAI_R$ is a definable if and only if it is closed under pure submodules and the class $(SAI_R)^+$ is a definable.*

**Corollary 3.7.** *If the class $SAI_R$ is a definable, then the following are equivalent.*
(1) *The class of flat left $R$-modules and the class $(SAI_R)^{\ominus}$ are coincide.*
(2) *Every module in $SAI_R$ is $FP$-injective.*
(3) *Every pure-injective module in $SAI_R$ is injective.*

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*Proof.* (1) ⇒ (2). Let $M \in SAI_R$, thus $M^* \in (SAI_R)^{\ominus}$ by Corollary 3.2. By hypothesis, $M^*$ is a flat left $R$-module and hence [10, Theorem, p. 239] implies that $M^{**}$ is injective. Since $M$ is a pure submodule in $M^{**}$, we have $M$ is $FP$-injective by [21, 35.8, p. 301].

(2) ⇒ (3). Let $M$ be any pure-injective module in $SAI_R$. Let $\mathcal{E} : 0 \to M \to N \to K \to 0$ be an exact sequence. By hypothesis, $M$ is $FP$-injective. By [17, Proposition 2.6], the sequence $\mathcal{E}$ is pure and hence pure-injectivity of $M$ implies that the sequence $\mathcal{E}$ is split by [21, 33.7, p. 279]. Therefore, $M$ is injective.

(3) ⇒ (1). Let $M$ be a flat left $R$-module, thus $\mathrm{Tor}_1(N, M) = 0$, for any right $R$-module $N$. By Lemma 3.4, $M \in (SAI_R)^{\ominus}$. Conversely, if $M \in (SAI_R)^{\ominus}$, then $M^* \in SAI_R$. By [16, Proposition 4.3.29, p. 149], $M^*$ is a pure injective module. By hypothesis, $M^*$ is injective and hence $M$ is flat by [10, Theorem, p. 239]. □

## 4. Relations between SA-injectivity and certain generalizations of injectivity

A right $R$-module $M$ is called quasi-injective if, for every submodule $N$ of $M$, every right $R$-homomorphism from $N$ to $M$ can be extended to a right $R$-endomorphism of $M$ [3, p. 169].

In general, if $M$ is $SA$-injective right $R$-module, then $M$ need not be quasi-injective, for example $\mathbb{Z}$ as $\mathbb{Z}$-module is $SA$-injective (by Example 2.2(1)) but it is not quasi-injective. Also, the converse is not true in general, for example in the ring $\mathbb{Z}_4$, the ideal $I =< \bar{2} >$ is a quasi-injective $\mathbb{Z}_4$-module but it is not $SA$-injective $\mathbb{Z}_4$-module.

The following theorem gives a relation between $SA$-injective modules and quasi-injective modules.

**Theorem 4.1.** *The following statements are equivalent for a ring $R$.*
(1) *$R$ is a right semi-artinian ring.*
(2) *Every $SA$-injective right $R$-module is injective.*
(3) *Every $SA$-injective right $R$-module is quasi-injective.*
(4) *Every cyclic $SA$-injective right $R$-module is quasi-injective.*

*Proof.* (1) ⇒ (2) Let $M$ be any $SA$-injective right $R$-module. Let $I$ be any right ideal of a ring $R$ and $f : I \to M$ be any right $R$-homomorphism. Since $R$ is a right semi-artinian ring (by hypothesis), it follows from [9, Exercise 7(8), p. 238] that $I$ is a semi-artinian right ideal of $R$. Since $M$ is an $SA$-injective right $R$-module (by hypothesis), $f$ extends to $R$ and hence $M$ is an injective right $R$-module.

(2) ⇒ (3) and (3) ⇒ (4) are clear.

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(4) ⇒ (1) Let $M$ be any nonzero cyclic right $R$-module. We will prove that $\mathrm{Soc}(M) \neq 0$. Assume that $\mathrm{Soc}(M) = 0$. Let $N$ be a nonzero submodule of $M$. Thus $\mathrm{Soc}(N) = 0$ and hence from Example 2.2(4) that $M$ and $N$ are $SA$-injective right $R$-modules. By Corollary 2.4, $N \oplus M$ is an $SA$-injective right $R$-module. By hypothesis, $N \oplus M$ is a quasi-injective right $R$-module. By [15, Proposition 1.17, p. 8], $N$ is an $M$-injective right $R$-module and hence $N$ is a direct summand of $M$. Thus $M$ is semisimple and hence $M = \mathrm{Soc}(M) = 0$ and this is a contradiction. Thus $\mathrm{Soc}(M) \neq 0$ for any nonzero cyclic right $R$-module $M$ and hence from [18, p. 183] we have that $R$ is a right semi-artinian ring. □

Since every left perfect ring is right semi-artinian [9, Theorem 11.6.3, p. 294], we have the following corollary immediately from Theorem 4.1.

**Corollary 4.2.** *If $R$ is a left perfect ring, then every $SA$-injective right $R$-module is injective (quasi-injective).*

In the following proposition, we give another connection between $SA$-injective modules and quasi-injective modules.

**Proposition 4.3.** *A commutative ring $R$ is semisimple if and only if $R$ is a semi-artinian ring and every quasi-injective $R$-module is $SA$-injective.*

*Proof.* (⇒) By Corollary 2.10.
(⇐) Let $M$ be any quasi-injective $R$-module. By hypothesis, $M$ is $SA$-injective. Since $R$ is a semi-artinian ring (by hypothesis), it follows from Theorem 4.1 that $M$ is injective and hence from [19, Corollary 2.2] we get that $R$ is a semisimple ring. □

The following corollary is immediately from Theorem 4.1 and Proposition 4.3.

**Corollary 4.4.** *The following statements are equivalent for a commutative ring $R$.*
(1) *$R$ is semisimple.*
(2) *For each $R$-module $M$, $M$ is $SA$-injective if and only if it is quasi-injective*
.

A right $R$-module $M$ is called P-injective (resp. F-injective) if, for every principally (resp. finitely generated) right ideal $I$ of $R$, every right $R$-homomorphism from $I$ to $M$ can be extended to a right $R$-homomorphism from $R$ into $M$ (see, for example [11] and [22]).

If $M$ is $SA$-injective right $R$-module, then $M$ need not be P-injective (resp. F-injective) in general, for example $\mathbb{Z}$ as $\mathbb{Z}$-module is $SA$-injective (by Example 2.2(1)) but it is not P-injective (resp. F-injective). Also, the converse is not true in general, for example: let $F = \mathbb{Z}_2$ be the field of two elements, $F_n = F$ for $n = 1, 2, ...$, $Q = \prod_{i=1}^{\infty} F_i$, $S = \oplus_{i=1}^{\infty} F_i$. If $R$ is the subring of $Q$ generated

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by 1 and $S$, then $R$ is a F-injective right $R$-module (by [1, Example 4.5]) and hence $R_R$ is a P-injective module. Thus Example 4.5 in [1] implies that $R$ is not a soc-injective right $R$-module and so $R$ is not a $SA$-injective module. Thus $R$ is F-injective (P-injective) right $R$-module but it is not $SA$-injective.

The following proposition gives a condition under which every F-injective right $R$-module is $SA$-injective.

**Proposition 4.5.** *Let $R$ be a ring. Then $Sa(R_R)$ is a noetherian right $R$-module if and only if every F-injective right $R$-module is $SA$-injective.*

*Proof.* ($\Rightarrow$) Let $M$ be any F-injective right $R$-module. Let $I$ be a semi-artinian right ideal of $R$ and let $f : I \to M$ be any right $R$-homomorphism. Since $\mathrm{Sa}(R_R)$ is noetherian and $I \subseteq \mathrm{Sa}(R_R)$, it follows that $I$ is a finitely generated right ideal. By F-injectivity of $M$, $f$ extends to a right $R$-homomorphism from $R$ into $M$ and hence $M$ is $SA$-injective.
($\Leftarrow$) Let $\{M_i\}_{i\in I}$ be a family of injective right $R$-modules. Thus $M_i$ are F-injective modules. By [22, Proposition 2.1(c)], $\bigoplus_{i\in I} M_i$ is an F-injective module. By hypothesis, $\bigoplus_{i\in I} M_i$ is a $SA$-injective module and hence from Proposition 2.14 we get that $\mathrm{Sa}(R_R)$ is a noetherian right $R$-module. □

Directly from Proposition 4.5 and Proposition 2.14, we have the following corollary.

**Corollary 4.6.** *Let $R$ be a ring. Then every F-injective right $R$-module is $SA$-injective if and only if every FP-injective right $R$-module is $SA$-injective.*

A ring $R$ is called (von Neumann) regular if for any $a \in R$, there is $b \in R$ such that $a = aba$ [9, p. 38].

**Proposition 4.7.** *The following statements are equivalent.*
(1) *$R$ is a (von Neumann) regular ring and every P-injective right $R$-module is $SA$-injective.*
(2) *$R$ is a (von Neumann) regular ring and $Sa(R_R)$ is a noetherian right $R$-module.*
(3) *Every $SA$-injective right $R$-module is P-injective and every semi-artinian right ideal of $R$ is a direct summand of $R_R$.*

*Proof.* (1) $\Rightarrow$ (2) Since every F-injective right $R$-module is P-injective, we have from hypothesis that every F-injective right $R$-module is $SA$-injective. By Proposition 4.5, $\mathrm{Sa}(R_R)$ is a noetherian right $R$-module.
(2) $\Rightarrow$ (3) Since $R$ is a (von Neumann) regular ring, it follows from [14, Lemma 2] that every $SA$-injective right $R$-module is P-injective. Let $I$ be any semi-artinian right ideal of $R$. Thus $I \subseteq \mathrm{Sa}(R_R)$. Since $\mathrm{Sa}(R_R)$ is a noetherian right $R$-module (by hypothesis), we have that $I$ is a finitely generated right ideal. By [9, Exercise 13, p. 38], $I$ is a direct summand of $R_R$.

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(3) $\Rightarrow$ (1) Since every semi-artinian right ideal of $R$ is a direct summand of $R_R$ (by hypothesis), it follows that from Corollary 2.8 that every right $R$-module is $SA$-injective and hence every P-injective right $R$-module is $SA$-injective. Since every $SA$-injective right $R$-module is P-injective (by hypothesis), we have that every right $R$-module is P-injective. By [14, Lemma 2], $R$ is a (von Neumann) regular ring. □

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