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Tracial Cyclic Rokhlin Property for Automorphisms of Non-unital Simple C*-algebras

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Abstract. The tracial cyclic Rokhlin property for automorphisms of simple not necessarily unital C*-algebras is investigated. We show that the tracial cyclic Rokhlin property is preserved by going to certain restrictions to subalgebras and taking direct limit or tensor products of actions. We also show that under certain conditions properties such as real rank zero, the tracial rank zero, stable rank one, (tracial) $\mathcal{Z}$-stability, Property (SP), strict comparison on projections are passed to crossed products under automorphisms with the tracial cyclic Rokhlin property.

Keywords: Tracial cyclic Rokhlin property, Crossed product, Tracial rank, Property (T_k).

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1. Introduction

The classical Rokhlin lemma in ergodic theory states that an aperiodic measure preserving dynamical system can be decomposed to an arbitrary high tower of measurable sets with a remainder of arbitrarily small measure. An analogue in the context of von Neumann algebras first appeared in the work of Alain Connes.

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Another definition of the Rokhlin property for finite group actions on unital C*-algebras along with a classification of finite group actions with Rokhlin property on some classes of unital C*-algebras was given by Izumi.

The notion gained its importance due to the related compatibility results for the nuclear dimension. However, examples of actions with the Rokhlin property are hard to find. In fact, many C*-algebras cannot admit such actions. On the other hand, some K-theoretical obstructions are imposed by the Rokhlin property. A tracial analogue of the Rokhlin property for finite group actions on simple unital C*-algebras was defined by Phillips [14] to deal with the obstructions mentioned above. A weak tracial Rokhlin property for actions on simple unital C*-algebras was defined and studied in [6].

Although the Rokhlin property was extended to the case of actions on non-unital C*-algebras, to the best of our knowledge, there is no study concerning the tracial Rokhlin property for automorphisms of simple non-unital C*-algebras.

This latter notion deserves further consideration, because of the importance of classes of simple non-unital C*-algebras (especially in the future of classification program) such as Razak-Jacelon algebras, non-unital Kirchberg algebras, and non-unital simple purely infinite C*-algebras. It is also important as the stably projectionless C*-algebras are attracting more attention in the Elliott's classification program (see [4]). A study of the actions (with the weak tracial Rokhlin property) on such algebras may also lead to further examples of classifiable C*-algebra crossed products.

In this paper, we define a notion of (weak) tracial cyclic Rokhlin property for automorphisms of simple not necessarily unital C*-algebras. Theorem 3.5 enables us to provide a broad range of examples of automorphisms with tracial cyclic Rokhlin property on not necessarily unital C*-algebras. Consider a unital UHF-algebra A with $K_0(A) = \mathbb{Q}$ and $\alpha \in \mathrm{Aut}(A)$ so that α^m is uniformly outer for all $m \neq 0$. According to [11, Theorem 3.5] α has the tracial cyclic Rokhlin property (in the sense of Lin and Osaka). Let B be a (not necessarily unital) simple C*-algebra with an approximate identity of projections. It follows from Theorem 3.5 that the automorphism $\alpha \otimes \mathrm{id} \in \mathrm{Aut}(A \otimes B)$ has the tracial cyclic Rokhlin property.

One of the main reasons to study a tracial version of the Rokhlin property was to investigate crossed products of simple unital C*-algebras with tracial rank zero by finite group actions. Lin defined the tracial rank for C*-algebras as a non-commutative analogue of the topological dimension. Golestani and Forough [5] investigated the tracial rank of non-unital C*-algebras and studied the crossed products of simple non-unital C*-algebras of tracial rank at most k by introducing *Property* (T_k) which covers both unital and non-unital cases in a unified manner.

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The notion of the tracial cyclic Rokhlin property was introduced by Lin and Osaka in [11]. It was proved that if A is a simple, unital C*-algebra with tracial rank zero and $\alpha \in \mathrm{Aut}(A)$ satisfies the tracial cyclic Rokhlin property, then $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) = 0$. As an application, the following conjecture of Kishimoto was proved: If A is a unital simple A$\mathbb{T}$-algebra of real rank zero and $\alpha \in \mathrm{Aut}(A)$ is approximately inner and satisfies the Rokhlin property, then $A \rtimes_\alpha \mathbb{Z}$ is again an A$\mathbb{T}$-algebra-algebra of real rank zero.

The paper is organized as follows. In Section 2, we define the tracial cyclic Rokhlin property for automorphisms of simple (not necessarily unital) C*-algebras and prove some basic results. In Section 3, we obtain some permanence properties, including passing to invariant (unital) hereditary C*-subalgebras, direct limits, and tensor products. In Section 4, we prove the properties of C*-algebras which are preserved under taking crossed products of automorphisms with the tracial cyclic Rokhlin property.

2. Tracial Cyclic Rokhlin Property

In this section, we introduce the notion of tracial cyclic Rokhlin property for automorphisms of simple not necessarily unital C*-algebras and study some of its basic properties. In the unital, simple, and finite case this notion coincides with that of Lin and Osaka [11, Definition 2.3] (cf. [1]).

Definition 2.1. Let A be a simple unital C*-algebra and let $\alpha \in \mathrm{Aut}(A)$. We say that α has the *tracial cyclic Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, every $n \in \mathbb{N}$, and every positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal projections $p_0, \ldots, p_n$ in A such that

(1) $\|\alpha(p_i) - p_{i+1}\| < \varepsilon$ for $0 \leq i \leq n$, where $p_{n+1} = p_0$.
(2) $\|p_i a - a p_i\| < \varepsilon$ for $0 \leq i \leq n$ and for all $a \in F$.
(3) With $p = \sum_{i=0}^{n} p_i$, $1 - p \precsim x$.
(4) $\|pxp\| > 1 - \varepsilon$.

We define the weak tracial cyclic Rokhlin property for automorphisms of simple not necessarily unital C*-algebras, similar to the non-unital tracial Rokhlin property.

Definition 2.2. Let A be a simple unital C*-algebra and let $\alpha \in \mathrm{Aut}(A)$. We say that α has the *weak tracial cyclic Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, every $n \in \mathbb{N}$, and every positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal positive contractions $e_0, \ldots, e_n$ in A such that

(1) $\|\alpha(e_i) - e_{i+1}\| < \varepsilon$ for $0 \leq i \leq n$, where $e_{n+1} = e_0$.
(2) $\|e_i a - a e_i\| < \varepsilon$ for $0 \leq i \leq n$ and for all $a \in F$.
(3) With $e = \sum_{i=0}^{n} e_i$, $1 - e \precsim x$.
(4) $\|exe\| > 1 - \varepsilon$.

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The non-unital analogues of these notions are defined as follows.

Definition 2.3. Let A be a simple (not necessarily unital) C*-algebra and let $\alpha \in \mathrm{Aut}(A)$. We say that α has the *tracial cyclic Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, every $n \in \mathbb{N}$, and every positive elements $x, y \in A$ with $\|x\| = 1$, there are mutually orthogonal projections $p_0, \ldots, p_n$ in A such that:

(1) $\|\alpha(p_i) - p_{i+1}\| < \varepsilon$ for $0 \leq i \leq n$, where $p_{n+1} = p_0$.
(2) $\|p_i a - a p_i\| < \varepsilon$ for $0 \leq i \leq n$ and all $a \in F$.
(3) With $p = \sum_{i=0}^{n} p_i$, $(y^2 - ypy - \varepsilon)_+ \precsim x$.
(4) $\|pxp\| > 1 - \varepsilon$.

The following definition is a slightly stronger version of the weak tracial Rokhlin property defined in [1, Definition 10.1].

Definition 2.4. Let A be a simple (not necessarily unital) C*-algebra and let $\alpha \in \mathrm{Aut}(A)$. We say that α has the weak tracial cyclic Rokhlin property if for every finite set $F \subset A$, every $\varepsilon > 0$, every $n \in \mathbb{N}$, and every non-zero positive elements $x, y \in A$ with $\|x\| = 1$, there are mutually orthogonal positive contractions $e_0, \ldots, e_n$ in A such that

(1) $\|\alpha(e_i) - e_{i+1}\| < \varepsilon$ for $0 \leq i \leq n$, where $e_{n+1} = e_0$.
(2) $\|e_i a - a e_i\| < \varepsilon$ for $0 \leq i \leq n$ and for all $a \in F$.
(3) With $e = \sum_{i=0}^{n} e_i$, $(y^2 - yey - \varepsilon)_+ \precsim x$.
(4) $\|exe\| > 1 - \varepsilon$.

Remark 2.5. Similar to the proof of [14, Lemma 1.16], it is easily seen that, if A is an infinite dimensional finite simple separable unital C*-algebra, and $\alpha \in \mathrm{Aut}(A)$ is an automorphism with the tracial cyclic Rokhlin property, Condition (4) could be omitted. Also, in non-unital case this condition is utilized with the purpose of the proof of pointwise outerness of α, (cf. Remark 2.12).

In the unital case, similar to the results of [5], Definition 2.1 is equivalent to Definition 2.3, and Definition 2.2 is equivalent to Definition 2.4. The main difference between Definitions 2.1–2.2 and Definitions 2.3–2.4 is Condition (3). Condition (3) implies that the projection $1 - p$ is small with respect to the Cuntz sub-equivalence relation.

In Examples 2.6, 2.7, and 2.8 we use a C*-algebra B and an automorphism $\beta \in \mathrm{Aut}(B)$ in which B^{β} has a not necessarily increasing approximate identity of projections for B. To see that such C*-algebras exist in abundance let A be a simple, separable C*-algebra with real rank zero (e.g., $K(l^2)$). Consider the C*-algebra $B = A \otimes A$ and the flip automorphism $\beta \in \mathrm{Aut}(B)$. As A has real rank zero, it has an approximate identity of projections $(p_i)_{i \in I}$. Set $q_i = p_i \otimes p_i$ for each $i \in I$. Then $(q_i)_{i \in I} \subseteq B^{\beta}$ is an approximate identity for B.

Now we provide examples of automorphisms with the tracial cyclic Rokhlin property of simple not necessarily unital C*-algebras.

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Example 2.6. Consider a unital UHF-algebra A with $K_0(A) = \mathbb{Q}$ and $\alpha \in \mathrm{Aut}(A)$ such that α^m is uniformly outer ([11, Definition 2.6]) for all $m \neq 0$. According to [11, Example 4.5] α has the tracial cyclic Rokhlin property (in the sense of Lin and Osaka). Let B be a (not necessarily unital), simple, separable, and purely infinite C*-algebra. It follows from Theorem 3.5 that the automorphism $\alpha \otimes \mathrm{id} \in \mathrm{Aut}(A \otimes B)$ has the tracial cyclic Rokhlin property.

Example 2.7. Let A be a unital, simple, separable C*-algebra and the order on projections in A is determined by traces ([16, Definition 1.3]). Let $\alpha \in \mathrm{Aut}(A)$ have the tracial Rokhlin property in which $\alpha^r \in \overline{\mathrm{Inn}(A)}$ for some $r \in \mathbb{N}$. Let B be a (not necessarily unital), simple, separable C*-algebra and $\beta \in \mathrm{Aut}(B)$ be an automorphism such that B^β has an approximate identity of projections. Then by Theorem 3.5 and [13, Theorem 4.4] the automorphism $\alpha \otimes \beta \in \mathrm{Aut}(A \otimes B)$ has the tracial cyclic Rokhlin property.

Example 2.8. Following [12, Remark 4.12], let $X = S^1$, $\theta \in \mathbb{R}\backslash\mathbb{Q}$, and $\varphi\colon X \to X$ where $\varphi(z) = (\exp 2\pi i\theta)z$. Theorem 4.10 and Remark 4.11 of [12] gives an automorphism α of a simple unital A$\mathbb{T}$-algebra A with the tracial cyclic Rokhlin property. Let B be a (not necessarily unital) simple separable C*-algebra and $\beta \in \mathrm{Aut}(B)$ be an automorphism in which B^β has an approximate identity of projections. Then by Theorem 3.5, the automorphism $\alpha \otimes \beta \in \mathrm{Aut}(A \otimes B)$ has the tracial cyclic Rokhlin property.

The next lemma will be used later. The proof is similar to that of [5, Lemma 3.3] and is omitted.

Lemma 2.9. *Let $\alpha \in \mathrm{Aut}(A)$ be an automorphism of a simple C*-algebra A. Let $x \in A_+$ with $\|x\| = 1$. Suppose that a positive element $y \in A$ has the following property. For every $\varepsilon > 0$, every $n \in \mathbb{N}$ and every finite subset $F \subseteq A$ there exist orthogonal positive contractions $\{e_i\colon 0 \leq i \leq n\}$ in A such that, with $e = \sum_{i=0}^{n} e_i$, the following hold:*

(1) $\|\alpha(e_i) - e_{i+1}\| < \varepsilon$ *for all* $0 \leq i \leq n$ *where* $e_{n+1} = e_0$.
(2) $\|e_i a - a e_i\| < \varepsilon$ *for all* $a \in F$ *and all* $0 \leq i \leq n$.
(3) $(y^2 - yey - \varepsilon)_+ \precsim x$.
(4) $\|exe\| > 1 - \varepsilon$.

Then every positive element $z \in \overline{Ay}$ also has the above property of y.

Remark 2.10. Let $\alpha \in \mathrm{Aut}(A)$ be an automorphism of a simple C*-algebra A.

(1) If the property stated in Definitions 2.4 and 2.3 holds for some $y \in A_+$ then it also holds for every positive element $z \in \overline{Ay}$ (by Lemma 2.9).
(2) If A is σ-unital, then α has the weak tracial cyclic Rokhlin property if *some* strictly positive element y in A has the property stated in Definitions 2.3 and 2.4. This follows from (1) and that $A = \overline{yAy} = \overline{Ay}$.
(3) In Definitions 2.3 and 2.4 if moreover, A is purely infinite then Condition (3) is redundant.

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In the following lemma, we give a (seemingly) stronger equivalent definition of automorphisms with the tracial cyclic Rokhlin property of simple C*-algebras. The proof is similar to that of [5, Lemma 3.7] and is omitted.

Lemma 2.11. *Let $\alpha \in \mathrm{Aut}(A)$ be an automorphism of a simple C*-algebra A. Then α has the tracial cyclic Rokhlin property if and only if the following holds. For every $\varepsilon > 0$, every finite subset $F \subseteq A$, every $n \in \mathbb{N}$, and every positive elements $x, y, z \in A$ with $x \neq 0$ and $\|z\| = 1$, there exist orthogonal projections $\{p_i : 0 \leq i \leq n\}$ in A such that, with $p = \sum_{i=0}^{n} p_i$, $\|\alpha(p) - p\| < \varepsilon$ and following hold:*

1. *$\|p_i a - a p_i\| < \varepsilon$ for all $a \in F$ and all $0 \leq i \leq n$.*
2. *$\|\alpha(p_i) - p_{i+1}\| < \varepsilon$ for all $0 \leq i \leq n$ where $p_{n+1} = p_0$.*
3. *$(y^2 - ypy - \varepsilon)_+ \precsim x$.*
4. *$\|pzp\| > 1 - \varepsilon$.*

The pointwise outerness could be considered as a form of freeness for actions on non-commutative C*-algebras. An action $\alpha \colon G \to \mathrm{Aut}(A)$ is called *pointwise outer* if for any $g \in G \setminus \{1\}$, the automorphism α_g is outer, i.e., is not of the form $\mathrm{Ad}\, u$ for any unitary u in the multiplier algebra of A.

Remark 2.12. Let α be an automorphism of a simple C*-algebra A. If α has the weak tracial cyclic Rokhlin property then α^m is outer for any $m \in \mathbb{Z} \setminus \{0\}$. Even in the case of the weak tracial Rokhlin property [1, Definition 8.1], this is proved in [1, Proposition 8.3].

By [8, Theorem 3.1] and Remark 2.12 we get the following result.

Proposition 2.13. *Let A be a simple C*-algebra and let $\alpha \in \mathrm{Aut}(A)$ have the (weak) tracial cyclic Rokhlin property. Then $A \rtimes_\alpha \mathbb{Z}$ is simple.*

Remark 2.14. (cf. Remark 3.4 of [9]) Let A be a simple C*-algebra, and $\alpha \in \mathrm{Aut}(A)$ be an automorphism with $\alpha^k = \mathrm{id}_A$ for some $k \in \mathbb{N}$, then α does not have the tracial cyclic Rokhlin property, since α is not pointwise outer.

3. Permanence Properties

In this section, we obtain some permanence properties, including passing to invariant (unital) hereditary C*-subalgebras, direct limits, and tensor products.

The proof of the following proposition is similar to the proof of [5, Proposition 4.13]. To avoid repetition, the proof is omitted.

Proposition 3.1. *Let A be a simple C*-algebra and $\alpha \in \mathrm{Aut}(A)$ be an automorphism with the weak tracial cyclic Rokhlin property. Let B be an α-invariant hereditary C*-subalgebra of A, and $\beta \in \mathrm{Aut}(B)$ be the restriction of α to B. Then β has the weak tracial cyclic Rokhlin property. If moreover, α has the tracial cyclic Rokhlin property and B is unital then β has the tracial Rokhlin property.*

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The following lemma follows immediately from Definition 2.4.

Lemma 3.2. *Let $\alpha \in \mathrm{Aut}(A)$ be an automorphism of a simple C*-algebra A. Suppose that for every finite set $F \subseteq A$ and every $\varepsilon > 0$ there is an α-invariant simple C*-subalgebra B of A such that $F \subseteq_{\varepsilon} B$ and the restriction of α to B has the (weak) tracial cyclic Rokhlin property. Then $\alpha \in \mathrm{Aut}(A)$ has the (weak) tracial cyclic Rokhlin property.*

The following corollary follows immediately from Lemma 3.2. See [15, Proposition 3.24] for the definition of the direct limit of actions.

Corollary 3.3. *Let $\left((\mathbb{Z}, A_i, \alpha^{(i)})_{i\in I}, (\varphi_{j,i})_{i\leq j}\right)$ be a directed system of simple $\mathbb{Z}$-algebras. Let A be the inductive limit of the A_i and let $\alpha \in \mathrm{Aut}(A)$ be the inductive limit of the $\alpha^{(i)}$. If each $\alpha^{(i)}$ has the (weak) tracial cyclic Rokhlin property then so does α.*

The next proposition gives a criterion for the non-unital weak tracial cyclic Rokhlin property in terms of the unital tracial cyclic Rokhlin property.

Proposition 3.4. *Let $\alpha \in \mathrm{Aut}(A)$ be an automorphism of a simple C*-algebra A. Suppose that A has an approximate identity $(p_i)_{i\in I}$ consisting of projections such that each p_i contains in A^{α}. Then $\alpha \in \mathrm{Aut}(A)$ has the (weak) tracial cyclic Rokhlin property if and only if the restriction of α to p_iAp_i has the (weak) tracial cyclic Rokhlin property for every $i \in I$.*

Proof. The "if" part follows from Lemma 3.2 and the "only if" part follows from Proposition 3.1. □

The next theorem enables us to make examples of automorphisms with the tracial cyclic Rokhlin property of simple not necessarily unital C*-algebras. The proof is similar to that of [5, Theorem 3.19] and is omitted.

Theorem 3.5. *Let $\alpha \in \mathrm{Aut}(A)$ and $\beta \in \mathrm{Aut}(B)$ be automorphisms of simple C*-algebras A and B. Let α has the (weak) tracial cyclic Rokhlin property and let B have a (not necessarily increasing) approximate identity of projections contained in B^{β}. Then the action $\alpha \otimes \beta \in \mathrm{Aut}(A \otimes B)$ has the (weak) tracial cyclic Rokhlin property.*

Corollary 3.6. *Let $\alpha \in \mathrm{Aut}(A)$ be an automorphism of a simple C*-algebra A. If α has the tracial cyclic Rokhlin property, then the induced automorphism of $M_n(A)$ has the tracial cyclic Rokhlin property for any $n \in \mathbb{N}$.*

Proof. This follows from Theorem 3.5 by taking $B = \mathbb{M}_n$ and β the trivial action.

□

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4. Crossed Product

In this section, we recall the notion of the tracial rank and Property (T_k) for simple C*-algebras (Definition 4.1), where k is a non-negative integer, which is studied in [5]. It is showed that under certain conditions properties such as real rank zero, the tracial rank zero, stable rank one, (tracial) $\mathcal{Z}$-stability, Property (SP), strict comparison on projections are passed to crossed products under automorphisms with the tracial cyclic Rokhlin property.

Definition 4.1 ([5]). Let A be a simple C*-algebra and let k be a non-negative integer. We say that A has *Property* (T_k), if A has an approximate identity (not necessarily increasing) consisting of projections and for every positive elements $x, y \in A$ with $x \neq 0$, every finite set $F \subseteq A$, and every $\varepsilon > 0$, there is a C*-subalgebra $E \subseteq A$ with $E \in \mathcal{I}^{(k)}$ such that, with $p = 1_E$, the following hold

(1) $\|pa - ap\| < \varepsilon$ for all $a \in F$.
(2) $pFp \subseteq_{\varepsilon} E$.
(3) $(y^2 - ypy - \varepsilon)_+ \precsim x$.
(4) $\|pxp\| > \|x\| - \varepsilon$.

We first need to recall Theorem 5.19 of [5].

Theorem 4.2 ([5]). *Let A be a simple (not necessarily unital) C*-algebra. Then A has Property* (T_0) *if and only if* $\mathrm{TR}(A) = 0$. *Also the tracial rank zero is preserved under the Morita equivalence in the class of simple C*-algebras.*

Proposition 4.3. *Let $\mathcal{C}$ be a class of simple (separable) C*-algebras satisfying the following conditions:*

(1) *if A is a simple (separable) C*-algebra and $p \in A$ is a non-zero projection, then $pAp \in \mathcal{C}$ if and only if $A \in \mathcal{C}$;*
(2) *if A is unital and $A \in \mathcal{C}$ and $\alpha \in \mathrm{Aut}(A)$ has the tracial cyclic Rokhlin property, then $A \rtimes_{\alpha} \mathbb{Z} \in \mathcal{C}$;*
(3) *if $A \in \mathcal{C}$ and B is a C*-algebra with $A \cong B$, then $B \in \mathcal{C}$.*

Then if $A \in \mathcal{C}$, $\alpha \in \mathrm{Aut}(A)$ has the tracial cyclic Rokhlin property, and A^{α} contains a non-zero projection, then $A \rtimes_{\alpha} \mathbb{Z} \in \mathcal{C}$.

Proof. Let $p \in A^{\alpha}$ be a non-zero projection. Let β be the restriction of α to pAp. We have $p(A \rtimes_{\alpha} \mathbb{Z})p \cong pAp \rtimes_{\beta} \mathbb{Z}$. By Proposition 3.1 and Condition (2), $pAp \rtimes_{\beta} \mathbb{Z} \in \mathcal{C}$. By Conditions (1) and (3), $A \rtimes_{\alpha} \mathbb{Z} \in \mathcal{C}$. □

Proposition 4.4. *Let A be a simple σ-unital tracially $\mathcal{Z}$-absorbing C*-algebra, $\alpha \in \mathrm{Aut}(A)$ be an automorphism with the tracial cyclic Rokhlin property, and $p \in A^{\alpha}$ be a nonzero projection. If $\beta \in \mathrm{Aut}(pAp)$ denotes the restriction of α to pAp and β^m acts trivially on $T(pAp)$ for some $m \in \mathbb{N}$, then $A \rtimes_{\alpha} \mathbb{Z}$ is tracially $\mathcal{Z}$-absorbing.*

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Proof. Denote by $\mathcal{C}$ the class of all simple σ-unital tracially $\mathcal{Z}$-absorbing C*-algebras. By [1], $\mathcal{C}$ satisfies Condition (1) of Proposition 4.3. By [6, Theorem 6.7], $\mathcal{C}$ satisfies Condition (2) of Proposition 4.3. Then According to Proposition 4.3, $A \rtimes_\alpha \mathbb{Z}$ belongs to $\mathcal{C}$. □

Theorem 4.5. *Let A be a simple separable nuclear $\mathcal{Z}$-absorbing C*-algebra, $\alpha \in \mathrm{Aut}(A)$ be an automorphism with the tracial cyclic Rokhlin property, and A^α contains a non-zero projection p. Let $\beta \in \mathrm{Aut}(pAp)$ denote the restriction of α to pAp. If the extreme boundary of the trace space of pAp is compact and finite dimensional, and that β fixes any tracial state of pAp, then $A \rtimes_\alpha \mathbb{Z}$ is $\mathcal{Z}$-absorbing.*

Proof. Denote by $\mathcal{C}$ the class of all simple separable nuclear $\mathcal{Z}$-absorbing C*-algebras. Remark 2.12 implies that β is pointwise outer. According to [17, Theorem 1.1], $pAp \rtimes_\beta \mathbb{Z} \cong p(A \rtimes_\alpha \mathbb{Z})p$ is $\mathcal{Z}$-absorbing. Since $\mathcal{Z}$-stability is preserved under Morita equivalence in the class of separable C*-algebras by [18, Corollary 3.2], we conclude that $\mathcal{C}$ satisfies Conditions (1) and (2) of Proposition 4.3. Then According to Proposition 4.3, $A \rtimes_\alpha \mathbb{Z}$ belongs to $\mathcal{C}$. □

Proposition 4.6. *Let A be a separable simple nuclear $\mathcal{Z}$-absorbing C*-algebra. Let $\alpha \in \mathrm{Aut}(A)$ be an automorphism with the tracial cyclic Rokhlin property, A^α contains a non-zero projection p, and pAp posses a unique tracial state. If $\beta \in \mathrm{Aut}(pAp)$ is the restriction of α to pAp and the trace space of $pAp \rtimes_\beta \mathbb{Z}$ is a Bauer simplex, then the nuclear dimension of $A \rtimes_\alpha \mathbb{Z}$ is at most one.*

Proof. Remark 2.12 implies that β is pointwise outer, so by [17, Corollary 1.2], the nuclear dimension of $pAp \rtimes_\beta \mathbb{Z}$ is at most one. Since $pAp \rtimes_\beta \mathbb{Z} \cong p(A \rtimes_\alpha \mathbb{Z})p$ is Morita equivalent to $A \rtimes_\alpha \mathbb{Z}$, [19, Corollary 2.8] implies that the nuclear dimension of $A \rtimes_\alpha \mathbb{Z}$ is at most one. □

The following results extend Theorem [9, Theorem 3.4] and [9, Theorem 4.5] to the non-unital case. The notions are the same as the ones used in [9]. The readers are referred to [10] for further details about $KL(A, A)$ for a C*-algebra A.

Proposition 4.7. *Let A be a simple separable nuclear C*-algebra with the tracial rank zero. Let $\alpha \in \mathrm{Aut}(A)$ be an automorphism with the tracial cyclic Rokhlin property, and let A^α contain a non-zero projection p. Assume that pAp satisfies the UCT and $\beta \in \mathrm{Aut}(pAp)$, the restriction of α to pAp, satisfies $[\beta^k] = [\mathrm{id}_{pAp}]$ in $KL(pAp, pAp)$ for some $k \in \mathbb{N}$. Then $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) = 0$.*

Proof. According to [9, Theorem 3.4], $pAp \rtimes_\beta \mathbb{Z}$ has the tracial rank zero. Since $pAp \rtimes_\beta \mathbb{Z} \cong p(A \rtimes_\alpha \mathbb{Z})p$ is Morita equivalent to $A \rtimes_\alpha \mathbb{Z}$, Corollary 5.26 of [5] implies that $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) = 0$. □

As noted in [9], the tracial state space of a unital C*-algebra A is denoted by $T(A)$. The normed space of all real affine continuous functions on $T(A)$, is

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denoted by $\mathrm{Aff}(T(A))$. Denote by $\rho_A\colon K_0(A) \to \mathrm{Aff}(T(A))$ the homomorphism induced by $\rho_A([p])(\tau) = \tau(p)$ for $\tau \in T(A)$.

Proposition 4.8. *Let A be a simple separable nuclear C*-algebra with the tracial rank zero, and $\alpha \in \mathrm{Aut}(A)$ be an automorphism with the tracial cyclic Rokhlin property. Assume that A^α contains a non-zero projection p, and $\beta \in \mathrm{Aut}(pAp)$ be the restriction of α to pAp. Suppose also that, pAp satisfies the UCT and there is $r > 0$ such that $\beta^r_{*0}|_G = \mathrm{id}_G$ for some subgroup $G \subseteq K_0(pAp)$ for which $\rho_{pAp}(G)$ is dense in $\rho_{pAp}(K_0(pAp))$. Then $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) = 0$.*

Proof. According to [9, Theorem 4.5], $pAp \rtimes_\beta \mathbb{Z}$ has the tracial rank zero. Since $pAp \rtimes_\beta \mathbb{Z} \cong p(A \rtimes_\alpha \mathbb{Z})p$ is Morita equivalent to $A \rtimes_\alpha \mathbb{Z}$, Corollary 5.26 of [5] implies that $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) = 0$. □

We give the following example to show that the assumptions of Proposition 4.7 may be satisfied.

Example 4.9. Let A be a simple, not necessarily unital, separable AF algebra, and $\alpha \in \mathrm{Aut}(A)$ be an automorphism with the tracial cyclic Rokhlin property. Let $p \in A^\alpha$ be a non-zero projection and pAp admits a unique trace, say τ. Let $\beta \in \mathrm{Aut}(pAp)$ be the restriction of α to pAp. Then $\tau \circ \mathrm{id}_{pAp} = \tau \circ \beta^k$ for each $k \in \mathbb{N}$. Then we have $[\beta^k] = [\mathrm{id}_{pAp}]$ in $KL(pAp, pAp)$. Now consider AF algebras containing an approximate identity of projections, $\mathcal{U}$, in which, every automorphism $\beta \in \mathrm{Aut}(pAp)$ is approximately inner automorphism, for each $p \in \mathcal{U}$ (e.g., $M_{2^\infty} \otimes K(l^2)$ possess $(1 \otimes p_i)_{i\in\mathbb{N}}$, as an approximate identity, where $(p_i)_{i\in\mathbb{N}}$ is an approximate identity for $K(l^2)$. Since any automorphism of a UHF-algebra is approximately inner, each automorphism of $(1 \otimes p_i)(M_{2^\infty} \otimes K(l^2))(1 \otimes p_i)$ is approximately inner.) Then by [9, Remark 3.5] we have $[\alpha_p] = [\mathrm{id}_{pAp}]$ in $KL(pAp, pAp)$ for all $p \in \mathcal{U}$, where α_p is the restriction of α to pAp.

Definition 4.10. ([7, Definition 2.2]) We say that a C*-algebra A has the *strict comparison of projections*, if $T(A)$ is non-empty and for any two projections p and q in A, the strict inequalities $\tau(p) < \tau(q)$ for all $\tau \in T(A)$ implies that $p \prec q$, namely $[p] < [q]$, the strict ordering induced by the Murray–von Neumann equivalence of projections.

Definition 4.11. ([16, Definition 1.3]) Let A be a unital C*-algebra. We say that *the order on projections over A is determined by traces* if, whenever $p, q \in M_\infty(A)$ are projections such that $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, then $p \prec q$.

Proposition 4.12. *Let A be a simple separable stably finite C*-algebra of real rank zero and $\alpha \in \mathrm{Aut}(A)$ be an automorphism with the tracial cyclic Rokhlin property. Let $p \in A^\alpha$ be a non-zero projection such that the order on projections*

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over pAp is determined by traces. Then $A \rtimes_\alpha \mathbb{Z}$ *has the strict comparison of projections.*

Proof. Let $\beta \in \mathrm{Aut}(pAp)$ be the restriction of α to pAp. It follows from [16, Theorem 3.5] that the order on projections over $pAp \rtimes_\beta \mathbb{Z}$ is determined by traces. Thus $pAp \rtimes_\beta \mathbb{Z}$ has the strict comparison of projections. Since $A \rtimes_\alpha \mathbb{Z}$ and $pAp \rtimes_\beta \mathbb{Z}$ are separable and Morita equivalent, they are stably isomorphic. Hence $(pAp \rtimes_\beta \mathbb{Z}) \otimes K(l^2) \cong (A \rtimes_\alpha \mathbb{Z}) \otimes K(l^2)$. Therefore by [7, Lemma 2.4], $A \rtimes_\alpha \mathbb{Z}$ has the strict comparison of projections. □

Proposition 4.13. *Let A be a simple separable stably finite C*-algebra with real rank zero and let* $\alpha \in \mathrm{Aut}(A)$ *be an automorphism with the tracial cyclic Rokhlin property. If* $p \in A^\alpha$ *is a non-zero projection and the order of projections over pAp is determined by traces, then* $A \rtimes_\alpha \mathbb{Z}$ *has real rank zero.*

Proof. Let $\beta \in \mathrm{Aut}(pAp)$ be the restriction of α to pAp. It follows from [16, Theorems 4.5] that $p(A \rtimes_\alpha \mathbb{Z})p \cong pAp \rtimes_\beta \mathbb{Z}$ has real rank zero. Since real rank zero is preserved under the Morita equivalence ([3, Theorem 3.8]), $A \rtimes_\alpha \mathbb{Z}$ has real rank zero. □

Proposition 4.14. *Let A be a simple separable C*-algebra with real rank zero and stable rank one. Let* $\alpha \in \mathrm{Aut}(A)$ *be an automorphism with the tracial cyclic Rokhlin property. Let* $p \in A^\alpha$ *be a non-zero projection and the order of projections over pAp is determined by traces. Then* $A \rtimes_\alpha \mathbb{Z}$ *has stable rank one.*

Proof. Let $\beta \in \mathrm{Aut}(pAp)$ be the restriction of α to pAp. It follows from [16, Theorems 5.3], that $p(A \rtimes_\alpha \mathbb{Z})p \cong pAp \rtimes_\beta \mathbb{Z}$ has stable rank one. By [2, Corollary 4.6], $\mathrm{tsr}(A \rtimes_\alpha \mathbb{Z}) \leq \mathrm{tsr}(pAp \rtimes_\beta \mathbb{Z})$. Hence $A \rtimes_\alpha \mathbb{Z}$ has stable rank one. □

Proposition 4.15. *Let A be a simple C*-algebra with Property (SP). Let* $\alpha \in \mathrm{Aut}(A)$ *be an automorphism with the weak tracial Rokhlin property. Then* $A \rtimes_\alpha \mathbb{Z}$ *has Property (SP).*

Proof. It follows from [5, Lemma 3.12] and Remark 2.12 that $A \rtimes_\alpha \mathbb{Z}$ has Property (SP). □

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References

1. M. Amini, N. Golestani, S. Jamali, N. C. Phillips, Simple Tracially $\mathcal{Z}$-absorbing C*-algebras, under preparation.

[DOI: 10.61186/ijmsi.19.1.51]

2. B. Blackadar, The Stable Rank of Full Corners in C*-algebras, *Proc. Amer. Math. Soc.*, **132**, (2004), 2945–2950.
3. L. G. Brown, G. K. Pedersen, C*-algebras of Real Rank Zero, *J. Funct. Anal.*, **99**, (1991), 131–149.
4. G. A. Elliott, G. Gong, H. Lin, Z. Niu, Simple Stably Projectionless C*-algebras with Generalized Tracial Rank One, arXiv:1711.01240, 2017.
5. M. Forough, N. Golestani, Tracial Rokhlin Property for Finite Group Action on Non-unital Simpe C*-algebras, arXiv:1711.10818v1, 2017.
6. I. Hirshberg, J. Orovitz, Tracially $\mathcal{Z}$-absorbing C*-algebras, *J. Func. Anal.*, **265**, (2013), 765–785.
7. V. Kaftal, P. W. Ng, S. Zhang, Commutators and Linear Spans of Projections in Certain Finite C*-algebras, *J. Funct. Anal.*, **266**, (2014), 1883–1912.
8. A. Kishimoto, Outer Automorphisms and Reduced Crossed Products of Simple C*-algebras, *Commun. Math. Phys.*, **81**, (1981), 429–435.
9. H. Lin, The Rokhlin Property for Automorphisms on Simple C*-algebras, arXiv:math/0602513, 2010.
10. H. Lin, Z. Niu, Lifting KK-elements, Asymptotic Unitary Equivalence and Classification of Simple C*-algebras, *Adv. Math.*, **219**, (2008), 1729–1769.
11. H. Lin, H. Osaka, The Rokhlin Property and the Tracial Topological Rank, *J. Funct. Anal.*, **218**, (2005), 475–494.
12. H. Lin, H. Osaka, Tracial Rokhlin Property for Automorphisms on Simple A$\mathbb{T}$-algebras, *Ergod. Theory Dyn. Syst.*, **28**, (2008), 1215–1241.
13. H. Matui, $\mathbb{Z}$-Actions on AH-Algebras and $\mathbb{Z}^2$-Actions on AF Algebras, *Comm. Math. Phys.*, **297**, (2010), 529–551.
14. N. C. Phillips, The Tracial Rokhlin Property for Actions of Finite Groups on C*-algebras, *Amer. J. Math.*, **133**(3), (2011), 581–636.
15. N. C. Phillips, An Introduction to Crossed Product C*-algebras and Minimal Dynamics, preprint, 2017.
16. N. C. Phillips, H. Osaka, Stable and Real Rank for Crossed Products by Automorphisms with the Tracial Rokhlin Property, *Ergod. Theory Dyn. Syst.*, **26**, (2006), 1579–1621.
17. Y. Sato, Actions of Amenable Groups and Crossed Products of $\mathcal{Z}$-absorbing C*-algebras, arXiv:1612.08529.
18. A. S. Toms, W. Winter, Strongly Self-absorbing C*-algebras, *Trans. Amer. Math. Soc.*, **359**, (2007), 3999–4029.
19. W. Winter, J. Zacharias, The Nuclear Dimension of C*-algebras, *Adv. Math.*, **224**, (2010), 461–498.

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