# Coincidence Quasi-Best Proximity Points for Quasi-Cyclic-Noncyclic Mappings in Convex Metric Spaces 

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#### Abstract

We introduce the notion of quasi-cyclic-noncyclic pair and its relevant new notion of coincidence quasi-best proximity points in a convex metric space. In this way we generalize the notion of coincidence-best proximity point already introduced by M. Gabeleh et al [14]. It turns out that under some circumstances this new class of mappings contains the class of cyclic-noncyclic mappings as a subclass. The existence and convergence of coincidence-best and coincidence quasi-best proximity points in the setting of convex metric spaces are investigated.


Keywords: Coincidence-best proximity point, Cyclic-noncyclic contraction, Quasi-cyclic-noncyclic contraction, Uniformly convex metric space.

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## 1. Introduction

Let $(X, d)$ be a metric space, and let $A, B$ be subsets of $X$. A mapping $T: A \cup B \rightarrow A \cup B$ is said to be cyclic provided that $T(A) \subseteq B$ and $T(B) \subseteq A$; similarly, a mapping $S: A \cup B \rightarrow A \cup B$ is said to be noncyclic if $S(A) \subseteq A$ and $S(B) \subseteq B$. The following theorem is an extension of Banach contraction principle.

[^0]Theorem 1.1. ([18]) Let $A$ and $B$ be nonempty closed subsets of a complete metric space $(X, d)$. Suppose that $T$ is a cyclic mapping such that

$$
d(T x, T y) \leq \alpha d(x, y)
$$

for some $\alpha \in(0,1)$ and for all $x \in A, y \in B$. Then $T$ has a unique fixed point in $A \cap B$.

Let $A$ and $B$ be nonempty subsets of a metric space $X$. A mapping $T$ : $A \cup B \rightarrow A \cup B$ is said to be a cyclic contraction if $T$ is cyclic and

$$
d(T x, T y) \leq \alpha d(x, y)+(1-\alpha) \operatorname{dist}(A, B)
$$

for some $\alpha \in(0,1)$ and for all $x \in A, y \in B$, where

$$
\operatorname{dist}(A, B):=\inf \{d(x, y):(x, y) \in A \times B\}
$$

For a cyclic mapping $T: A \cup B \rightarrow A \cup B$, a point $x \in A \cup B$ is said to be a best proximity point provided that

$$
d(x, T x)=\operatorname{dist}(A, B)
$$

The following existence, uniqueness and convergence result of a best proximity point for cyclic contractions is the main result of [8].

Theorem 1.2. ([8]) Let $A$ and $B$ be nonempty closed convex subsets of $a$ uniformly convex Banach space $X$ and let $T: A \cup B \rightarrow A \cup B$ be a cyclic contraction map. For $x_{0} \in A$, define $x_{n+1}:=T x_{n}$ for each $n \geq 0$. Then there exists a unique $x \in A$ such that $x_{2 n} \rightarrow x$ and

$$
\|x-T x\|=\operatorname{dist}(A, B)
$$

In the theory of best proximity points, one usually considers a cyclic mapping $T$ defined on the union of two (closed) subsets of a given metric space. Here the objective is to minimize the expression $d(x, T x)$ where $x$ runs through the domain of $T$; that is $A \cup B$. In other words, we want to find

$$
\min \{d(x, T x): x \in A \cup B\} .
$$

If $A$ and $B$ intersect, the solution is clearly a fixed point of $T$; otherwise we have

$$
d(x, T x) \geq \operatorname{dist}(A, B), \quad \forall x \in A \cup B
$$

so that the point at which the equality occurs is called a best proximity point of $T$. This point of view dominates the literature.

Very recently, M. Gabeleh, O. Olela Otafudu, and N. Shahzad [14] considered two mappings $T$ and $S$ simultaneously and established interesting results. For technical reasons, the first map should be cyclic and the second one should be noncyclic. According to [14], for a nonempty pair of subsets $(A, B)$, and a cyclic-noncyclic pair $(T ; S)$ on $A \cup B$ (that is, $T: A \cup B \rightarrow A \cup B$ is cyclic and
$S: A \cup B \rightarrow A \cup B$ is noncyclic); they called a point $p \in A \cup B$ a coincidence best proximity point for $(T ; S)$ provided that

$$
d(S p, T p)=\operatorname{dist}(A, B)
$$

Note that if $S=I$, the identity map on $A \cup B$, then $p \in A \cup B$ is a best proximity point for $T$. Also, if $\operatorname{dist}(A, B)=0$, then $p$ is called a coincidence point for $(T ; S)$ (see [12] and [15] for more information). With the definition just given, and depending on the situation as to whether $S$ equals the identity map, or if the distance between the underlying sets is zero, one obtains a best proximity point for $T$, or a coincidence point for the pair $(T ; S)$. This was in fact the philosophy behind the phrase coincidence-best proximity point coined by Gabeleh et al. They then defined the notion of a cyclic-noncyclic contraction.

Definition 1.3. ([14]) Let $(A, B)$ be a nonempty pair of subsets of a metric space $(X, d)$ and $T, S: A \cup B \rightarrow A \cup B$ be two mappings. The pair $(T ; S)$ is called a cyclic-noncyclic contraction pair if it satisfies the following conditions:
(1) $(T ; S)$ is a cyclic-noncyclic pair on $A \cup B$.
(2) For some $r \in(0,1)$ we have

$$
d(T x, T y) \leq r d(S x, S y)+(1-r) \operatorname{dist}(A, B), \forall(x, y) \in A \times B
$$

To state the main result of [14], we need to recall the notion of convexity in the framework of metric spaces. In [26], Takahashi introduced the notion of convexity in metric spaces as follows (see also [24]).

Definition 1.4. Let $(X, d)$ be a metric space and $I:=[0,1]$. A mapping $\mathcal{W}: X \times X \times I \rightarrow X$ is said to be a convex structure on $X$ provided that for each $(x, y ; \lambda) \in X \times X \times I$ and $u \in X$,

$$
d(u, \mathcal{W}(x, y ; \lambda)) \leq \lambda d(u, x)+(1-\lambda) d(u, y)
$$

A metric space $(X, d)$ together with a convex structure $\mathcal{W}$ is called a convex metric space, and is denoted by $(X, d, \mathcal{W})$. A Banach space and each of its convex subsets are convex metric spaces.

A subset $K$ of a convex metric space $(X, d, \mathcal{W})$ is said to be a convex set provided that $\mathcal{W}(x, y ; \lambda) \in K$ for all $x, y \in K$ and $\lambda \in I$.

Similarly, a convex metric space $(X, d, \mathcal{W})$ is said to be uniformly convex if for any $\varepsilon>0$, there exists $\alpha=\alpha(\varepsilon)$ such that for all $r>0$ and $x, y, z \in X$ with $d(z, x) \leq r, d(z, y) \leq r$ and $d(x, y) \geq r \varepsilon$,

$$
d\left(z, \mathcal{W}\left(x, y ; \frac{1}{2}\right)\right) \leq r(1-\alpha)<r
$$

For example every uniformly convex Banach space is a uniformly convex metric space.

Definition 1.5. ([14]) Let $(A, B)$ be a nonempty pair of subsets of a metric space $(X, d)$. A mapping $S: A \cup B \rightarrow A \cup B$ is said to be a relatively antiLipschitzian mapping if there exists $c>0$ such that

$$
d(x, y) \leq c d(S x, S y), \forall(x, y) \in A \times B
$$

The main result of M. Gabeleh et al reads as follows:
Theorem 1.6. ([14]) Let $(A, B)$ be a nonempty, closed pair of subsets of a complete uniformly convex metric space $(X, d, \mathcal{W})$ such that $A$ is convex. Let $(T ; S)$ be a cyclic-noncyclic contraction pair defined on $A \cup B$ such that $T(A) \subseteq$ $S(B)$ and $T(B) \subseteq S(A)$ and that $S$ is continuous on $A$ and relatively antiLipschitzian on $A \cup B$. Then $(T ; S)$ has a coincidence best proximity point in $A$. Further, if $x_{0} \in A$ and $S x_{n+1}:=T x_{n}$, then $\left(x_{2 n}\right)$ converges to the coincidence-best proximity point of $(T ; S)$.

Existence of best proximity pairs was first studied in [9] by using a geometric property on a nonempty pair of subsets of a Banach space, called proximal normal structure, for noncyclic relatively nonexpansive mappings (Theorem 2.2 of [9]). Some existence results of best proximity pairs can be found in $[1,2,5,6,7,10,11,13,17,23,25]$.

In the current paper, we study sufficient conditions which ensure the existence and convergence of coincidence-best and quasi-best proximity point for a pair of quasi-cyclic-noncyclic contraction mappings in the setting of convex metric spaces.

## 2. Coincidence quasi-Best proximity point

In this section, we introduce the class of quasi-cyclic-noncyclic mappings that contains the class of cyclic-noncyclic mappings as a subclass. Next, we introduce the new notion of quasi-best proximity points for this mappings. Finally, we study the existence and convergence of coincidence quasi-best proximity points for quasi-cyclic-noncyclic contraction mappings in the setting of convex metric spaces.

Definition 2.1. Let $(A, B)$ be a nonempty pair of subsets of a metric space $(X, d)$ and $T, S: X \rightarrow X$ be two mappings. The pair $(T ; S)$ is called a quasi-cyclic-noncyclic $(\mathbf{Q C N})$ contraction pair if it satisfies the following conditions:
(1) $(T ; S)$ is a quasi-cyclic-noncyclic pair on $X$; that is,

$$
T(A) \subseteq S(B), T(B) \subseteq S(A)
$$

(2) For some $\alpha \in(0,1)$ and for each $(x, y) \in A \times B$ we have

$$
d(T x, T y) \leq \alpha d(S x, S y)+(1-\alpha) \operatorname{dist}(S(A), S(B))
$$

Note that if $S(A)=A$ and $S(B)=B$, then the above definition reduces to Definition 1.3; that is, the pair $(T ; S)$ is a cyclic-noncyclic pair.

Example 2.2. Let $X:=\mathbb{R}$ with the usual metric. For $A=(-\infty,-1]$ and $B=[1,+\infty)$ define $T, S: X \rightarrow X$ by

$$
T x:=\left\{\begin{array}{l}
-x, \text { if } x \in A \cup B \\
0, \text { ow. }
\end{array} \quad \text { and } \quad S x:=\left\{\begin{array}{l}
2 x+1, \text { if } x \in A \\
2 x-1, \text { if } x \in B \\
0, \text { ow }
\end{array}\right.\right.
$$

Then $(T ; S)$ is a QCN contraction pair with $\alpha=\frac{1}{2}$. Indeed, for all $(x, y) \in A \times B$ we have

$$
\begin{aligned}
|T x-T y| & =(y-x) \leq \frac{1}{2}(2 y-2 x-2)+\frac{1}{2}(2) \\
& =\alpha|S x-S y|+(1-\alpha) \operatorname{dist}(S(A), S(B))
\end{aligned}
$$

Also, $T(A)=B \subseteq S(B)$ and $T(B)=A \subseteq S(A)$.
The next example shows that there is a QCN mapping that is not a cyclicnoncyclic mapping.

Example 2.3. Let $X:=\mathbb{R}$ with the usual metric. For $A=(-\infty,-1]$ and $B=[1,+\infty)$ define $T, S: X \rightarrow X$ by

$$
T x:=\left\{\begin{array}{l}
-x, \text { if } x \in A \cup B \\
0, \text { ow. }
\end{array} \quad \text { and } \quad S x:=\left\{\begin{array}{l}
x+1, \text { if } x \in A \\
x-1, \text { if } x \in B \\
0, \text { ow } .
\end{array}\right.\right.
$$

Then $(T ; S)$ is a quasi-cyclic-noncyclic pair that is not a cyclic-noncyclic pair.
Remark 2.4. Notice that (2) implies that

$$
d(T x, T y) \leq d(S x, S y), \forall(x, y) \in A \times B
$$

Moreover, if $S$ is a noncyclic relatively nonexpansive mapping; meaning that

$$
d(S x, S y) \leq d(x, y), \forall(x, y) \in A \times B
$$

then $T$ is a cyclic contraction. In addition, if in the above definition $S$ is assumed to be continuous, then $T$ would be continuous too.

Definition 2.5. Let $(A, B)$ be a nonempty pair of subsets of a metric space $(X, d)$ and $T, S: X \rightarrow X$ be a quasi-cyclic-noncyclic pair on $X$. A point $p \in A \cup B$ is said to be a coincidence quasi-best proximity point for $(T ; S)$ provided that

$$
d(S p, T p)=\operatorname{dist}(S(A), S(B))
$$

Note that if $S=I$, then $p$ reduces to a coincidence-best proximity point for $(T ; S)$.

To prove the main result of this section, we need some preparations.

Lemma 2.6. Let $(A, B)$ be a nonempty pair of subsets of a metric space $(X, d)$ and let $(T ; S)$ be a quasi-cyclic-noncyclic pair defined on $X$. Then there exists a sequence $\left\{x_{n}\right\}$ in $X$ such that for all $n \geq 0$ we have $T x_{n}=S x_{n+1}$ where $\left\{x_{2 n}\right\},\left\{x_{2 n+1}\right\}$ are subsequences in $A$ and $B$ respectively.

Proof. Let $x_{0} \in A$. Since $T x_{0} \in S(B)$, there exists $x_{1} \in B$ such that $T x_{0}=$ $S x_{1}$. Again, since $T x_{1} \in S(A)$, there exists $x_{2} \in A$ such that $T x_{1}=S x_{2}$.

Continuing this process, we obtain a sequence $\left\{x_{n}\right\}$, such that $\left\{x_{2 n}\right\},\left\{x_{2 n+1}\right\}$ are in $A$ and $B$ respectively and $T x_{n}=S x_{n+1}$ for all $n \in \mathbb{N} \cup\{0\}$.

Lemma 2.7. Let $(A, B)$ be a nonempty pair of subsets of a metric space $(X, d)$ and let $(T ; S)$ be a $Q C N$ contraction pair defined on $X$. For $x_{0} \in A$, define $T x_{n}=S x_{n+1}$ for each $n \geq 0$. Then we have

$$
d\left(S x_{2 n}, S x_{2 n+1}\right) \rightarrow \operatorname{dist}(S(A), S(B))
$$

Proof.

$$
\begin{aligned}
d\left(S x_{2 n+1}, S x_{2 n+2}\right) & =d\left(T x_{2 n}, T x_{2 n+1}\right) \\
& \leq \alpha d\left(S x_{2 n}, S x_{2 n+1}\right)+(1-\alpha) \operatorname{dist}(S(A), S(B)) \\
& =\alpha d\left(T x_{2 n-1}, T x_{2 n}\right)+(1-\alpha) \operatorname{dist}(S(A), S(B)) \\
& \leq \alpha\left[\alpha d\left(S x_{2 n-1}, S x_{2 n}\right)+(1-\alpha) \operatorname{dist}(S(A), S(B))\right] \\
& +(1-\alpha) \operatorname{dist}(S(A), S(B)) \\
& =\alpha^{2} d\left(S x_{2 n-1}, S x_{2 n}\right)+\left(1-\alpha^{2}\right) \operatorname{dist}(S(A), S(B)) \\
& =\alpha^{2} d\left(T x_{2 n-2}, T x_{2 n-1}\right)+\left(1-\alpha^{2}\right) \operatorname{dist}(S(A), S(B)) \\
& \leq \cdots \\
& \leq \alpha^{2 n} d\left(T x_{0}, T x_{1}\right)+\left(1-\alpha^{2}\right) \operatorname{dist}(S(A), S(B))
\end{aligned}
$$

Now, if $n \rightarrow \infty$ in above relation, we conclude that

$$
d\left(S x_{2 n}, S x_{2 n+1}\right) \rightarrow \operatorname{dist}(S(A), S(B))
$$

Theorem 2.8. Let $(A, B)$ be a nonempty pair of subsets of a metric space $(X, d)$ and let $(T ; S)$ be a $Q C N$ contraction pair defined on $X$. Assume that $S$ is continuous on $A$. For $x_{0} \in A$, define $T x_{n}=S x_{n+1}$ for each $n \geq 0$. If $\left\{x_{2 n}\right\}$ has a convergent subsequence in $A$, then the pair $(T ; S)$ has a coincidence quasi-best proximity point in $A$.

Proof. Let $\left\{x_{2 n_{k}}\right\}$ be a subsequence of $\left\{x_{2 n}\right\}$ such that $x_{2 n_{k}} \rightarrow p \in A$. We have

$$
\begin{aligned}
\operatorname{dist}(S(A), S(B)) & \leq d\left(T x_{2 n_{k}-1}, T p\right) \leq d\left(S x_{2 n_{k}-1}, S p\right) \\
& \leq d\left(S p, S x_{2 n_{k}}\right)+d\left(S x_{2 n_{k}}, S x_{2 n_{k}-1}\right)
\end{aligned}
$$

By Lemma 2.7, if $k \rightarrow \infty$, we obtain that

$$
d\left(T x_{2 n_{k}-1}, T p\right) \rightarrow \operatorname{dist}(S(A), S(B))
$$

Moreover, we have

$$
\begin{aligned}
\operatorname{dist}(S(A), S(B)) & \leq d(S p, T p) \\
& \leq d\left(S p, T x_{2 n_{k}-1}\right)+d\left(T x_{2 n_{k}-1}, T p\right) \\
& =d\left(S p, S x_{2 n_{k}}\right)+d\left(T x_{2 n_{k}-1}, T p\right) \\
& \rightarrow \operatorname{dist}(S(A), S(B))
\end{aligned}
$$

that is,

$$
d(S p, T p)=\operatorname{dist}(S(A), S(B))
$$

Lemma 2.9. Let $(A, B)$ be a nonempty pair of subsets of a metric space $(X, d)$ and let $(T ; S)$ be a $Q C N$ contraction pair defined on $X$. For $x_{0} \in A$, define $T x_{n}=S x_{n+1}$ for each $n \geq 0$. Then $\left\{S x_{2 n}\right\}$, and $\left\{S x_{2 n+1}\right\}$ are bounded sequences in $S(A)$ and $S(B)$ respectively.

Proof. Since

$$
d\left(S x_{2 n}, S x_{2 n+1}\right) \rightarrow \operatorname{dist}(S(A), S(B))
$$

it suffices to show that $\left\{S x_{2 n}\right\}$ is bounded in $S(A)$. Assume to the contrary that there exists $N_{0} \in \mathbb{N}$ such that

$$
d\left(S x_{2}, S x_{2 N_{0}+1}\right)>M, d\left(S x_{2}, S x_{2 N_{0}-1}\right) \leq M
$$

where,

$$
M>\max \left\{\frac{\alpha^{2}}{1-\alpha^{2}} d\left(S x_{0}, S x_{2}\right)+\operatorname{dist}(S(A), S(B)), d\left(S x_{1}, S x_{0}\right)\right\}
$$

By the above assumption, we have

$$
\begin{aligned}
\frac{M-\operatorname{dist}(S(A), S(B))}{\alpha^{2}} & +\operatorname{dist}(S(A), S(B)) \\
& <\frac{d\left(S x_{2}, S x_{2 N_{0}+1}\right)-\operatorname{dist}(S(A), S(B))}{\alpha^{2}} \\
& +\operatorname{dist}(S(A), S(B)) \\
& \leq \frac{d\left(S x_{2}, S x_{2 N_{0}+1}\right)+\left(\alpha^{2}-1\right) d\left(S x_{2}, S x_{2 N_{0}+1}\right)}{\alpha^{2}} \\
& =d\left(S x_{2}, S x_{2 N_{0}+1}\right)=d\left(T x_{1}, T x_{2 N_{0}}\right) \\
& \leq d\left(S x_{1}, S x_{2 N_{0}}\right)=d\left(T x_{0}, T x_{2 N_{0}-1}\right) \\
& =d\left(S x_{0}, S x_{2 N_{0}-1}\right) \\
& \leq d\left(S x_{0}, S x_{2}\right)+d\left(S x_{2}, S x_{2 N_{0}-1}\right) \\
& \leq d\left(S x_{0}, S x_{2}\right)+M .
\end{aligned}
$$

This implies that

$$
\frac{M-\operatorname{dist}(S(A), S(B))}{\alpha^{2}}+\operatorname{dist}(S(A), S(B))<d\left(S x_{0}, S x_{2}\right)+M
$$

hence,

$$
M-\left(1-\alpha^{2}\right) \operatorname{dist}(S(A), S(B))<\alpha^{2}\left[d\left(S x_{0}, S x_{2}\right)+M\right]
$$

and,

$$
\left(1-\alpha^{2}\right) M<\alpha^{2} d\left(S x_{0}, S x_{2}\right)+\left(1-\alpha^{2}\right) \operatorname{dist}(S(A), S(B))
$$

Now, it follows that

$$
M<\frac{\alpha^{2}}{1-\alpha^{2}} d\left(S x_{0}, S x_{2}\right)+\operatorname{dist}(S(A), S(B))
$$

which contradicts the choice of $M$.
Before we state the following theorem, we recall that a subset $A \subseteq X$ is said to be boundedly compact if the closure of every bounded subset of $A$ is compact and is contained in $A$.

Theorem 2.10. Let $(A, B)$ be a nonempty pair of subsets of a metric space $(X, d)$ such that $S(A)$ is boundedly compact and let $(T ; S)$ be a $Q C N$ contraction pair defined on $X$. If $S$ is relatively anti-Lipschitzian and continuous on $A$, then there exists $p \in A$ such that

$$
d(S p, T p)=\operatorname{dist}(S(A), S(B))
$$

Proof. For $x_{0} \in A$, define $T x_{n}=S x_{n+1}$ for each $n \geq 0$. By Lemma 2.9, $\left\{S x_{2 n}\right\}$ is bounded in $S(A)$. On the other hand, $S(A)$ is boundedly compact, so that there exists a subsequence $\left\{S x_{2 n_{k}}\right\}$ of $\left\{S x_{2 n}\right\}$ such that

$$
S x_{2 n_{k}} \rightarrow S p
$$

for some $p \in A$. We know that $S$ is relatively anti-Lipschitzian, therefore

$$
d\left(x_{2 n_{k}}, p\right) \leq c d\left(S x_{2 n_{k}}, S p\right) \rightarrow 0, k \rightarrow \infty
$$

This implies that $\left\{x_{2 n_{k}}\right\}$ is a convergent subsequence of $\left\{x_{2 n}\right\}$. Now, the result follows from Theorem 2.8.

Example 2.11. Let $X:=\mathbb{R}$ with the usual metric. For $A=(-\infty, 0]$ and $B=[0,+\infty)$ define $T, S: X \rightarrow X$ by

$$
T x:=\left\{\begin{array}{l}
-x, \text { if } x \in A \cup B \\
0, \text { ow. }
\end{array} \quad \text { and } \quad S x:=\left\{\begin{array}{l}
2 x, \text { if } x \in A \cup B \\
0, \text { ow. }
\end{array}\right.\right.
$$

Then $(T ; S)$ is a QCN contraction pair with $\alpha=\frac{1}{2}$. Indeed, for all $(x, y) \in A \times B$ we have

$$
\begin{aligned}
|T x-T y| & =(y-x) \leq \frac{1}{2}(2 y-2 x)+\frac{1}{2}(0) \\
& =\alpha|S x-S y|+(1-\alpha) \operatorname{dist}(S(A), S(B))
\end{aligned}
$$

Also, $T(A)=B \subseteq S(B)$ and $T(B)=A \subseteq S(A)$. Moreover, $S$ is continuous on $A$ and $S(A)$ is boundedly compact in $X$. Besides, $S$ is relatively antiLipschitzian on $A \cup B$ with $c=1$. In fact, for all $(x, y) \in A \times B$ we have

$$
|S x-S y|=2 y-2 x \geq|x-y| .
$$

Finally, the existence of coincidence quasi-best proximity point of the pair $(T ; S)$ follows from Theorem 2.10; that is, there exists $p \in A$ such that

$$
|T p-S p|=\operatorname{dist}(S(A), S(B))=0 \text { or }-p-2 p=0
$$

which implies that $p=0$. In this case, $p$ is a fixed point of $S$.
In the following we supply an example which shows that there exists a coincidence quasi-best proximity point that is not a fixed point of $S$.

Example 2.12. Let $X:=\mathbb{R}$ with the usual metric. For $A=(-\infty, 0]$ and $B=[0,+\infty)$ define $T, S: X \rightarrow X$ by

$$
T x:=\left\{\begin{array}{l}
-(x+1), \text { if } x \in A \cup B \\
0, \text { ow. }
\end{array} \quad \text { and } \quad S x:=\left\{\begin{array}{l}
2 x, \text { if } x \in A \cup B \\
0, \text { ow. }
\end{array}\right.\right.
$$

Then $(T ; S)$ is a QCN contraction pair with $\alpha=\frac{1}{2}$. Indeed, for all $(x, y) \in A \times B$ we have

$$
\begin{aligned}
|T x-T y| & =(y-x) \leq \frac{1}{2}(2 y-2 x)+\frac{1}{2}(0) \\
& =\alpha|S x-S y|+(1-\alpha) \operatorname{dist}(S(A), S(B))
\end{aligned}
$$

Also, $T(A)=[1,+\infty) \subseteq S(B)$ and $T(B)=(-\infty,-1] \subseteq S(A)$. Moreover, $S$ is continuous on $A$ and $S(A)$ is boundedly compact in $X$. Besides, $S$ is relatively anti-Lipschitzian on $A \cup B$ with $c=1$. In fact, for all $(x, y) \in A \times B$ we have

$$
|S x-S y|=2 y-2 x \geq|x-y| .
$$

Finally, the existence of coincidence quasi-best proximity point of the pair ( $T ; S$ ) follows from Theorem 2.10; that is, there exists $p \in A$ such that

$$
|T p-S p|=\operatorname{dist}(S(A), S(B))=0 \text { or }-(p+1)-2 p=0
$$

which implies that $p=-\frac{1}{3}$.
Lemma 2.13. Let $(A, B)$ be a nonempty pair of subsets of a uniformly convex metric space $(X, d, \mathcal{W})$ such that $S(A)$ is convex. Let $(T ; S)$ be a $Q C N$ contraction pair defined on $X$. For $x_{0} \in A$, define $T x_{n}=S x_{n+1}$ for each $n \geq 0$. Then

$$
d\left(S x_{2 n+2}, S x_{2 n}\right) \rightarrow 0, d\left(S x_{2 n+3}, S x_{2 n+1}\right) \rightarrow 0
$$

Proof. We prove that $d\left(S x_{2 n+2}, S x_{2 n}\right) \rightarrow 0$. To the contrary, assume that there exists $\varepsilon_{0}>0$ such that for each $k \geq 1$, there exists $n_{k} \geq k$ such that

$$
d\left(S x_{2 n_{k}+2}, S x_{2 n_{k}}\right) \geq \varepsilon_{0} .
$$

Choose $0<\gamma<1$ such that $\frac{\varepsilon_{0}}{\gamma}>\operatorname{dist}(S(A), S(B))$ and choose $\varepsilon>0$ such that

$$
0<\varepsilon<\min \left\{\frac{\varepsilon_{0}}{\gamma}-\operatorname{dist}(S(A), S(B)), \frac{\operatorname{dist}(S(A), S(B)) \alpha(\gamma)}{1-\alpha(\gamma)}\right\}
$$

By Lemma 2.7, since $d\left(S x_{2 n_{k}}, S x_{2 n_{k}+1}\right) \rightarrow \operatorname{dist}(S(A), S(B))$, there exists $N \in$ $\mathbb{N}$ such that

$$
\begin{gathered}
d\left(S x_{2 n_{k}}, S x_{2 n_{k}+1}\right) \leq \operatorname{dist}(S(A), S(B))+\varepsilon \\
d\left(S x_{2 n_{k}+2}, S x_{2 n_{k}+1}\right) \leq \operatorname{dist}(S(A), S(B))+\varepsilon
\end{gathered}
$$

and

$$
d\left(S x_{2 n_{k}}, S x_{2 n_{k}+2}\right) \geq \varepsilon_{0}>\gamma(\operatorname{dist}(S(A), S(B))+\varepsilon)
$$

It now follows from the uniform convexity of $X$ and the convexity of $S(A)$ that

$$
\begin{aligned}
\operatorname{dist}(S(A), S(B)) & \leq d\left(S x_{2 n_{k}+1}, \mathcal{W}\left(S x_{2 n_{k}}, S x_{2 n_{k}+2}, \frac{1}{2}\right)\right) \\
& \leq(\operatorname{dist}(S(A), S(B))+\varepsilon)(1-\alpha(\gamma)) \\
& <\operatorname{dist}(S(A), S(B))+\frac{\operatorname{dist}(S(A), S(B)) \alpha(\gamma)}{1-\alpha(\gamma)}(1-\alpha(\gamma)) \\
& =\operatorname{dist}(S(A), S(B))
\end{aligned}
$$

which is a contradiction. Similarly, we see that $d\left(S x_{2 n+3}, S x_{2 n+1}\right) \rightarrow 0$.
The following Theorem guarantees the existence and convergence of coincidence quasi-best proximity points for QCN contraction mappings in the setting of uniformly convex metric spaces.

Theorem 2.14. Let $(A, B)$ be a nonempty, closed pair of subsets of a complete uniformly convex metric space $(X, d ; \mathcal{W})$ such that $S(A)$ is convex. Let $(T ; S)$ be a $Q C N$ contraction pair defined on $X$ such that $S$ is continuous on $A$ and relatively anti-Lipschitzian on $A \cup B$. Then there exists $p \in A$ such that

$$
d(S p, T p)=\operatorname{dist}(S(A), S(B))
$$

Further, if $x_{0} \in A$ and $T x_{n}=S x_{n+1}$, then $\left\{x_{2 n}\right\}$ converges to the coincidence quasi-best proximity point of $(T ; S)$.
Proof. For $x_{0} \in A$ define $T x_{n}=S x_{n+1}$ for each $n \geq 0$. We prove that $\left\{S x_{2 n}\right\}$ and $\left\{S x_{2 n+1}\right\}$ are Cauchy sequences. First, we verify that for each $\varepsilon>0$ there exists $N_{0} \in \mathbb{N}$ such that

$$
\begin{equation*}
d\left(S x_{2 l}, S x_{2 n+1}\right)<\operatorname{dist}(S(A), S(B))+\varepsilon, \forall l>n \geq N_{0} \tag{*}
\end{equation*}
$$

Assume to the contrary that there exists $\varepsilon_{0}>0$ such that for each $k \geq 1$ there exists $l_{k}>n_{k} \geq k$ satisfying

$$
d\left(S x_{2 l_{k}}, S x_{2 n_{k}+1}\right) \geq \operatorname{dist}(S(A), S(B))+\varepsilon_{0}
$$

and

$$
d\left(S x_{2 l_{k}-2}, S x_{2 n_{k}+1}\right)<\operatorname{dist}(S(A), S(B))+\varepsilon_{0}
$$

We have

$$
\begin{aligned}
\operatorname{dist}(S(A), S(B))+\varepsilon_{0} & \leq d\left(S x_{2 l_{k}}, S x_{2 n_{k}+1}\right) \\
& \leq d\left(S x_{2 l_{k}}, S x_{2 l_{k}-2}\right)+d\left(S x_{2 l_{k}-2}, S x_{2 n_{k}+1}\right) \\
& \leq d\left(S x_{2 l_{k}}, S x_{2 l_{k}-2}\right)+\operatorname{dist}(S(A), S(B))+\varepsilon_{0} .
\end{aligned}
$$

Letting $k \rightarrow \infty$, we obtain

$$
d\left(S x_{2 l_{k}}, S x_{2 n_{k}+1}\right) \rightarrow \operatorname{dist}(S(A), S(B))+\varepsilon_{0}
$$

Moreover, we have

$$
\begin{aligned}
\operatorname{dist}(S(A), S(B))+\varepsilon_{0} & \leq d\left(S x_{2 l_{k}}, S x_{2 n_{k}+1}\right)=d\left(T x_{2 l_{k}-1}, T x_{2 n_{k}}\right) \\
& \leq \alpha d\left(S x_{2 l_{k}-1}, S x_{2 n_{k}}\right)+(1-\alpha) \operatorname{dist}(S(A), S(B)) \\
& =\alpha d\left(T x_{2 l_{k}-2}, T x_{2 n_{k}-1}\right)+(1-\alpha) \operatorname{dist}(S(A), S(B)) \\
& \leq \alpha d\left(S x_{2 l_{k}-2}, S x_{2 n_{k}-1}\right)+(1-\alpha) \operatorname{dist}(S(A), S(B)) .
\end{aligned}
$$

Therefore, by letting $k \rightarrow \infty$ we obtain

$$
\begin{aligned}
\operatorname{dist}(S(A), S(B))+\varepsilon_{0} & \leq \alpha\left(\operatorname{dist}(S(A), S(B))+\varepsilon_{0}\right)+(1-\alpha) \operatorname{dist}(S(A), S(B)) \\
& \leq \operatorname{dist}(S(A), S(B))+\varepsilon_{0}
\end{aligned}
$$

This implies that $\alpha=1$, which is a contradiction. That is, $(*)$ holds. Now, assume $\left\{S x_{2 n}\right\}$ is not a Cauchy sequence. Then there exists $\varepsilon_{0}>0$ such that for each $k \geq 1$ there exists $l_{k}>n_{k} \geq k$ such that

$$
d\left(S x_{2 l_{k}}, S x_{n_{k}}\right) \geq \varepsilon_{0}
$$

Choose $0<\gamma<1$ such that $\frac{\varepsilon_{0}}{\gamma}>\operatorname{dist}(S(A), S(B))$ and choose $\varepsilon>0$ such that

$$
0<\varepsilon<\min \left\{\frac{\varepsilon_{0}}{\gamma}-\operatorname{dist}(S(A), S(B)), \frac{\operatorname{dist}(S(A), S(B)) \alpha(\gamma)}{1-\alpha(\gamma)}\right\}
$$

Let $N \in \mathbb{N}$ be such that

$$
d\left(S x_{2 n_{k}}, S x_{2 n_{k}+1}\right) \leq \operatorname{dist}(S(A), S(B))+\varepsilon, \forall n_{k} \geq N
$$

and

$$
d\left(S x_{2 l_{k}}, S x_{2 n_{k}+1}\right) \leq \operatorname{dist}(S(A), S(B))+\varepsilon, \forall l_{k}>n_{k} \geq N
$$

Uniform convexity of $X$ implies that

$$
\begin{aligned}
\operatorname{dist}(S(A), S(B)) & \leq d\left(S x_{2 n_{k}+1}, \mathcal{W}\left(S x_{2 n_{k}}, S x_{2 l_{k}}, \frac{1}{2}\right)\right) \\
& \leq(\operatorname{dist}(S(A), S(B))+\varepsilon)(1-\alpha(\gamma))<\operatorname{dist}(S(A), S(B))
\end{aligned}
$$

which is a contradiction. Therefore, $\left\{S x_{2 n}\right\}$ is a Cauchy sequence in $S(A)$. By the fact that $S$ is relatively anti-Lipschitzian on $A \cup B$, we have

$$
d\left(x_{2 l}, x_{2 n}\right) \leq c d\left(S x_{2 l}, S x_{2 n}\right) \rightarrow 0, l, n \rightarrow \infty
$$

that is, $\left\{x_{2 n}\right\}$ is a Cauchy sequence. Since $A$ is complete, there exists $p \in A$ such that $x_{2 n} \rightarrow p$. Now, the result follows from a similar argument as in Theorem 2.8.

## 3. QUASI-CYCLIC-NONCYCLIC RELATIVELY CONTRACTION MAPPINGS

In this section, we introduce the class of quasi-cyclic-noncyclic relatively contraction mappings that contains the class of cyclic-noncyclic contraction mappings as a subclass. Next, we study the existence and convergence of coincidence best proximity points in the setting of convex metric spaces for quasi-cyclic-noncyclic relatively contraction mappings.

Definition 3.1. Let $(A, B)$ be a nonempty pair of subsets of a metric space $(X, d)$ and $T, S: X \rightarrow X$ be two mappings. The pair $(T ; S)$ is called a quasi-cyclic-noncyclic relatively contraction pair if it satisfies the following conditions:
(1) $(T ; S)$ is a quasi-cyclic-noncyclic pair on $X$; that is,

$$
T(A) \subseteq S(B), T(B) \subseteq S(A)
$$

(2) For some $\alpha \in(0,1)$ and for each $(x, y) \in A \times B$ we have

$$
d(T x, T y) \leq \alpha d(S x, S y)+(1-\alpha) \operatorname{dist}(A, B)
$$

Note that in the above definition we do not have the inequality

$$
\operatorname{dist}(A, B) \leq d(S x, S y)
$$

that is,

$$
d(T x, T y) \leq d(S x, S y)
$$

is not always true.
We emphasize that if $S=I$ or if $S(A)=A$ and $S(B)=B$, then the above definition reduces to Definition 1.3.

Example 3.2. Let $X:=\mathbb{R}$ with the usual metric. For $A=(-\infty,-3]$ and $B=[3,+\infty)$ define $T, S: X \rightarrow X$ by

$$
T x:=\left\{\begin{array}{l}
-(x+1), \text { if } x \in A \cup B \\
0, \text { ow. }
\end{array} \quad \text { and } \quad S x:=\left\{\begin{array}{l}
3 x+5, \text { if } x \in A \\
3 x-7, \text { if } x \in B \\
0, \text { ow. }
\end{array}\right.\right.
$$

Then $(T ; S)$ is a QCN relatively contraction pair with $\alpha=\frac{1}{3}$. Indeed, for all $(x, y) \in A \times B$ we have

$$
\begin{aligned}
|T x-T y| & =(y-x) \leq \frac{1}{3}(3 y-3 x-12)+\frac{2}{3}(6) \\
& =\alpha|S x-S y|+(1-\alpha) \operatorname{dist}(A, B)
\end{aligned}
$$

Also, $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$.

Lemma 3.3. Let $(A, B)$ be a nonempty pair of subsets of a metric space $(X, d)$ and let $(T ; S)$ be a $Q C N$ relatively contraction pair defined on $X$ and $\operatorname{dist}(A, B) \leq \operatorname{dist}(S(A), S(B))$. For $x_{0} \in A$, define $T x_{n}=S x_{n+1}$ for each $n \geq 0$. Then we have

$$
d\left(S x_{2 n}, S x_{2 n+1}\right) \rightarrow \operatorname{dist}(A, B)
$$

Proof. We note that

$$
\begin{aligned}
\operatorname{dist}(A, B) \leq \operatorname{dist}(S(A), S(B)) & \leq d\left(S x_{2 n+1}, S x_{2 n+2}\right)=d\left(T x_{2 n}, T x_{2 n+1}\right) \\
& \leq \alpha d\left(S x_{2 n}, S x_{2 n+1}\right)+(1-\alpha) \operatorname{dist}(A, B) \\
& =\alpha d\left(T x_{2 n-1}, T x_{2 n}\right)+(1-\alpha) \operatorname{dist}(A, B) \\
& \leq \alpha\left[\alpha d\left(S x_{2 n-1}, S x_{2 n}\right)+(1-\alpha) \operatorname{dist}(A, B)\right] \\
& +(1-\alpha) \operatorname{dist}(A, B) \\
& =\alpha^{2} d\left(S x_{2 n-1}, S x_{2 n}\right)+\left(1-\alpha^{2}\right) \operatorname{dist}(A, B) \\
& =\alpha^{2} d\left(T x_{2 n-2}, T x_{2 n-1}\right)+\left(1-\alpha^{2}\right) \operatorname{dist}(A, B) \\
& \leq \cdots \\
& \leq \alpha^{2 n} d\left(T x_{0}, T x_{1}\right)+\left(1-\alpha^{2}\right) \operatorname{dist}(A, B) .
\end{aligned}
$$

Now, if $n \rightarrow \infty$, we conclude that

$$
d\left(S x_{2 n}, S x_{2 n+1}\right) \rightarrow \operatorname{dist}(A, B) .
$$

Remark 3.4. If the pair $(T ; S)$ is a QCN relatively contraction pair such that

$$
S(A) \subseteq A \text { and } S(B) \subseteq B
$$

then we have

$$
\operatorname{dist}(A, B) \leq \operatorname{dist}(S(A), S(B))
$$

Thus, by this assumption, the Lemma holds true.
Theorem 3.5. Let $(A, B)$ be a nonempty pair of subsets of a metric space $(X, d)$ and let $(T ; S)$ be a $Q C N$ relatively contraction pair defined on $X$ and $\operatorname{dist}(A, B) \leq \operatorname{dist}(S(A), S(B))$. Assume $S$ is continuous on $A$. For $x_{0} \in A$, define $T x_{n}=S x_{n+1}$ for each $n \geq 0$. If $\left\{x_{2 n}\right\}$ has a convergent subsequence in A, then the pair $(T ; S)$ has a coincidence best proximity point in $A$.

Proof. Let $\left\{x_{2 n_{k}}\right\}$ be a subsequence of $\left\{x_{2 n}\right\}$ such that $x_{2 n_{k}} \rightarrow p \in A$. we have

$$
\begin{aligned}
\operatorname{dist}(A, B) \leq \operatorname{dist}(S(A), S(B)) & \leq d\left(T x_{2 n_{k}-1}, T p\right) \leq d\left(S x_{2 n_{k}-1}, S p\right) \\
& \leq d\left(S p, S x_{2 n_{k}}\right)+d\left(S x_{2 n_{k}}, S x_{2 n_{k}-1}\right)
\end{aligned}
$$

By Lemma 3.3, if $k \rightarrow \infty$, we obtain that

$$
d\left(T x_{2 n_{k}-1}, T p\right) \rightarrow \operatorname{dist}(A, B)
$$

Moreover,

$$
\begin{aligned}
\operatorname{dist}(A, B) \leq \operatorname{dist}(S(A), S(B)) & \leq d(S p, T p) \\
& \leq d\left(S p, T x_{2 n_{k}-1}\right)+d\left(T x_{2 n_{k}-1}, T p\right) \\
& =d\left(S p, S x_{2 n_{k}}\right)+d\left(T x_{2 n_{k}-1}, T p\right) \\
& \rightarrow \operatorname{dist}(A, B)
\end{aligned}
$$

that is,

$$
d(S p, T p)=\operatorname{dist}(A, B)
$$

Lemma 3.6. Let $(A, B)$ be a nonempty pair of subsets of a metric space $(X, d)$. Suppose $(T ; S)$ is a $Q C N$ relatively contraction pair defined on $X$ and $\operatorname{dist}(A, B) \leq \operatorname{dist}(S(A), S(B))$. For $x_{0} \in A$, define $T x_{n}=S x_{n+1}$ for each $n \geq 0$. Then $\left\{S x_{2 n}\right\}$, and $\left\{S x_{2 n+1}\right\}$ are bounded sequences in $S(A)$ and $S(B)$ respectively.

Proof. Since

$$
d\left(S x_{2 n}, S x_{2 n+1}\right) \rightarrow \operatorname{dist}(A, B)
$$

it suffices to verify that $\left\{S x_{2 n}\right\}$ is bounded in $S(A)$. Assume to the contrary that there exists $N_{0} \in \mathbb{N}$ such that

$$
d\left(S x_{2}, S x_{2 N_{0}+1}\right)>M, d\left(S x_{2}, S x_{2 N_{0}-1}\right) \leq M
$$

where,

$$
M>\max \left\{\frac{\alpha^{2}}{1-\alpha^{2}} d\left(S x_{0}, S x_{2}\right)+\operatorname{dist}(A, B), d\left(S x_{1}, S x_{0}\right)\right\}
$$

By the above assumption, we have

$$
\begin{aligned}
\frac{M-\operatorname{dist}(A, B)}{\alpha^{2}} & +\operatorname{dist}(A, B)<\frac{d\left(S x_{2}, S x_{2 N_{0}+1}\right)-\operatorname{dist}(A, B)}{\alpha^{2}}+\operatorname{dist}(A, B) \\
& \leq \frac{d\left(S x_{2}, S x_{2 N_{0}+1}\right)+\left(\alpha^{2}-1\right) d\left(S x_{2}, S x_{2 N_{0}+1}\right)}{\alpha^{2}} \\
& =d\left(S x_{2}, S x_{2 N_{0}+1}\right)=d\left(T x_{1}, T x_{2 N_{0}}\right) \\
& \leq d\left(S x_{1}, S x_{2 N_{0}}\right)=d\left(T x_{0}, T x_{2 N_{0}-1}\right) \\
& =d\left(S x_{0}, S x_{2 N_{0}-1}\right) \\
& \leq d\left(S x_{0}, S x_{2}\right)+d\left(S x_{2}, S x_{2 N_{0}-1}\right) \\
& \leq d\left(S x_{0}, S x_{2}\right)+M
\end{aligned}
$$

This implies that

$$
\frac{M-\operatorname{dist}(A, B)}{\alpha^{2}}+\operatorname{dist}(A, B)<d\left(S x_{0}, S x_{2}\right)+M
$$

or,

$$
M-\left(1-\alpha^{2}\right) \operatorname{dist}(A, B)<\alpha^{2}\left[d\left(S x_{0}, S x_{2}\right)+M\right]
$$

and finally,

$$
\left(1-\alpha^{2}\right) M<\alpha^{2} d\left(S x_{0}, S x_{2}\right)+\left(1-\alpha^{2}\right) \operatorname{dist}(A, B)
$$

Now, we conclude that

$$
M<\frac{\alpha^{2}}{1-\alpha^{2}} d\left(S x_{0}, S x_{2}\right)+\operatorname{dist}(A, B)
$$

which is a contradiction by the choice of $M$.
Theorem 3.7. Let $(A, B)$ be a nonempty pair of subsets of a metric space $(X, d)$ such that $S(A)$ is boundedly compact. Suppose $(T ; S)$ is a QCN relatively contraction pair defined on $X$ and $\operatorname{dist}(A, B) \leq \operatorname{dist}(S(A), S(B))$. If $S$ is relatively anti-Lipschitzian and continuous on $A$, then there exists $p \in A$ such that

$$
d(S p, T p)=\operatorname{dist}(A, B)
$$

Proof. For $x_{0} \in A$, define $T x_{n}=S x_{n+1}$ for each $n \geq 0$. According to Lemma 3.6, $\left\{S x_{2 n}\right\}$ is bounded in $S(A)$, on the other hand $S(A)$ is boundedly compact, so that there exists a subsequence $\left\{S x_{2 n_{k}}\right\}$ of $\left\{S x_{2 n}\right\}$ such that

$$
S x_{2 n_{k}} \rightarrow S p
$$

for some $p \in A$. We know that $S$ is relatively anti-Lipschitzian, therefore

$$
d\left(x_{2 n_{k}}, p\right) \leq c d\left(S x_{2 n_{k}}, S p\right) \rightarrow 0, k \rightarrow \infty .
$$

This implies that $\left\{x_{2 n_{k}}\right\}$ is a convergent subsequence of $\left\{x_{2 n}\right\}$, hence the result follows from Theorem 3.5.

In the following we give examples to show that there exists a coincidence best proximity point that is not a fixed point for $S$.

Example 3.8. Let $X:=\mathbb{R}$ with the usual metric. For $A=(-\infty,-3]$ and $B=[3,+\infty)$ define $T, S: X \rightarrow X$ by

$$
T x:=\left\{\begin{array}{l}
3-x, \text { if } x \in A \cup B \\
0, \text { ow. }
\end{array} \quad \text { and } \quad S x:=\left\{\begin{array}{l}
2 x+6, \text { if } x \in A \\
2 x, \text { if } x \in B \\
0, \text { ow } .
\end{array}\right.\right.
$$

Then $(T ; S)$ is a QCN relatively contraction pair with $\alpha=\frac{1}{2}$. Indeed, for all $(x, y) \in A \times B$ we have

$$
\begin{aligned}
|T x-T y| & =(y-x) \leq \frac{1}{2}(2 y-2 x-6)+\frac{1}{2}(6) \\
& =\alpha|S x-S y|+(1-\alpha) \operatorname{dist}(A, B) .
\end{aligned}
$$

Also, $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. Finally, the existence of coincidence best proximity point of the pair $(T ; S)$ follows from Theorem 3.7 ; that is, there exists $p \in A$ such that

$$
|T p-S p|=\operatorname{dist}(A, B)=0 \text { or } 3-p-2 p-6=6,
$$

which implies that $p=-3$.
Example 3.9. Let $X:=\mathbb{R}$ with the usual metric. For $A=(-\infty,-4]$ and $B=[4,+\infty)$ define $T, S: X \rightarrow X$ by

$$
T x:=\left\{\begin{array}{l}
4-x, \text { if } x \in A \cup B \\
0, \text { ow. }
\end{array} \quad \text { and } \quad S x:=\left\{\begin{array}{l}
4 x+16, \text { if } x \in A \\
4 x-8, \text { if } x \in B \\
0, \text { ow } .
\end{array}\right.\right.
$$

Then $(T ; S)$ is a QCN relatively contraction pair with $\alpha=\frac{1}{4}$. Indeed, for all $(x, y) \in A \times B$ we have

$$
\begin{aligned}
|T x-T y| & =(y-x) \leq \frac{1}{4}(4 y-4 x-24)+\frac{3}{4}(8) \\
& =\alpha|S x-S y|+(1-\alpha) \operatorname{dist}(A, B)
\end{aligned}
$$

Also, $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. Finally, the existence of coincidence best proximity point of the pair $(T ; S)$ follows from Theorem 3.7 ; that is, there exists $p \in A$ such that

$$
|T p-S p|=\operatorname{dist}(A, B)=8 \text { or } 4-p-4 p-16=8
$$

which implies that $p=-4$.
Lemma 3.10. Let $(A, B)$ be a nonempty pair of subsets of a uniformly convex metric space $(X, d, \mathcal{W})$ such that $S(A)$ is convex. Suppose $(T ; S)$ is a $Q C N$ relatively contraction pair defined on $X$ and $\operatorname{dist}(A, B) \leq \operatorname{dist}(S(A), S(B))$. For $x_{0} \in A$, define $T x_{n}=S x_{n+1}$ for each $n \geq 0$. Then

$$
d\left(S x_{2 n+2}, S x_{2 n}\right) \rightarrow 0, d\left(S x_{2 n+3}, S x_{2 n+1}\right) \rightarrow 0
$$

Proof. We prove that $d\left(S x_{2 n+2}, S x_{2 n}\right) \rightarrow 0$. Assume to the contrary that there exists $\varepsilon_{0}>0$ such that for each $k \geq 1$, there exists $n_{k} \geq k$ such that

$$
d\left(S x_{2 n_{k}+2}, S x_{2 n_{k}}\right) \geq \varepsilon_{0}
$$

Choose $0<\gamma<1$ such that $\frac{\varepsilon_{0}}{\gamma}>\operatorname{dist}(A, B)$ and choose $\varepsilon>0$ such that

$$
0<\varepsilon<\min \left\{\frac{\varepsilon_{0}}{\gamma}-\operatorname{dist}(A, B), \frac{\operatorname{dist}(A, B) \alpha(\gamma)}{1-\alpha(\gamma)}\right\}
$$

By Lemma 3.3, we know that $d\left(S x_{2 n_{k}}, S x_{2 n_{k}+1}\right) \rightarrow \operatorname{dist}(A, B)$, so there exists $N \in \mathbb{N}$ such that

$$
\begin{gathered}
d\left(S x_{2 n_{k}}, S x_{2 n_{k}+1}\right) \leq \operatorname{dist}(A, B)+\varepsilon \\
d\left(S x_{2 n_{k}+2}, S x_{2 n_{k}+1}\right) \leq \operatorname{dist}(A, B)+\varepsilon
\end{gathered}
$$

and

$$
d\left(S x_{2 n_{k}}, S x_{2 n_{k}+2}\right) \geq \varepsilon_{0}>\gamma(\operatorname{dist}(A, B)+\varepsilon) .
$$

It now follows from the uniformly convexity of $X$ and the convexity of $S(A)$ that

$$
\begin{aligned}
\operatorname{dist}(A, B) \leq \operatorname{dist}(S(A), S(B)) & \leq d\left(S x_{2 n_{k}+1}, \mathcal{W}\left(S x_{2 n_{k}}, S x_{2 n_{k}+2}, \frac{1}{2}\right)\right) \\
& \leq(\operatorname{dist}(A, B)+\varepsilon)(1-\alpha(\gamma)) \\
& <\operatorname{dist}(A, B)+\frac{\operatorname{dist}(A, B) \alpha(\gamma)}{1-\alpha(\gamma)}(1-\alpha(\gamma)) \\
& =\operatorname{dist}(A, B)
\end{aligned}
$$

which is a contradiction. Similarly, we see that $d\left(S x_{2 n+3}, S x_{2 n+1}\right) \rightarrow 0$.
The following Theorem guarantees the existence and convergence of coincidence best proximity points for QCN relatively contraction mappings in the setting of uniformly convex metric spaces.

Theorem 3.11. Let $(A, B)$ be a nonempty, closed pair of subsets of a complete uniformly convex metric space $(X, d ; \mathcal{W})$ such that $S(A)$ is convex. Suppose $(T ; S)$ is a $Q C N$ relatively contraction pair defined on $X$ such that $S$ is continuous on $A$ and relatively anti-Lipschitzian on $A \cup B$. Assume that $\operatorname{dist}(A, B) \leq \operatorname{dist}(S(A), S(B))$. Then there exists $p \in A$ such that

$$
d(S p, T p)=\operatorname{dist}(A, B)
$$

Further, if $x_{0} \in A$ and $T x_{n}=S x_{n+1}$, then $\left\{x_{2 n}\right\}$ converges to the coincidence best proximity point of $(T ; S)$.

Proof. For $x_{0} \in A$ define $T x_{n}=S x_{n+1}$ for each $n \geq 0$. We prove that $\left\{S x_{2 n}\right\}$ and $\left\{S x_{2 n+1}\right\}$ are Cauchy sequences. First, we verify that for each $\varepsilon>0$ there exists $N_{0} \in \mathbb{N}$ such that

$$
\begin{equation*}
d\left(S x_{2 l}, S x_{2 n+1}\right)<\operatorname{dist}(A, B)+\varepsilon, \forall l>n \geq N_{0} \tag{*}
\end{equation*}
$$

Assume the contrary. Then there exists $\varepsilon_{0}>0$ such that for each $k \geq 1$ there exists $l_{k}>n_{k} \geq k$ satisfying

$$
d\left(S x_{2 l_{k}}, S x_{2 n_{k}+1}\right) \geq \operatorname{dist}(A, B)+\varepsilon_{0}, d\left(S x_{2 l_{k}-2}, S x_{2 n_{k}+1}\right)<\operatorname{dist}(A, B)+\varepsilon_{0}
$$

Note that

$$
\begin{aligned}
\operatorname{dist}(A, B)+\varepsilon_{0} & \leq d\left(S x_{2 l_{k}}, S x_{2 n_{k}+1}\right) \\
& \leq d\left(S x_{2 l_{k}}, S x_{2 l_{k}-2}\right)+d\left(S x_{2 l_{k}-2}, S x_{2 n_{k}+1}\right) \\
& \leq d\left(S x_{2 l_{k}}, S x_{2 l_{k}-2}\right)+\operatorname{dist}(A, B)+\varepsilon_{0} .
\end{aligned}
$$

Letting $k \rightarrow \infty$, we obtain

$$
d\left(S x_{2 l_{k}}, S x_{2 n_{k}+1}\right) \rightarrow \operatorname{dist}(A, B)+\varepsilon_{0}
$$

Moreover, we have

$$
\begin{aligned}
\operatorname{dist}(A, B)+\varepsilon_{0} & \leq d\left(S x_{2 l_{k}}, S x_{2 n_{k}+1}\right)=d\left(T x_{2 l_{k}-1}, T x_{2 n_{k}}\right) \\
& \leq \alpha d\left(S x_{2 l_{k}-1}, S x_{2 n_{k}}\right)+(1-\alpha) \operatorname{dist}(A, B) \\
& =\alpha d\left(T x_{2 l_{k}-2}, T x_{2 n_{k}-1}\right)+(1-\alpha) \operatorname{dist}(A, B) \\
& \leq \alpha d\left(S x_{2 l_{k}-2}, S x_{2 n_{k}-1}\right)+(1-\alpha) \operatorname{dist}(A, B)
\end{aligned}
$$

Therefore, by letting $k \rightarrow \infty$ we obtain

$$
\operatorname{dist}(A, B)+\varepsilon_{0} \leq \alpha\left(\operatorname{dist}(A, B)+\varepsilon_{0}\right)+(1-\alpha) \operatorname{dist}(A, B) \leq \operatorname{dist}(A, B)+\varepsilon_{0}
$$

This implies that $\alpha=1$, which is a contradiction. That is, $(*)$ holds. Now, assume that $\left\{S x_{2 n}\right\}$ is not a Cauchy sequence. Then there exists $\varepsilon_{0}>0$ such that for each $k \geq 1$ there exists $l_{k}>n_{k} \geq k$ such that

$$
d\left(S x_{2 l_{k}}, S x_{n_{k}}\right) \geq \varepsilon_{0}
$$

Choose $0<\gamma<1$ such that $\frac{\varepsilon_{0}}{\gamma}>\operatorname{dist}(A, B)$ and choose $\varepsilon>0$ such that

$$
0<\varepsilon<\min \left\{\frac{\varepsilon_{0}}{\gamma}-\operatorname{dist}(A, B), \frac{\operatorname{dist}(A, B) \alpha(\gamma)}{1-\alpha(\gamma)}\right\}
$$

Let $N \in \mathbb{N}$ be such that

$$
d\left(S x_{2 n_{k}}, S x_{2 n_{k}+1}\right) \leq \operatorname{dist}(A, B)+\varepsilon, \forall n_{k} \geq N
$$

and

$$
d\left(S x_{2 l_{k}}, S x_{2 n_{k}+1}\right) \leq \operatorname{dist}(A, B)+\varepsilon, \forall l_{k}>n_{k} \geq N
$$

Uniformly convexity of $X$ implies that

$$
\begin{aligned}
\operatorname{dist}(A, B) \leq \operatorname{dist}(S(A), S(B)) & \leq d\left(S x_{2 n_{k}+1}, \mathcal{W}\left(S x_{2 n_{k}}, S x_{2 l_{k}}, \frac{1}{2}\right)\right) \\
& \leq(\operatorname{dist}(A, B)+\varepsilon)(1-\alpha(\gamma))<\operatorname{dist}(A, B)
\end{aligned}
$$

which is a contradiction. Therefore, $\left\{S x_{2 n}\right\}$ is a Cauchy sequence in $S(A)$. By the fact that $S$ is relatively anti-Lipschitzian on $A \cup B$, we have

$$
d\left(x_{2 l}, x_{2 n}\right) \leq c d\left(S x_{2 l}, S x_{2 n}\right) \rightarrow 0, l, n \rightarrow \infty
$$

that is, $\left\{x_{2 n}\right\}$ is Cauchy. Since $A$ is complete, there exists $p \in A$ such that $x_{2 n} \rightarrow p$. Now, the result follows from a similar argument as in the proof of Theorem 3.5.

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