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Recognition of $L_2(q)$ by the Main Supergraph

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Abstract. Let G be a finite group. The main supergraph $\mathcal{S}(G)$ is a graph with vertex set G in which two vertices x and y are adjacent if and only if $o(x) \mid o(y)$ or $o(y) \mid o(x)$. In this paper, we will show that $G \cong L_2(q)$ if and only if $\mathcal{S}(G) \cong \mathcal{S}(L_2(q))$, where q is a prime power. This work implies that there is not a solvable group that has the same order type as the simple group $L_2(q)$.

Keywords: Graph, Main supergraph, Thompson's problem.

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1. Introduction

Let G be a finite group and $x \in G$. The order of x is denoted by $o(x)$. The set of all element orders of G is denoted by $\pi_e(G)$ and the set of all prime factors of $|G|$ is denoted by $\pi(G)$. It is clear that the set $\pi_e(G)$ is closed and partially ordered by divisibility, and hence it is uniquely determined by $\mu(G)$, the subset of its maximal elements. We set $M_i = M_i(G) = |\{g \in G|$ the order of g is $i\}|$.

We define the graph $\mathcal{S}(G)$ with the vertex set G such that two vertices x and y are adjacent if and only if $o(x) \mid o(y)$ or $o(y) \mid o(x)$. This graph is called

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the *main supergraph* of *power graph* G and was introduced in [8]. The power graph $\mathcal{P}(G)$ is a graph with the vertex set G, in which two distinct elements are adjacent if one is a power of the other. The main properties of this graph were investigated by Cameron [1] and Chakrabarty et al. [2]. The *proper main supergraph* $\mathcal{S}^*(G)$ is the graph constructed from $\mathcal{S}(G)$ by removing the identity element of G. We write $x \sim y$ when two vertices x and y are adjacent.

We say that groups G_1 and G_2 are of the *same order type* if and only if $M_t(G_1) = M_t(G_2)$ for all t. By the definition of the main supergraph, it is clear that if G_1 and G_2 are groups with the same order type, then $\mathcal{S}(G_1) \cong \mathcal{S}(G_2)$. The converse of this result is not generally correct. To prove, we consider $G_1 = Z_4 \times Z_4$ and $G_2 = Z_4 \times Z_2 \times Z_2$. Since G_1 and G_2 are 2-groups, we have $\mathcal{S}(G_1) \cong \mathcal{S}(G_2)$. But $M_4(G_1) = 12 > 8 = M_4(G_2)$ and $M_2(G_1) = 3 < 7 = M_2(G_2)$.
In 1987, J. G. Thompson [16, Problem 12.37] posed the following problem:
Thompson's Problem. Suppose that G_1 and G_2 are two groups of the same order type. If G_1 is solvable, is it true that G_2 is also necessarily solvable?

Let $\mathrm{nse}(G)$ be the set of the number of elements of the same order in G. If G_1 and G_2 are the same type, then $\mathrm{nse}(G_1) = \mathrm{nse}(G_2)$ and $|G_1| = |G_2|$. Therefore, if a group G has been uniquely determined by its order and $\mathrm{nse}(G)$, then Thompson's problem is true for G. In [11], the authors proved that no solvable group has the same order type as $L_2(p)$, where p is a prime number.

Clearly, for two groups G_1 and G_2 that are the same order type, $\mathcal{S}(G_1) \cong \mathcal{S}(G_2)$. So, if a group G has been uniquely determined by $\mathcal{S}(G)$, then Thompson's problem is true for G. In [12], the authors of this paper proved that alternating group of degree p, $p+1$, $p+2$ and symmetric group of degree p are uniquely determined by their main supergraph. Also, in [13], [14] and [15], it is proved that the groups $L_2(p)$, $\mathrm{PGL}_2(p)$, where p is prime, all of the sporadic simple groups, the small Ree group ${}^2G_2(3^{2n+1})$, where n is a natural number and Suzuki group are uniquely determined by their main supergraph. In this paper, we will show that $L_2(q)$, where q is a prime power uniquely determined by their main supergraph. It follows that no solvable group has the same order type as $L_2(q)$. In fact, the main theorem of our paper is as follow.

Theorem 1.1. *Let $\mathcal{S}(G) \cong \mathcal{S}(L_2(q))$, where q is a prime power. Then $G \cong L_2(q)$.*

As noted above, as an immediate consequence of Main Theorem, we have that

Corollary 1.2. *If G is a finite group with the same order type as $L_2(q)$, where q is a prime power, then G is isomorphic to $L_2(q)$.*

We construct the *prime graph* of G, which is denoted by $\Gamma(G)$, as follows: the vertex set is $\pi(G)$ and two distinct vertices p and q are joined by an edge if and only if G has an element of order pq $(p \neq q)$. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, \ldots, \pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$.

Given a finite group G, we can express $|G|$ as a product of integers $m_1, m_2, \ldots, m_{t(G)}$, where $\pi(m_i) = \pi_i$ for each i. These numbers m_i are called the order components of G. In particular, if m_i is odd, then we call it an odd order component of G (see [5]).

According to the classification theorem of finite simple groups and [10, 17, 18], we can list the order components of finite simple groups with disconnected prime graphs as in Tables 1-4 in [4].

Let p be a prime. A group G is called a C_{pp}-group if $p \in \pi(G)$ and p is an isolated vertex of the prime graph of G, in other words, the centralizers of its elements of order p in G are p-groups.

Throughout this paper we denote by $\phi(n)$, where n is a natural number, Euler's totient function. We denote by P_q a Sylow q-subgroup of G. The other notations and terminologies in this paper are standard, and the reader is referred to [6] if necessary.

2. Preliminary Results

We first quote some lemmas that are used in deducing the theorem of this paper.

Lemma 2.1. [7] *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m \mid |L_m(G)|$.*

Remark 2.2. Let M_n be the number of elements of order n in G. We note that $M_n = k\phi(n)$, where k is the number of cyclic subgroups of order n in G. If $n \mid |G|$, then by Lemma 2.1 we have

$$\begin{cases} \phi(n) \mid M_n \\ \\ n \mid \sum_{d|n} M_d \end{cases}.$$

Definition 2.3. A group G is a 2-*Frobenius* group if there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively.

We quote some known results about Frobenius group and 2-Frobenius group, which are useful in the sequel.

Lemma 2.4. [3] *Let G be a 2-Frobenius group of even order. Then:*
(a) $t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;
(b) G/K and K/H are cyclic, $|G/K| \mid (|K/H| - 1)$, $(|G/K|, |K/H|) = 1$ and $G/K \lesssim \mathrm{Aut}(K/H)$.

Lemma 2.5. [3] *Suppose that G is a Frobenius group of even order and H, K are the Frobenius kernel and the Frobenius complement of G, respectively. Then $t(G) = 2$, and the prime graph components of G are $\pi(H)$ and $\pi(K)$.*

Lemma 2.6. [18] *If G is a finite group such that $t(G) \geq 2$, then G has one of the following structures:*
(a) G is a Frobenius group or a 2-Frobenius group;
(b) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1$ and K/H is a non-abelian simple group. In particular, H is nilpotent, $G/K \lesssim \mathrm{Out}(K/H)$ and the odd order components of G are the odd order components of K/H.

3. Proof of Main Theorem

By the definition of the main supergraph and our assumption, we have $|G| = |L_2(q)|$ and $\mathcal{S}^*(L_2(q)) \cong \mathcal{S}^*(G)$. Let $q = p^n$, where p is a prime number. By [9, pp. 213] , we have $\mu(L_2(q)) = \{(q-1)/2,\ p,\ (q+1)/2\}$. Thus $L_2(q)$ has not any element of order rs, where $r \mid (q-1)/2$ and $s \mid (q+1)/2$ and kp, where $k \in \pi(G)\backslash\{p\}$. It follows that $\mathcal{S}^*(G)$ is a disconnected graph with three connected components. One of the connected components is a complete graph, we denote it by K_1 and the other of the connected components denoted by K_2 and K_3. Since $L_2(q)$ has not any element of order kp, where $k \in \pi(G)\backslash\{p\}$, the order of complete connected component is $M_p(L_2(q))$. On the other hand, by [9, Theorems 8.2–8.5], $M_p(L_2(q)) = q^2 - 1$. Thus order of K_1 is $q^2 - 1$. We prove the vertices of K_1 are elements of order p^k, where $k \geq 1$ is an integer.

First, let x and y be two vertices of K_1 such that $o(x) = r$, $o(y) = s$ where $r, s \in \pi(G)$ and $r \neq s$. Since K_1 is a complete graph, we have $x \sim y$, a contradiction. So, the vertices of K_1 are elements of order r^k, where r is prime and $k \geq 1$ is an integer. We will show that $r = p$. Let the vertices of K_1 be all of $x \in G$ such that $o(x) = r$, r^2, ..., or r^k (note that $\exp(P_r) = r^k$). Then with considering $n = |P_r|$ in Remark 2.2, $|P_r| \mid (1 + M_r + M_{r^2} + \ldots + M_{r^k}) = 1 + q^2 - 1 = q^2$. It follows that $r = p$. Hence, the vertices of K_1 are $x \in G$ such that $o(x) = p^k$, where $k \geq 1$ is an integer. It follows that p is an isolated vertex of the prime graph of G.

Let x, y be two arbitrary vertices of K_2 and K_3, respectively such that $o(x) = r$ and $o(y) = s$, where r and s are primes. We prove that r and s are not joined by an edge in the prime graph of G. Let r and s are joined by an edge in the prime graph of G. Then $rs \in \pi_e(G)$. So, there exists an element of order rs in G. Assume $z \in G$ and $o(z) = rs$. By the definition of the main supergraph $x \sim z$ and $y \sim z$. Thus K_2 and K_3 are connected, a contradiction. It follows that $t(G) \geq 3$.

Since $t(G) \geq 3$, Lemmas 2.4(a) and 2.5 show that G is neither a Frobenius group nor a 2-Frobenius group. By Lemma 2.6, G has a normal series $1 \trianglelefteq N \triangleleft G_1 \trianglelefteq G$ such that N is a nilpotent π_1 -group, G/G_1 is a solvable π_1-group and

G_1/N is a simple C_{pp}-group. Since G is a C_{pp}-group, the odd order component q of G is equal to a certain odd order component of G_1/N (by the prime graph components of G). In particular, $t(G_1/N) \geq 3$. Furthermore, $G_1/N \lesssim G/N \lesssim \mathrm{Aut}(G_1/N)$ by Lemma 2.6.

Now using the classification of finite simple groups and the results in Tables 1–4 in [4], we consider the following steps.

Step 1. We prove that G_1/N can not be an alternating group $A_{n'}$.

If $G_1/N \cong A_{n'}$, then since the odd order components of $A_{n'}$ are primes, say $p^{'}$ or $p^{'}-2$, we conclude that $q = p^{'}$ or $q = p^{'}-2$. In both cases, q is a prime number. By Tables 1-4 in [4], we have $G_1/N \cong A_q$, A_{q+1} or A_{q+2}. Suppose $G_1/N \cong A_q$. It follows that $\frac{q!}{2} \leq \frac{q(q^2-1)}{2|N|}$ since $G/N \lesssim \mathrm{Aut}(G_1/N)$, or equivalently, $|N|(q-2)! \leq q+1 \leq 2|N|(q-2)!$. Since $q \geq 5$, we conclude that $2(q-2) \leq (q-2)(q-3)! \leq |N|(q-2)! \leq q+1$, which implies that $q \leq 5$, and so $q = 5$. We have already considered the case q is prime. Thus the case $G_1/N \cong A_q$ can be ruled out. The cases $G_1/N \cong A_{q+1}$ and A_{q+2} can be ruled out similarly.

Step 2. If $G_1/N \cong L_{r+1}(q^{'})$, then since $t(G_1/N) \geq 3$ we distinguish the following four cases.

2.1. $G_1/N \cong L_2(q^{'})$, where $4 \mid (q^{'}+1)$ and $q^{'}$ is a prime power. Then $q = q^{'}$ or $\frac{q^{'}-1}{2}$. Moreover, $\frac{q^{'}(q^{'2}-1)}{2} \leq \frac{q(q^2-1)}{2|N|}$ in both cases. If $q = q^{'}$, then $\frac{q(q^2-1)}{2} \leq \frac{q(q^2-1)}{2|N|}$, which implies that $|N| = 1$. It follows that $G \cong L_2(q)$.

If $q = \frac{q^{'}-1}{2}$, then $q^{'} = 2q+1$. Since $\frac{q^{'}(q^{'2}-1)}{2} \mid \frac{q(q^2-1)}{2|N|}$, we have that $(2q+1)[(2q+1)^2-1] \leq \frac{q(q^2-1)}{|N|}$. It follows that $(2q+1)[(2q+1)^2-1] \leq q(q^2-1)$, which implies that $7q \leq -1$, a contradiction.

2.2. $G_1/N \cong L_2(q^{'})$, where $4 \mid (q^{'}-1)$ and $q^{'}$ is a prime power. Then $q = q^{'}$ or $\frac{q^{'}+1}{2}$. Moreover, $\frac{q^{'}(q^{'2}-1)}{2} \leq \frac{q(q^2-1)}{2|N|}$ in both cases. If $q = q^{'}$, then $q(q^2-1) \leq \frac{q(q^2-1)}{|N|}$, which implies that $|N| = 1$. It follows that $G \cong L_2(q)$.

If $q = \frac{q^{'}+1}{2}$, then $q^{'} = 2q-1$. Since $\frac{q^{'}(q^{'2}-1)}{2} \mid \frac{q(q^2-1)}{2|N|}$, we have that $(2q-1)[(2q-1)^2-1] \leq \frac{q(q^2-1)}{|N|}$. It follows that $q[(2q-1)^2-1] \leq (2q-1)[(2q-1)^2-1] \leq q(q^2-1)$, which implies that $3q \leq 1$, a contradiction.

2.3. $G_1/N \cong L_2(q^{'})$, where $4 \mid q^{'}$ and $q^{'}$ is a prime power. First, let q be a power of $p \neq 2$. Then $q = q^{'}+1$ or $q^{'}-1$, and $q^{'}(q^{'2}-1) \mid \frac{q(q^2-1)}{2|N|}$. If $q = q^{'}+1$, then $q^{'} = q-1$. It follows that $(q-1)[(q-1)^2-1] \leq \frac{q(q^2-1)}{2|N|}$, which implies that $q \leq 5$. Hence, $q = 5$, which implies that $|N| = 1$ and $G \cong L_2(5)$.

If $q = q^{'}-1$, then $q^{'} = q+1$. Since $q^{'}(q^{'2}-1) \mid \frac{q(q^2-1)}{2|N|}$, we have that $(q+1)[(q+1)^2-1] \mid \frac{q(q^2-1)}{2|N|}$. It follows that $q^2+2q \mid q(q^2-1)$, which implies that $q+2 \mid q-1$, a contradiction.

Now, let q be a power of 2. Then $q = q' + 1$, or q', and $q'(q'^2 - 1) \mid \frac{q(q^2-1)}{|N|}$. If $q = q' + 1$, then $q' = q - 1$. It follows that $(q-1)[(q-1)^2 - 1] \leq \frac{q(q^2-1)}{|N|}$, which implies that $q \leq 5$. Hence, $q = 4$, which implies that $q' = 3$, a contradiction.

If $q = q'$, then $q(q^2 - 1) \leq \frac{q(q^2-1)}{|N|}$, which implies that $|N| = 1$. It follows that $G \cong L_2(q)$.

2.4. $G_1/N \cong L_3(2)$ or $L_3(4)$. If $G_1/N \cong L_3(2) \cong L_2(7)$, then q must be equal to 3, 7. Since $q > 3$, $q = 7$, which implies that $|N| = 1$ and $G \cong L_2(7)$, as desired.

If $G_1/N \cong L_3(4)$, then q must be equal to 3, 5, 7 or 9. So, $q = 5$, 7, or 9. Since $|L_3(4)| \mid |G|$, we get a contradiction.

Step 3. If $G_1/N \cong F_4(q')$, where q' is a prime power, then we distinguish the following two cases.

3.1. Suppose $G_1/N \cong F_4(q')$, where q' is an odd prime power. Then $q = q'^4 - q'^2 + 1$ and $q'^{24}(q'^8 - 1)(q'^6 - 1)^2(q'^4 - 1) \mid \frac{q^2-1}{2}$(or $q^2 - 1$ when q is even). Thus $q^2 = (q'^4 - q'^2 + 1)^2 \leq q'^8$ and $q'^{24} < q'^{24}(q'^8 - 1)(q'^6 - 1)^2(q'^4 - 1) \leq \frac{q^2-1}{2} < q^2$. Hence, $q'^{24} < q'^8$, which implies that $q' < 1$, a contradiction.

3.2. Suppose $G_1/N \cong F_4(q')$, where $2 \mid q'$ and $q' > 2$. Then $q = q'^4 + 1$ or $q'^4 - q'2 + 1$. If $q = q'^4 + 1$, then $q'^{24}(q'^6 - 1)^2(q'^4 - 1)^2(q'^4 - q'^2 + 1) \mid \frac{q^2-1}{2}$(or $q^2 - 1$ when q is even). Thus $q^2 = (q'^4 + 1)^2 < q'^{10}$ and $q'^{24} < q'^{24}(q'^6 - 1)^2(q'^4 - 1)^2(q'^4 - q'^2 + 1) \leq \frac{q^2-1}{2} < q^2$. Hence, $q'^{24} < q'^{10}$, which implies that $q' < 1$, a contradiction. If $q = q'^4 - q'^2 + 1$, then $q'^{24}(q'^6 - 1)^2(q'^4 - 1)^2(q'^4 + 1) \mid \frac{q^2-1}{2}$(or $q^2 - 1$ when q is even). Thus $q^2 = (q'^4 - q'^2 + 1)^2 < q'^8$ and $q'^{24} < q'^{24}(q'^6 - 1)^2(q'^4 - 1)^2(q'^4 + 1) \leq \frac{q^2-1}{2} < q^2$. Hence $q'^{24} < q'^8$, which implies that $q' < 1$, a contradiction.

Step 4. If $G_1/N \cong {}^2F_4(q')$, where $q' = 2^{2t+1} > 2$, then $q = q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'} + 1$ and $q'^{12}(q'^4 - 1)(q'^3 + 1)(q'^2 + 1)(q' - 1)(q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'} + 1) \mid \frac{q^2-1}{2}$ (or $q^2 - 1$ when q is even). Thus $q^2 = (q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'} + 1)^2 \leq q'^{10}$ and $q'^{12} < q'^{12}(q'^4 - 1)(q'^3 + 1)(q'^2 + 1)(q' - 1)(q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'} + 1) \leq \frac{q^2-1}{2} < q^2$. Hence, $q'^{12} < q'^{10}$, which implies that $q' < 1$, a contradiction.

Step 5. If $G_1/N \cong G_2(q')$, where $3 \mid q'$. Then $q = q'^2 + q' + 1$ or $q'^2 - q' + 1$.

If $q = q'^2 + q' + 1$, then $q'^6(q'^2 - 1)^2(q'^2 - q' + 1) \mid \frac{q^2-1}{2}$(or $q^2 - 1$ when q is even). Thus $q^2 = (q'^2 + q' + 1)^2 \leq (q'^3 - 1)^2 \leq q'^6$ and $q'^6(q'^2 - 1) < q'^6 (q'^2 - 1)^2(q'^2 - q' + 1) \leq \frac{q^2-1}{2} < q^2$. Hence, $q'^6(q'^2 - 1) < q'^6$, which implies that $q' < 2$, a contradiction.

If $q = q'^2 - q' + 1$, then $q'^6(q'^2 - 1)^2(q'^2 + q' + 1) \mid \frac{q^2-1}{2}$(or $q^2 - 1$ when q is even). Thus $q^2 = (q'^2 - q' + 1)^2 \leq q'^4$ and $q'^6 < q'^6 (q'^2 - 1)^2(q'^2 + q' + 1) \leq \frac{q^2-1}{2} < q^2$. Hence, $q'^6 < q'^4$, which implies that $q' < 1$, a contradiction.

Step 6. If $G_1/N \cong {}^2G_2(q')$, where $q' = 3^{2t+1} > 3$, then $q = q' \pm \sqrt{3q'} + 1$ and $q'^3(q'^2 - 1)(q' \pm \sqrt{3q'} + 1) \mid \frac{q^2-1}{2}$(or $q^2 - 1$ when q is even). Thus $q^2 =$

$(q' \pm \sqrt{3q'}+1)^2 \leq [(q'+1)^2 - 3q']^2 = (q'^2 - q' + 1)^2 < q'^4$ and $q'^3(q'^2-1) < q'^3(q'^2-1)(q' \pm \sqrt{3q'}+1) \leq \frac{q^2-1}{2} < q^2$. Hence, $q'^3(q'^2-1) < q'^4$, which implies that $q' < 2$, a contradiction.

Step 7. If $G_1/N \cong {}^2B_2(q')$, where $q' = 2^{2t+1} > 2$, then we distinguish the following three cases.

7.1. Suppose $q = q' - 1$. Then $q' = q+1$. Since $q'^2(q' - \sqrt{2q'}+1)(q' + \sqrt{2q'}+1) \mid \frac{q^2-1}{2}$(or q^2-1 when q is even), it follows that $(q+1)^2[(q+1)^2+1] \leq \frac{q^2-1}{2} < q^2$, a contradiction.

7.2. Suppose $q = q' - \sqrt{2q'}+1$. Since $q'^2(q'-1)(q'+\sqrt{2q'}+1) \mid \frac{q^2-1}{2}$(or q^2-1 when q is even) and $q' > 2$, it follows that $q'^2(q' - \sqrt{2q'}+1)(q' + \sqrt{2q'}+1) \leq q'^2(q'-1)(q' + \sqrt{2q'}+1) \leq (q^2-1)/2 < q^2 = (q' - \sqrt{2q'}+1)^2$. Therefore $q'^2(q' + \sqrt{2q'}+1) < q' - \sqrt{2q'}+1 < q' + \sqrt{2q'}+1$, which shows that $q'^2 < 1$, a contradiction.

7.3. Suppose $q = q' + \sqrt{2q'}+1$. Since $q'^2(q'-1)(q' - \sqrt{2q'}+1) \mid \frac{q^2-1}{2}$, it follows that $q'^2(q' - \sqrt{2q'}+1)^2 \leq q'^2(q'-1)(q' - \sqrt{2q'}+1) \leq \frac{q^2-1}{2} < q^2 = (q' + \sqrt{2q'}+1)^2$. Therefore $q'(q' - \sqrt{2q'}) < q'(q' - \sqrt{2q'}+1) < q' + \sqrt{2q'}+1 < 2q' + \sqrt{2q'}$, which shows that $q'(q' - \sqrt{2q'}) < 2q' + \sqrt{2q'}$. Thus $\sqrt{q'}(q' - \sqrt{2q'}) < 2\sqrt{q'} + \sqrt{2} < 3\sqrt{q'}$. Hence, $q' - \sqrt{2q'} < 3$. It follows that $4 - \sqrt{7} < q' < 4 + \sqrt{7}$, which shows that $1 < q' < 7$. This is a contradiction since $q' = 2^{2t+1} \geq 8$.

Step 8. If $G_1/N \cong E_7(2)$, $E_7(3)$, or ${}^2E_6(2)$.

8.1. If $G_1/N \cong E_7(2)$, then $|G_1/N| = |E_7(2)| = 2^{63} \cdot 3^{11} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 43 \cdot 73 \cdot 127$ and $q = 73$ or 127. Because $|G_1/N| \nmid |G| = |L_2(q)|$, we get a contradiction.

8.2. If $G_1/N \cong E_7(3)$, then $|G_1/N| = |E_7(3)| = 2^{23} \cdot 3^{63} \cdot 5^2 \cdot 7^3 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 37 \cdot 41 \cdot 61 \cdot 73 \cdot 547 \cdot 757 \cdot 1093$ and $q = 757$ or 1093. Because $|G_1/N| \nmid |G| = |L_2(q)|$, we get a contradiction.

8.3. If $G_1/N \cong {}^2E_6(2)$, then $|G_1/N| = |{}^2E_6(2)| = 2^{36} \cdot 3^9 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$ and $q = 13$, 17 or 19. We get a contradiction by $|G_1/N| \nmid |G| = |L_2(q)|$.

Step 9. If G_1/N is a sporadic simple group, then $q = 5$, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 59, 67, or 71. It is easy to check that $|G_1/N| \nmid |G| = |L_2(q)|$, we get a contradiction.

The other steps are very similar and we omit them.

Now, we have just seen if $G_1/N \cong L_2(q')$, where $4 \mid (q'-1)$ and q' is a prime power, $G_1/N \cong L_2(q')$, where $4 \mid (q'+1)$ and q' is a prime power or $G_1/N \cong L_2(q')$, where $4 \mid q'$ and q' is a prime power, then $q = q'$ and $G \cong L_2(q)$. In the other cases we get a contradiction.

This completes the proof of the main theorem.

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