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Some Properties of Vector-valued Lipschitz Algebras

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Abstract. Let (X, d) be a metric space and $J \subseteq (0, \infty)$ be a nonempty set. We study the structure of the arbitrary intersection of vector-valued Lipschitz algebras, and define a special Banach subalgebra of $\cap\{Lip_\gamma(X, E) : \gamma \in J\}$, where E is a Banach algebra, denoted by $ILip_J(X, E)$. Mainly, we investigate $C-$character amenability of $ILip_J(X, E)$.

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1. Introduction

Let (X, d) be a metric space and $B(X)$ indicates the Banach space consisting of all bounded complex valued functions on X, endowed with the norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)| \qquad (f \in B(X)).$$

Take $\alpha \in \mathbb{R}$ with $\alpha > 0$, then $Lip_\alpha X$ is a subspace of $B(X)$ consisting of all bounded complex-valued functions f on X such that

$$p_\alpha(f) := \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)^\alpha} : x, y \in X, \ \ x \neq y \right\} < \infty. \tag{1.1}$$

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It is well known that $Lip_\alpha X$ endowed with the norm $\|.\|_\alpha$ given by

$$\|f\|_\alpha = p_\alpha(f) + \|f\|_\infty;$$

and pointwise product is a unital commutative Banach algebra, called Lipschitz algebra.

In [1], the authors showed that $\{Lip_\alpha X\}_\alpha$ is a decreasing net respect to relation " $\subseteq$ ". They investigated intersections of Lipschitz algebras and obtained a necessary and sufficient condition for equality of Lipschitz algebras and $B(X)$. They did a detailed study, concerning the structure of Lipschitz spaces $Lip_\alpha X$. Moreover, they investigated arbitrary intersections of Lipschitz algebras, denoted by $\cap_{\alpha\in J} Lip_\alpha X$, where J is an arbitrary subset of $(0,\infty)$. Then they introduced a special subset of $\cap_{\alpha\in J} Lip_\alpha X$, denoted by $ILip_J X$, which is defined as the set of all functions f in $\cap_{\alpha\in J} Lip_\alpha X$ such that

$$\|f\|_J = \sup_{\alpha\in J} \|f\|_\alpha < \infty.$$

They proved that if $M_J = \sup\{\alpha : \alpha \in J\} < \infty$, then $ILip_J X = Lip_{M_J} X$, and for each $f \in ILip_J X$

$$\frac{\|f\|_J}{3} \leq \|f\|_{M_J} \leq 3\|f\|_J.$$

In fact $\|.\|_J$ defines a norm on $ILip_J X$, equivalent to the norm $\|.\|_{M_J}$. They also studied $\cap_{\alpha\in J} Lip_\alpha X$, for the case where $M_J = \infty$ and introduced an especial subspace of $\cap_{\alpha\in J} Lip_\alpha X$, denoted by $Lip_\infty X$, as

$$Lip_\infty X = \{f \in \cap_{\alpha\in J} Lip_\alpha X : \ \|f\|_{Lip_\infty X} < \infty\}$$

,for which

$$\|f\|_{Lip_\infty X} = \sup_{\alpha>0} \|f\|_\alpha.$$

They showed that $Lip_\infty X$ is a Banach space, endowed with the norm $\|.\|_{Lip_\infty X}$. Furthermore, they considered Lipschitz spaces as Banach algebras associated with pointwise product, and studied $C-$character amenability of Lipschitz algebras. In [2], they fully investigated the structure of $lip_\alpha X$, for any metric space (X,d) and $\alpha > 0$. They showed that if $0 < \alpha < \beta < \infty$, then

$$lip_\beta X \subseteq Lip_\beta X \subseteq lip_\alpha X \subseteq Lip_\alpha X, \tag{1.2}$$

and all these inclusions can be proper. The inclusions (1.2) lead them to obtain the structure of arbitrary intersections of $lip_\alpha X$, whenever α runs into $J \subseteq (0,\infty)$. They also introduced and studied $Ilip_J X$ and $lip_\infty X$, analogous to $ILip_J X$ and $Lip_\infty X$.

Moreover, Hu, Monfared and Traynor investigated character amenability of Lipschitz algebras, see [11]. They showed that if X is an infinite compact metric space and $0 < \alpha < 1$, then $Lip_\alpha X$ is not character amenable. Moreover, recently, $C-$character amenability of Lipschitz algebras were studied by Dashti, Nasr Isfahani and Soltani for each $\alpha > 0$, see [7]. In fact, as a generalization of [13], they showed that for $\alpha > 0$ and any, locally compact metric space X, the algebra $Lip_\alpha X$ is $C-$character amenable, for some $C > 0$ if and only if X is uniformly discrete. In [3] they investigated the extensions of Lipschitz functions. In fact they found conditions that a function can extend such that its norm have least increasing. Also they showed that under some conditions every $f \in Lip_\alpha X_0$ ($X_0 \subseteq X$), can be extended to a function $f \in Lip_\alpha X$, preserving Lipschitz norm. In another part of the paper they studied the Lipschitz version of Urysohn‘s lemma.

Let (X, d) be a metric space and $(E, \|.\|)$ be a Banach space over the scalar field $\mathbb{F}(= \mathbb{R}\ or\ \mathbb{C})$. For a constant $\alpha > 0$ and a function $f : X \longrightarrow E$, set

$$p_{\alpha,E}(f) := \sup_{x\neq y} \frac{\|f(x) - f(y)\|_E}{d(x, y)^\alpha},$$

is called the Lipschitz constant of f . For any metric space (X, d), any Banach algebra E and any $\alpha > 0$, we define the Lipschitz algebra $Lip_\alpha(X, E)$ by

$$Lip_\alpha(X, E) := \{f \in B(X, E) : \ p_{\alpha,E}(f) < \infty\},$$

with pointwise multiplication and norm

$$\|f\|_{\alpha,E} := p_{\alpha,E}(f) + \|f\|_{\infty,E}.$$

where

$$B(X, E) = \{f : X \to E \ : \ \|f\|_{\infty,E} < \infty\}$$

and

$$\|f\|_{\infty,E} = \sup_{x\in X} \|f(x)\|_E.$$

The Lipschitz algebra $lip_\alpha(X, E)$ is the subalgebra of $Lip_\alpha(X, E)$ defined by

$$lip_\alpha(X, E) = \{f \in Lip_\alpha(X, E) \ : \ \frac{\|f(x) - f(y)\|_E}{d(x, y)^\alpha} \longrightarrow 0 \text{ as } d(x, y) \longrightarrow 0\}.$$

If X is a locally compact metric space, then $lip^0_\alpha(X, E)$ is the subalgebra of $lip_\alpha(X, E)$ consisting of those functions whose are zero at infinity. In [6], they showed that $lip^0_\alpha(X, E)^{**} = Lip_\alpha(X, E^{**})$ as Banach algebra, whenever the linear space generated by character space $\Delta(E)$ in normed-dense in E^*. Note that for every Banach algebra A, $\Delta(A)$ denotes the spectrum (character space) of A consisting of all nonzero multiplicative linear functionals on A.

It is clear that the Lipschitz algebra $Lip_\alpha(X, E)$ contains the space $Cons(X, E)$ consisting of all constant $E-$valued functions on X. The Lipschitz algebras were first considered by [4, 12, 15]. There are valuable works related to some notions of amenability of Lipschitz algebras. Gourdeau [9, 10] discussed amenability of vector-valued Lipschitz algebras.

In [5], they studied approximate and character amenability of vector-valued Lipschitz algebras.

In this paper, we study an arbitrary intersection of vector-valued Lipschitz algebras, denoted by $ILip_J(X,E)$. In fact for an arbitrary subset J of $(0,\infty)$ let

$$\|f\|_{J,E} = \sup_{\alpha\in J} \|f\|_{\alpha,E}$$

$$ILip_J(X,E) := \{f \in \cap_{\alpha\in J} Lip_\alpha(X,E) \ : \ \|f\|_{J,E} < \infty\}$$

and

$$Ilip_J(X,E) := \{f \in \cap_{\alpha\in J} lip_\alpha(X,E) \ : \ \|f\|_{J,E} < \infty\}.$$

Now suppose that $M_J = \sup\{\alpha : \alpha \in J\}$, then we show that if $M_J < \infty$, then

$$ILip_J(X,E) = Lip_{M_J}(X,E)$$

and if $M_J = \infty$, then

$$ILip_J(X,E) = Lip_\infty(X,E)$$

for which

$$Lip_\infty(X,E) := \{f \in \cap_{\alpha>0} Lip_\alpha(X,E) \ : \ \|f\|_{Lip_{\infty,E}(X,E)} < \infty\}$$

,where

$$\|f\|_{Lip_{\infty,E}(X,E)} = \sup_{\alpha>0} \|f\|_{\alpha,E}.$$

We obtain a necessary and sufficient condition for amenability of $Lip_\infty(X,E)$, as Banach algebra under pointwise multiplication.
Also we state that whenever the Lipschitz algebras are equal. In the rest of the paper, we show that if E is a Banach algebra and f be an arbitrary function, then $f \in Lip_\infty(X,E)$ if and only if $\sigma\circ f \in Lip_\infty X$ for every $\sigma \in E^*$. In the last section we study $Lip_\infty(X,E)$. In fact we show that $Lip_\infty(X,E) = B(X,E)$ with equivalent norms if and only if X is $\epsilon-$uniformly discrete, for some $\epsilon \geq 1$. Recall that (X,d) is called $\varepsilon-$uniformly discrete, for some $\varepsilon > 0$, if

$$d(x,y) \geq \varepsilon \qquad (x,y \in X, x \neq y).$$

2. The structure of Lipschitz algebra $Lip_\alpha(X,E)$

Let (X,d) be a metric space and $\alpha > 0$. It is easy to show that $Lip_\alpha(X,E)$, $lip_\alpha(X,E)$ and $lip^0_\alpha(X,E)$ are vector spaces, Banach space and Banach algebra, whenever E is so, respectively.
The purpose of this section is studying the structure of $Lip_\alpha(X,E)$, where $\alpha > 0$. We investigate conditions related to equality of two Lipschitz algebras. We show that if $0 < \alpha < \beta$, then

$$lip_\beta(X,E) \subseteq Lip_\beta(X,E) \subseteq lip_\alpha(X,E) \subseteq Lip_\alpha(X,E).$$

Also we obtain another criteria for the norms of $Lip_\alpha(X,E)$ and $B(X,E)$ by considering dual space E^*. Finally we find a necessary and sufficient condition for which a function be in Lipschitz algebra $Lip_\alpha(X,E)$.

Lemma 2.1. *Let (X, d) be a metric space, E be a Banach algebra and $0 \leq \alpha, \beta \leq 1$. Then the following statements are equivalent:*

(1) $Lip_\alpha(X, E) = Lip_\beta(X, E)$*, with equivalent norms.*
(2) $lip_\alpha(X, E) = lip_\beta(X, E)$*, with equivalent norms.*
(3) X *is uniformly discrete or* $\alpha = \beta$*.*

Proof. (1) $\Rightarrow$ (3): Suppose that $Lip_\alpha(X, E) = Lip_\beta(X, E)$ and $\alpha \neq \beta$. We show that X is uniformly discrete. Without less of generality suppose that $\alpha < \beta$. By using [5, Corollary 2.3] we have

$$Lip_\beta(X, E) \subseteq lip_\alpha(X, E) \subseteq Lip_\alpha(X, E) = Lip_\beta(X, E).$$

Consequently $Lip_\alpha(X, E) = lip_\alpha(X, E)$. By using [5, Lemma 2.8], we have $Lip_\alpha X = lip_\alpha X$. Now by using [12, Lemma 2.5], it follows that X is uniformly discrete space.
(3) $\Rightarrow$ $(1, 2)$: It is obtained By using [5, Theorem 2.10].
(2) $\Rightarrow$ (3): Suppose that $lip_\alpha(X, E) = lip_\beta(X, E)$ and $\alpha \neq \beta$. We show that X is uniformly discrete. Without less of generality, suppose that $\alpha < \beta$. By using [5, Corollary 2.3] we have

$$Lip_\beta(X, E) \subseteq lip_\alpha(X, E) = lip_\beta(X, E) \subseteq Lip_\beta(X, E).$$

Consequently $Lip_\beta(X, E) = lip_\beta(X, E)$. By using [5, Lemma 2.8], we have $Lip_\beta X = lip_\beta X$. Now by using [12, Lemma 2.5], it follows that X is uniformly discrete space. □

If we eliminate the condition $0 < \alpha, \beta \leq 1$, then the above lemma is not valid. For instance note that to the following example:

Example 2.2. Let $X := \mathbb{R}$ with $d(x, y) = |x - y|$, for every $x, y \in \mathbb{R}$. $\alpha = 2$, $\beta = 3$ and $E := \mathbb{C}$. Then by using [5, Example 2.5], we have $Lip_\alpha(X, E) = Lip_\beta(X, E)$ but neither $\alpha = \beta$ nor X is uniformly discrete.

Remark 2.3. Note that if (X, d) is a metric space ,with at least two elements, and $0 < \alpha \leq 1$, then $Cons(X) \neq Lip_\alpha X$. Suppose $f : X \to \mathbb{C}$ be defined by $f(x) = \min\{1, d^\alpha(x, a_0)\}$, such that a_0 is a fixed element of X. Then $f \in Lip_\alpha X - Cons(X)$. In fact $Lip_\alpha X$ separates points of X.

If X is a metric space, then $D(X)$ denotes the set of all cluster points of X.

Lemma 2.4. *Let X be a nonzero normed space. Then*

(1) $D(X) = X$*,*
(2) X *is not uniformly discrete,*
(3) $Lip_\alpha(X, E) \subsetneqq B(X, E)$*, for each Banach space* $E \neq \{0\}$ *and* $\alpha > 0$*.*

Proof. (1): Let $x \in X$ and $\epsilon > 0$. Then there exists $n \in \mathbb{N}$ such that $1/n < \epsilon$. Put

$$y = x + \frac{1}{2n} . \frac{y_0}{1 + \|y_0\|}$$

for some $0 \neq y_0 \in X$. Hence $\|x - y\| < \epsilon$. So $x \in D(X)$.
(2), (3) are obtained by definition and [5, Theorem 2.10]. □

By [5, Example 2.5, Theorem 2.10], remark (2.3) and lemma (2.4), the following corollary is immediate.

Corollary 2.5. *Let X be a nonzero normed space, E be a Banach algebra and $\alpha > 0$. Then*

(1) *If $\alpha > 1$, then $Lip_\alpha(X,E) = Cons(X,E)$.*
(2) *If $0 < \alpha \leq 1$, then $Cons(X,E) \subsetneqq Lip_\alpha(X,E) \subsetneqq B(X,E)$.*

Proposition 2.6. *Let X be a nonzero normed space and α, β be two distinct positive numbers. Then $Lip_\alpha X = Lip_\beta X$ if and only if $\alpha, \beta > 1$.*

Proof. If $\alpha, \beta > 1$, then [2, Proposition 2.9] follows that
$Lip_\alpha X = Lip_\beta X = Cons(X)$. Conversely, suppose that $Lip_\alpha X = Lip_\beta X$. Then according to [2, Proposition 2.9] and remark (2.3), we obtain that the case where one of α, β is less than or equal 1 and another is greater than 1, can not be hold. Also if $\alpha, \beta \leq 1$, then by lemma (2.1), X is uniformly discrete. In other hand by lemma (2.4), X is not uniformly discrete. That is a contradiction. Therefore we must have $\alpha, \beta > 1$. □

The following example shows that Proposition (2.6) does not hold for metric space.

Example 2.7. Let X be as defined in [2, Theorem 2.10], $\alpha = 4$ and $\beta = 3$. Then by [2, Proposition 3.1] and [2, Theorem 2.10],

$$Lip_4 X \subseteq lip_3 X \subsetneqq Lip_3 X.$$

So $Lip_4 X \neq Lip_3 X$.

Proposition 2.8. *Let (X,d) be a metric space, E be a Banach algebra and $0 < \alpha < \beta$. Then*

$$lip_\beta(X,E) \subseteq Lip_\beta(X,E) \subseteq lip_\alpha(X,E) \subseteq Lip_\alpha(X,E)$$

and

$$\|f\|_{\alpha,E} \leq 3\|f\|_{\beta,E} \quad (f \in Lip_\beta(X,E)).$$

Proof. At first, we prove the inequality of norms. Suppose that $f \in Lip_\beta(X,E)$. Consider two following cases:

(i) If $d(x,y) \geq 1$, then

$$\|f(x) - f(y)\|_E \leq 2\|f\|_{\infty,E} d^\alpha(x,y) \leq 2\|f\|_{\beta,E} d^\alpha(x,y).$$

(ii) If $d(x,y) < 1$, then

$$\|f(x) - f(y)\|_E \leq p_{\beta,E}(f) d^\beta(x,y) \leq 2\|f\|_{\beta,E} d^\alpha(x,y).$$

Therefore in each case we have:

$$\frac{\|f(x) - f(y)\|_E}{d^\alpha(x,y)} \leq 2\|f\|_{\beta,E}$$

Consequently $p_{\alpha,E}(f) \leq 2\|f\|_{\beta,E}$. And finally

$$\|f\|_{\alpha,E} \leq 3\|f\|_{\beta,E}.$$

.

By using a similar argument as in [2, Proposition 3.1] one can show that, $Lip_\beta(X,E) \subseteq lip_\alpha(X,E)$. Other inclusions are obvious by using definition. □

Lemma 2.9. *Let (X,d) be a metric space, E be a Banach algebra, $\alpha > 0$ and $f : X \to E$ be an arbitrary function. Then*

(1) $\|f\|_{\infty,E} = \sup\{\|\sigma \circ f\|_\infty : \sigma \in E^*$ *and* $\|\sigma\| \leq 1\}$.
(2) $p_{\alpha,E}(f) = \sup\{p_\alpha(\sigma \circ f) : \sigma \in E^*$ *and* $\|\sigma\| \leq 1\}$.
(3) $\|f\|_{\alpha,E} = \sup\{\|\sigma \circ f\|_\alpha : \sigma \in E^*$ *and* $\|\sigma\| \leq 1\}$.

Proof. Suppose that $\sigma \in E^*$.

(1) For every $x \in X$,

$$|\sigma \circ f(x)| \leq \|\sigma\|\|f(x)\|_E.$$

Therefore whenever $\|\sigma\| \leq 1$, we have $\|\sigma \circ f\|_\infty \leq \|f\|_{\infty,E}$. Consequently

$$\sup\{\|\sigma \circ f\|_\infty : \|\sigma\| \leq 1\} \leq \|f\|_{\infty,E}.$$

Conversely if $x \in X$, then by using the Hahn-Banach Theorem [8, Theorem 5.7], there exists $\sigma_x \in E^*$ such that $\|\sigma_x\| \leq 1$ and $\sigma_x(f(x)) = \|f(x)\|_E$. Therefore

$$\begin{aligned} \|f\|_{\infty,E} &= \sup\{\|f(x)\|_E : x \in X\} \\ &= \sup\{\sigma_x(f(x)) : x \in X\} \\ &\leq \sup\{|\sigma(f(x))| : \|\sigma\| \leq 1 \text{ and } x \in X\} \\ &= \sup\{\|\sigma \circ f\|_\infty : \|\sigma\| \leq 1\}. \end{aligned}$$

Hence the equality is hold.

(2) Suppose that $\alpha > 0$. Then we have

$$\begin{aligned} p_\alpha(\sigma \circ f) &= \sup_{x \neq y} \frac{|\sigma \circ f(x) - \sigma \circ f(y)|}{d^\alpha(x,y)} \\ &\leq \sup_{x \neq y} \frac{|\sigma(f(x) - f(y))|}{d^\alpha(x,y)} \\ &\leq \|\sigma\| \sup_{x \neq y} \frac{\|f(x) - f(y)\|_E}{d^\alpha(x,y)} \\ &= \|\sigma\| p_{\alpha,E}(f). \end{aligned}$$

Therefore $\sup\{p_\alpha(\sigma\circ f) : \|\sigma\| \leq 1\} \leq p_{\alpha,E}(f)$. Conversely for every $x, y \in X$ there exists $\sigma_{x,y} \in E^*$ such that
$\sigma_{x,y}(f(x) - f(y)) = \|f(x) - f(y)\|$ and $\|\sigma_{x,y}\| \leq 1$. Therefore

$$\begin{aligned}
p_{\alpha,E}(f) &= \sup\{\frac{\|f(x)-f(y)\|_E}{d^\alpha(x,y)} : x \neq y\} \\
&= \sup\{\frac{\sigma_{x,y}(f(x)-f(y))}{d^\alpha(x,y)} : x \neq y\} \\
&\leq \sup\{\frac{|\sigma(f(x)) - \sigma(f(y))|}{d^\alpha(x,y)} : \|\sigma\| \leq 1 \text{ and } x \neq y\} \\
&= \sup\{p_\alpha(\sigma\circ f) : \|\sigma\| \leq 1\}.
\end{aligned}$$

(3) By using (1) and (2),

$$\|f\|_{\alpha,E} = \sup\{\|\sigma\circ f\|_\alpha : \sigma \in E^*, \|\sigma\| \leq 1\}.$$

□

By using the principle of uniform boundedness theorem, the following lemma is immediate.

Lemma 2.10. *Let (X, d) be a metric space and E be a Banach algebra. For every $\sigma \in E^*$, define T_σ*

$$T_\sigma : B(X, E) \to B(X) \text{ (resp. } T_\sigma : Lip_\alpha(X, E) \to Lip_\alpha(X))$$

such that for every $f \in B(X, E)$ (resp. $Lip_\alpha(X, E)$)

$$T_\sigma(f) := \sigma\circ f.$$

Then $\{T_\sigma\}_{\|\sigma\|\leq 1}$ is a family of continuous linear functionals such that

$$\sup_{\|\sigma\|\leq 1} \|T_\sigma\| < \infty$$

In fact for every $\sigma \in E^*$, we have

$$\|T_\sigma\| \leq \|\sigma\|.$$

Proposition 2.11. *Let (X, d) be a metric space, E be a Banach algebra, $\alpha > 0$ and $f : X \to E$ be an arbitrary function. Then the following statements are equivalent:*

(1) *$f \in B(X, E)$ (resp. $Lip_\alpha(X, E)$),*
(2) *$\sigma\circ f \in B(X)$ (resp. $Lip_\alpha X$), for each $\sigma \in E^*$.*

Proof. If $f = 0$, then the conclusion is obvious. Now suppose that $f \neq 0$ and consider two following cases:
(i) If $f \in B(X, E)$ and $\sigma \in E^*$, then it is obvious that $\sigma\circ f \in B(X)$.

Conversely, suppose that $\sigma \in E^*$ such that $\sigma \circ f \neq 0$. Therefore $\sigma \circ f \in B(X)$. Also let $p \in X$ such that $f(p) \neq 0$. Define

$$0 \neq z := \frac{f(p)}{\|f(p)\|_E \|\sigma \circ f\|_\infty} \in E.$$

Therefore by Hahn-Banach Theorem there exists $\overline{\sigma} \in E^*$ such that $\|\overline{\sigma}\| \leq 1$ and $\overline{\sigma}(z) = 1$. Now define the function $\overline{f} : X \to E$ as following:

$$\overline{f}(x) := (\sigma \circ f(x)).z \quad (x \in X).$$

Clearly

$$\|\overline{f}\|_{\infty,E} \leq \|\sigma \circ f\|_\infty \|z\|_E = \frac{\|\sigma \circ f\|_\infty \|f(p)\|_E}{\|f(p)\|_E \|\sigma \circ f\|_\infty} = 1 \tag{2.1}$$

So $\overline{f} \in B(X, E)$. Also obviously we have $\overline{\sigma} \circ \overline{f} = \sigma \circ f$. Consequently by using lemmas (2.9) and (2.10),

$$\begin{aligned}
\|f\|_{\infty,E} &= \sup\{\|\sigma \circ f\|_\infty : \|\sigma\| \leq 1\} \\
&= \sup\{|\sigma \circ f(x)| : \|\sigma\| \leq 1 \text{ and } x \in X\} \\
&\leq \sup\{|\sigma \circ h(x)| : \|\sigma\| \leq 1, \|h\|_{\infty,E} \leq 1 \text{ and } x \in X\} \\
&= \sup\{\|\sigma \circ h\|_\infty : \|\sigma\| \leq 1 \text{ and } \|h\|_{\infty,E} \leq 1\} \\
&= \sup\{\|T_\sigma(h)\|_\infty : \|\sigma\| \leq 1 \text{ and } \|h\|_{\infty,E} \leq 1\} \\
&\leq \sup\{\|T_\sigma\| : \|\sigma\| \leq 1\} \\
&< \infty.
\end{aligned}$$

Therefore $f \in B(X, E)$.

(ii) If $f \in Lip_\alpha(X, E)$ and $\sigma \in E^*$, then it is obvious that $\sigma \circ f \in Lip_\alpha X$. Conversely suppose that $\sigma \in E^*$, such that $\sigma \circ f \neq 0$. Therefore $\sigma \circ f \in Lip_\alpha X$. Also let $p \in X$ such that $f(p) \neq 0$. Put

$$0 \neq z := \frac{f(p)}{\|f(p)\|_E \|\sigma \circ f\|_\alpha} \in E.$$

Let $\overline{\sigma}$ and $\overline{f}$ be such as case (i). Then:

$$\begin{aligned}
p_{\alpha,E}(\overline{f}) &= \sup_{x \neq y} \frac{\|\overline{f}(x) - \overline{f}(y)\|_E}{d^\alpha(x, y)} \\
&= \sup_{x \neq y} \frac{\|(\sigma \circ f(x)).z - (\sigma \circ f(y)).z\|_E}{d^\alpha(x, y)} \\
&= \|z\|_E \sup_{x \neq y} \frac{\|\sigma \circ f(x) - \sigma \circ f(y)\|_E}{d^\alpha(x, y)} \\
&= \|z\|_E \; p_\alpha(\sigma \circ f).
\end{aligned}$$

Thus the inequality (2.1) and equality (2.2) follow that $\|\overline{f}\|_{\alpha,E} \leq 1$. So $\overline{f} \in Lip_\alpha(X, E)$. Clearly as in (i), $\overline{\sigma} \circ \overline{f} = \sigma \circ f$. Since for every $\sigma \in E^*$,

$\sigma \circ f \in Lip_\alpha X \subseteq B(X)$, thus by using (i), $f \in B(X,E)$. Also we have by using lemmas (2.9) and (2.10),

$$\begin{aligned} p_{\alpha,E}(f) &= \sup\{p_\alpha(\sigma \circ f) : \|\sigma\| \leq 1\} \\ &= \sup\{\frac{|\sigma \circ f(x) - \sigma \circ f(y)|}{d^\alpha(x,y)} \ : \ x \neq y \text{ and } \|\sigma\| \leq 1\} \\ &\leq \sup\{\frac{|\sigma \circ h(x) - \sigma \circ h(y)|}{d^\alpha(x,y)} \ : \ \|\sigma\| \leq 1, \|h\|_{\alpha,E} \leq 1 \text{ and } x \neq y\} \\ &= \sup\{\frac{|T_\sigma(h)(x) - T_\sigma(h)(y)|}{d^\alpha(x,y)} \ : \ \|\sigma\| \leq 1, \|h\|_{\alpha,E} \leq 1 \text{ and } x \neq y\} \\ &= \sup\{p_\alpha(T_\sigma(h)) \ : \ \|\sigma\| \leq 1 \text{ and } \|h\|_{\alpha,E} \leq 1\} \\ &\leq \sup\{\|T_\sigma(h)\| \ : \ \|\sigma\| \leq 1 \text{ and } \|h\|_{\alpha,E} \leq 1\} \\ &\leq \sup\{\|\sigma\|\|h\|_{\alpha,E} \ : \ \|\sigma\| \leq 1 \text{ and } \|h\|_{\alpha,E} \leq 1\} \\ &\leq 1. \end{aligned}$$

So $f \in Lip_\alpha(X,E)$. □

3. The structure of the algebra $Lip_\infty(X,E)$

Let (X,d) be a metric space, E be a Banach algebra and $J \subseteq (0,\infty)$ be a nonempty set. In this section we study the structure and properties of $ILip_J(X,E)$, whenever $M_J = \infty$. For this purpose, we define $Lip_\infty(X,E)$ as following. Let

$$Lip_\infty(X,E) = \{f \in \cap_{\alpha>0} Lip_\alpha(X,E) \ : \ \|f\|_{Lip_\infty(X,E)} < \infty\}$$

where

$$\|f\|_{Lip_\infty(X,E)} := \sup_{\alpha>0} \|f\|_{\alpha,E} = p_{\infty,E}(f) + \|f\|_{\infty,E}$$

such that

$$p_{\infty,E}(f) := \sup_{\alpha>0} p_{\alpha,E}(f).$$

Note that by definition,

$$Lip_\infty(X,E) = \{f : X \to E \ : \ \|f\|_{Lip_\infty(X,E)} < \infty\}.$$

We obtain two necessary and sufficient conditions for that a function belongs in $Lip_\infty(X,E)$. Also we find conditions related to equality $Lip_\infty(X,E)$ and $B(X,E)$ or $Cons(X,E)$. Finally we show that whenever $Lip_\infty(X,E)$ is amenable. We begin this section with an elementary proposition.
By a similar argument as used in [1, Theorem 3.3], the following is immediate.

Proposition 3.1. *Let (X,d) be a metric space and E be a Banach algebra. Then $Lip_\infty(X,E)$ is a Banach algebra, endowed with the norm $\|.\|_{Lip_\infty(X,E)}$ and pointwise multiplication.*

Such as lemma (2.9) we have the next lemma for $Lip_\infty(X,E)$. Its proof is obtained by taking supremum over $\alpha > 0$ by using (3) of that lemma.

Lemma 3.2. *Let (X,d) be a metric space, E be a Banach algebra and $f : X \to E$ be a function. Then*

$$\|f\|_{Lip_\infty(X,E)} = \sup\{\|\sigma \circ f\|_{Lip_\infty X} : \sigma \in E^* \text{ and } \|\sigma\| \leq 1\}.$$

The following lemma is obtained by a similar argument as is used in [1, Corollary 2.4, Theorem 2.5, Proposition 3.1].

Lemma 3.3. *Let (X,d) be a metric space, E be a Banach algebra and $J \subseteq (0,+\infty)$. Then*

(1) *If $M_J < \infty$, then:*

$$\frac{\|f\|_{J,E}}{3} \leq \|f\|_{M_J,E} \leq 3\|f\|_{J,E}.$$

(2) *If $M_J = \infty$, then:*

$$\|f\|_{J,E} \leq \|f\|_{Lip_\infty(X,E)} \leq 3\|f\|_{J,E}.$$

And

$$\cap_{\alpha \in J} Lip_\alpha(X,E) = \cap_{\alpha>0} Lip_\alpha(X,E).$$

We know state the main result of this section. The following theorem is immediate by using lemma (3.3).

Theorem 3.4. *Let (X,d) be a metric space, E be a Banach algebra and $J \subseteq (0,+\infty)$. Then:*

(1) *If $M_J < \infty$, then:*

$$ILip_J(X,E) = Lip_{M_J}(X,E), \quad \text{with equivalent norms.}$$

(2) *If $M_J = \infty$, then:*

$$ILip_J(X,E) = Lip_\infty(X,E), \quad \text{with equivalent norms.}$$

The next proposition is very useful in calculating $Lip_\infty(X,E)$. The proof of two next propositions is similar to [1, Propositions 3.5, 3.7].

Proposition 3.5. *Let (X,d) be a metric space and E be a Banach algebra. Then:*

$$Lip_\infty(X,E) = \{f \in B(X,E) \ : \ d(x,y) < 1 \Rightarrow f(x) = f(y)\}.$$

Example 3.6. If (X,d) is a metric space such that

$$diam(X) := \sup\{d(x,y) \ : \ x,y \in X\} < 1$$

and E is Banach algebra, then $Lip_\infty(X,E) = Cons(X,E)$.

Corollary 3.7. *Let (X,d) be a metric space and E be a Banach algebra. Then*

(1) *If X is σ-compact and $f \in Lip_\infty(X,E)$, then f has countable range.*
(2) *If X is compact and $f \in Lip_\infty(X,E)$, then f has finite range.*

EXAMPLE 3.8. (1) If X is σ-compact, $\alpha > 0$ and $f \in Lip_\alpha(X, E)$, then it is not necessary that f has countable range. Suppose $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$. Then it is obvious that $f \in Lip_1(\mathbb{R})$ and X is $\sigma-$compact, but f has not countable range.

(2) If X is compact, $\alpha > 0$ and $f \in Lip_\alpha(X, E)$, then it is not necessary that f has finite range. Suppose $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = x$. Then $f \in Lip_1(\mathbb{R})$ and X is compact, but f has not finite range.

Proposition 3.9. *Let (X, d) be a metric space, E be a Banach algebra and $f : X \to E$ be a function. Then the following statements are equivalent:*

(1) $f \in Lip_\infty(X, E)$,
(2) $\sigma \circ f \in Lip_\infty(X)$ *for each* $\sigma \in E^*$.

Proof. (2)$\Rightarrow$ (1): Suppose that for every $\sigma \in E^*$, $\sigma \circ f \in Lip_\infty X$. Therefore for every $\alpha > 0$, $\sigma \in E^*$ we have $\sigma \circ f \in Lip_\alpha X$. Proposition (2.11) follows that $f \in Lip_\alpha(X, E)$ for every $\alpha > 0$. So $f \in \cap_{\alpha>0} Lip_\alpha(X, E)$. Now suppose that $f \notin Lip_\infty(X, E)$, by using Proposition (3.5), there exist $x, y \in X$ such that $d(x, y) < 1$ and $f(x) \neq f(y)$. Hence by Hahn-Banach theorem, there exists $\sigma_0 \in E^*$ such that $\sigma_0(f(x)) \neq \sigma_0(f(y))$. By hypothesis, $\sigma_0 \circ f \in Lip_\infty X$, therefore [1, proposition 3.5] follows that $\sigma_0(f(x)) = \sigma_0(f(y))$. That is a contradiction. Consequently $f(x) = f(y)$ and by using proposition (3.5), $f \in Lip_\infty(X, E)$.
(1)$\Rightarrow$ (2): Suppose that $f \in Lip_\infty(X, E)$ and $\sigma \in E^*$. By definition $f \in Lip_\alpha(X, E)$ for every $\alpha > 0$. Therefore by using proposition (2.11), $\sigma \circ f \in Lip_\alpha X$ for every $\alpha > 0$. Consequently $\sigma \circ f \in \cap_{\alpha>0} Lip_\alpha X$. Also if $d(x, y) < 1$, then by using proposition (3.5), $f(x) = f(y)$. So $\sigma \circ f(x) = \sigma \circ f(y)$. Now [1, proposition 3.5] follows that $\sigma \circ f \in Lip_\infty X$. □

The next theorem provides a necessary and sufficient condition for equality of $Lip_\infty(X, E)$ with $B(X, E)$.

Theorem 3.10. *Let (X, d) be a metric space, $E \neq \{0\}$ be a Banach algebra. Then $Lip_\infty(X, E) = B(X, E)$, with equivalent norms if and only if X is $\varepsilon-$uniformly discrete, for some $\varepsilon \geq 1$.*

Proof. Suppose that X is not $\epsilon-$uniformly discrete for each $\epsilon \geq 1$. Thus there exist two distinct elements x_0 and x_1 in X such that $d(x_0, x_1) < 1$. Take z to be a nonzero element of E and define the function $g : X \to E$ by

$$g(x) = \begin{cases} 0 & \text{if } x = x_0 \\ z & \text{if } x \neq x_0. \end{cases}$$

Then for each $\alpha > 0$, we have

$$p_{\alpha,E}(g) = \sup_{x \neq x_0} \frac{\|g(x) - g(x_0)\|_E}{d(x, x_0)^\alpha} \geq \frac{\|z\|_E}{d(x_1, x_0)^\alpha}.$$

Consequently

$$\sup_{\alpha>0} p_{\alpha,E}(g) \geq \sup_{\alpha>0} \frac{\|z\|_E}{(d(x_1,x_0))^\alpha} = \infty,$$

and so $g \notin Lip_\infty(X,E)$. Therefore $Lip_\infty(X,E) \subsetneqq B(X,E)$. Conversely, suppose that X is $\varepsilon-$uniformly discrete, for some $\varepsilon \geq 1$. Thus for each $f \in B(X,E)$ we have

$$p_{\infty,E}(f) = \sup_{\alpha\geq 0}\sup_{x\neq y} \frac{\|f(x)-f(y)\|_E}{d(x,y)^\alpha} \leq \sup_{\alpha\geq 0} \frac{2\|f\|_{\infty,E}}{\varepsilon^\alpha} \leq 2\|f\|_{\infty,E}.$$

It follows that $f \in Lip_\infty(X,E)$ and

$$\|f\|_{\infty,E} \leq \|f\|_{Lip_\infty(X,E)} \leq 3\|f\|_{\infty,E}.$$

This completes the proof. □

We know state a criteria for amenability of $Lip_\infty(X,E)$.

Theorem 3.11. *Let (X,d) be a metric space, E be a Banach algebra with $\Delta(E) \neq \emptyset$ and $J \subseteq (0,\infty)$. Also suppose that $ILip_J(X,E)$ separates the points of X. Then:*

1. *If $M_J < \infty$, then $ILip_J(X,E)$ is amenable if and only if E is amenable and X is uniformly discrete.*
2. *If $M_J = \infty$, then $ILip_J(X,E)$ is amenable if and only if E is amenable and X is $\epsilon-$uniformly discrete for some $\epsilon \geq 1$.*

Proof. (1) It is obvious by using Theorem (3.4) and [5, Theorem 4.3].

(2) By using Theorem (3.4), we know that $ILip_J(X,E) = Lip_\infty(X,E)$. Suppose $Lip_\infty(X,E)$ is amenable and $x_0 \in X$. Define the function $\phi : Lip_\infty(X,E) \to E$ by $f \to f(x_0)$. Then ϕ is a linear and epimorphism. by using [14, Proposition 2.3.1], E is amenable. Suppose that $\sigma \in \Delta(E)$, then for every $x \in X$ define $\phi_x : Lip_\infty(X,E) \to \mathbb{C}$ by $\phi_x(f) = \sigma \circ f(x)$. Therefore ϕ_x is a nonzero linear multiplicative functional. Thus $\phi_x \in \Delta(Lip_\infty(X,E))$. Also since $Lip_\infty(X,E)$ separates the points of X it follows that $\phi_x \neq \phi_y$ whenever $x \neq y$. Now [9, Corollary 2] follows that $\Delta(Lip_\infty(X,E))$ is uniformly discrete. Therefore there exists $\epsilon > 0$ such that $0 < \epsilon \leq \|\phi_x - \phi_y\|_{A^*}$ and $A = Lip_\infty(X,E)$. Otherwise for every $\alpha > 0$,

$$\begin{aligned} |\phi_x(f) - \phi_y(f)| &= |\sigma \circ f(x) - \sigma \circ f(y)| \\ &\leq \|\sigma\|\|f(x) - f(y)\| \\ &\leq \|\sigma\| p_{\alpha,E}(f) d^\alpha(x,y). \end{aligned}$$

Furthermore for every $\alpha > 0$,

$$\|f\|_A = \sup_{\beta>0} \|f\|_{\beta,E} \geq \|f\|_{\alpha,E} \geq p_{\alpha,E}(f).$$

Therefore

$$\begin{aligned}\epsilon &\leq \|\phi_x - \phi_y\|_{A^*}\\ &= \sup_{\|f\|_A\leq 1} |\phi_x(f) - \phi_y(f)|\\ &\leq \sup_{\|f\|_A\leq 1} \|\sigma\| p_{\alpha,E}(f) d^{\alpha}(x,y).\end{aligned}$$

Hence for every $\alpha > 0$, we have

$$d(x,y) \geq (\frac{\epsilon}{\|\sigma\|})^{\frac{1}{\alpha}}.$$

By tending α to infinity, we obtain $d(x,y) \geq 1$. Therefore X is ϵ-uniformly discrete, for some $\epsilon \geq 1$.
Conversely, since X is uniformly discrete, by theorem (3.10),

$$Lip_{\infty}(X,E) = B(X,E) = \overline{B(X)\hat{\otimes}E}.$$

Since E and $B(X)$ are amenable, therefore $Lip_{\infty}(X,E)$ is too.

□

Note that the above theorem is true when we change amenability by character amenability.

Remark 3.12. All results of this paper are valid for Banach algebras $lip_{\alpha}(X,E)$ or $Ilip_J(X,E)$, except Theorem (3.4) and Proposition (2.6).

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