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Sharply $(n-2)$-transitive Sets of Permutations

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Abstract. Let S_n be the symmetric group on the set $[n] = \{1, 2, \ldots, n\}$. For $g \in S_n$ let $fix(g)$ denote the number of fixed points of g. A subset S of S_n is called *t-transitive* if for any two t-tuples $(x_1, x_2, \ldots, x_t)$ and $(y_1, y_2, \ldots, y_t)$ of distinct elements of $[n]$, there exists $g \in S$ such that $x_i^g = y_i$ for any $1 \leq i \leq t$ and additionally S is called *sharply t-transitive* if for any given pair of t-tuples, exactly one element of S carries the first to the second. In addition, a subset S of S_n is called t-intersecting if $fix(h^{-1}g) \geq t$ for any two distinct permutations h and g of S. In this paper, we prove that there are only two sharply $(n-2)$-transitive subsets of S_n and finally we establish some relations between sharply k-transitive subsets and t-intersecting subsets of S_n where $k, t \in \mathbb{Z}$ and $0 \leq t \leq k \leq n$.

Keywords: Symmetric group, Sharply transitive set of permutations, Cayley graph.

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1. Introduction

Let S_n be the symmetric group on the set $[n] = \{1, 2, \ldots, n\}$. For each $g \in S_n$, a point x is called a *fixed point* of g if $g(x) = x$ and $fix(g)$ denotes the number of fixed points of g. In addition for each non-empty subset L of $\{0, 1, 2, \ldots, n-2\}$, a subset S of S_n is called *L-intersecting* if, for any two distinct permutations h and g of S we have $fix(h^{-1}g) \in L$ and any L-intersecting subset S of S_n is called t-intersecting where $L = \{t, t+1, t+2, \ldots, n-2\}$.

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Furthermore each subset S of S_n is called *t-transitive* if for any two t-tuples $(x_1, x_2, \ldots, x_t)$ and $(y_1, y_2, \ldots, y_t)$ of distinct elements of $[n]$, there exists $g \in S$ such that

$$1 \leq i \leq t : x_i^g = y_i.$$

Additionally S is called *sharply t-transitive* if for any given pair of t-tuples, exactly one element of S carries the first to the second. It is not difficult to prove that for each t-transitive subset S and each sharply t-transitive subset S_1 of S_n we have

$$|S| \geq \frac{n!}{(n-t)!} = |S_1|.$$

The following theorem is proved by P. J. Cameron, M. Deza and P. Frankl in [3]:

Theorem 1.1. *Let $L = \{0, 1, 2, \ldots, t-1\}$ and S be an L-intersecting subset of S_n . Then*

$$|S| \leq \frac{n!}{(n-t)!}$$

and the equality holds if and only if S is a sharply t-transitive subset of S_n.

In [1], the authors studied *invertible* sharply t-transitive subsets of S_n. A subset of S_n is said to be invertible if it contains the identity and if whenever it contains a permutation it also contains its inverse. The following theorem is one of the main results of [1]:

Theorem 1.2. *Let G be an invertible sharply d-transitive permutation set on a finite set X. If $d \geq 6$ then G is either S_d, S_{d+1} or A_{d+2}. If $d = 5$ then G is either S_5, S_6, A_7 or the Mathieu group of degree 12. If $d = 4$ then G is either S_4, S_5, A_6 or the Mathieu group of degree 11.*

Our main result in this paper is about the structure of a sharply $(n-2)$-transitive subset of S_n. Let S be a sharply $(n-2)$-transitive subset of S_n. By Theorem 1.1, S is an L-intersecting subset of S_n , where $L = \{0, 1, 2, \ldots, n-3\}$, and S has $\dfrac{n!}{2}$ elements. In Theorem 3.1 we will prove that there exist only two sharply $(n-2)$-transitive subsets of S_n.

2. Preliminary Notes

To prove the main results in the next section we need some definitions and lemmas about *Cayley graphs.*

Definition 2.1. For each $g \in S_n$, the set $Fix(g) = \{x \in [n] : g(x) = x\}$ is the *fixed point set* of g and the set $Supp(g) = \{x \in [n] : g(x) \neq x\} = [n] - Fix(g)$ is called the *support set* of g.

Let f and g be two permutations of $[n]$, $k \in \mathbb{N}$ and $k \leq n$. Then by Definition 2.1 we have $\{x \in [n] : f(x) = g(x)\} = Fix(g^{-1}f)$ and $\{x \in [n] : f(x) \neq g(x)\} = Supp(g^{-1}f)$.

Definition 2.2. Let S be a subset of a group G such that $1 \notin S$ and $S = S^{-1}$. The *Cayley graph* $\Gamma(G,S)$ associated with G and S is defined to have vertex set G and a vertex f is joined to a vertex g if $g^{-1}f \in S$ or equivalently $f^{-1}g \in S$ (Since $S = S^{-1}$ it follows that $g^{-1}f \in S \Longleftrightarrow (g^{-1}f)^{-1} = f^{-1}g \in S$).

The following properties of Cayley graphs are well known and proofs can be found, for example, in page 241 of [2].

Lemma 2.3. *(i)* $\Gamma = \Gamma(G,S)$ *is connected if and only if* S *generates* G.

(ii) For each $g \in G$, *the map* $\rho_g : x \mapsto g^{-1}x$ *is an automorphism of* Γ *and* $\{\rho_g : g \in G\} \cong G$.

(iii) $d_\Gamma(f,g) = d_\Gamma(hf,hg)$ *where* $h \in G$, *and* f *and* g *are two vertices of* Γ.

Lemma 2.4. *Let* n *be a positive integer and* S *be the set of all transpositions of* $[n]$. *Then*

(i) $\Gamma = \Gamma(S_n, S)$ *is a bipartite graph.*

(ii) $diam(\Gamma) = n-1$ *and so* Γ *is connected and* $d_\Gamma(f,g) \leq n-1$ *for any two vertices* f *and* g *of* Γ.

(iii) Γ *is Hamiltonian when* $n \geq 3$.

Proof. (i) It is easy to check that Γ is bipartite with bipartition $(A_n, S_n - A_n)$.

(ii) Suppose f and g are distinct and $|Supp(f^{-1}g)| = k$. We claim that

$$d_\Gamma(f,g) \leq k-1.$$

By part (iii) of Lemma 2.3 we have $d_\Gamma(f,g) = d_\Gamma(1, f^{-1}g)$.

Since $|Supp(f^{-1}g)| = k$, $f^{-1}g$ can be written as a product of at most $k-1$ transpositions . So $d_\Gamma(1, f^{-1}g) \leq k-1$. Part (ii) follows.

(iii) This is proved by mathematical induction. Clearly $\Gamma(S_3, S) = K_{3,3}$ is Hamiltonian. Suppose that $k \geq 3$ and that $\Gamma_k = \Gamma(S_k, S)$ is Hamiltonian, so Γ_k has a cycle of length $k!$ that contains all the vertices of Γ_k. We can represent any permutation π^i of $[k]$ by a k-tuple $\pi^i = (\pi^i_1, \pi^i_2, \ldots, \pi^i_k)$ where π^i maps the point j to π^i_j for each j. Now suppose π^1, π^2, ...,$\pi^{k!}$ are consecutive vertices of a Hamiltonian cycle in Γ_k. We construct a Hamiltonian cycle of $\Gamma_{k+1} = \Gamma(S_{k+1}, S)$ as follows:

For every permutation π^i of Γ_k, we construct a path $path(\pi^i)$ with $k+1$ vertices in Γ_{k+1} as follows:

$$\begin{aligned} \text{path}(\pi^i) : (\pi^i_1, \pi^i_2, \ldots, \pi^i_k, k+1), (\pi^i_1, \pi^i_2, \ldots, k+1, \pi^i_k), \\ (\pi^i_1, \pi^i_2, \ldots, k+1, \pi^i_{k-1}, \pi^i_k), \ldots, (k+1, \pi^i_1, \pi^i_2, \ldots, \pi^i_k). \end{aligned}$$

The number of these paths is even ($k!$ paths) and any two of them have no common vertex. Thus each vertex of Γ_{k+1} occurs in exactly one of these paths.

In addition the first vertices of $path(\pi^i)$ and $path(\pi^{i+1})$, $(\pi_1^i, \pi_2^i, \ldots, \pi_k^i, k+1)$ and $(\pi_1^{i+1}, \pi_2^{i+1}, \ldots, \pi_k^{i+1}, k+1)$, are adjacent in Γ_{k+1}, and also the first vertices of $path(\pi^1)$ and $path(\pi^{k!})$ are adjacent.

Also the last vertices of $path(\pi^i)$ and $path(\pi^{i+1})$, $(k+1, \pi_1^i, \pi_2^i, \ldots, \pi_k^i)$ and

$$(k+1, \pi_1^{i+1}, \pi_2^{i+1}, \ldots, \pi_k^{i+1}),$$

are adjacent. Thus we can construct a Hamiltonian cycle of length $(k+1)!$ in Γ_{k+1} by chaining $path(\pi^1)$, $path(\pi^2)$, $\ldots$, $path(\pi^{k!-1})$, and $path(\pi^{k!})$ consecutively as in Figure1. $\square$

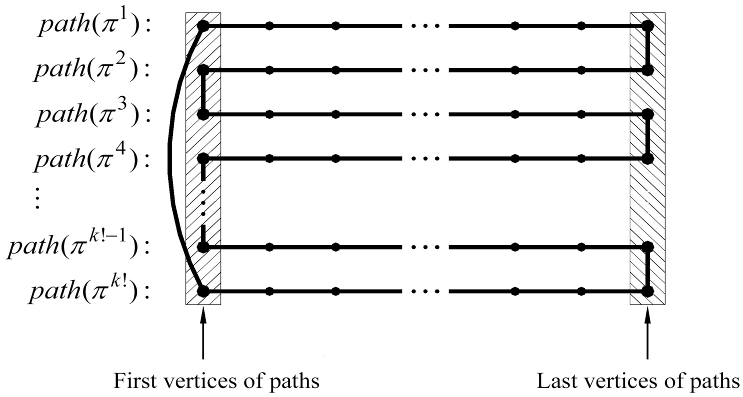

Figure1

3. Main Results

Theorem 3.1. *Let S be a sharply $(n-2)$-transitive subset of S_n. Then $S = A_n$ or $S_n - A_n$.*

We have two proofs:

Suppose S is a sharply $(n-2)$-transitive subset of S_n. So S is an L-intersecting subset of S_n where $L = \{0, 1, 2, \ldots, n-3\}$ and has $\frac{n!}{2}$ elements.

First proof. In this proof we use parts (i) and (ii) of Lemma 2.4. Suppose that $S_n - A_n \neq S \neq A_n$. Then $S \cap A_n \neq \varnothing \neq S \cap (S_n - A_n)$ (since $|S| = |A_n| = |S_n - A_n|$).

Let $|S \cap A_n| = m$. Then $1 \leq m \leq \frac{n!}{2}$ and $|S \cap (S_n - A_n)| = \frac{n!}{2} - m$. By defining $V_1 = S \cap A_n$, $V_2 = A_n - S$, $V_3 = S \cap (S_n - A_n)$ and $V_4 = (S_n - A_n) - S$, we can say that S is partitioned into V_1 and V_3. Any permutations f and g of S are not adjacent in the graph Γ_n of Lemma 2.4 because they disagree on at least 3 points of $[n]$ and hence $d_\Gamma(f, g) \geq 2$; so there is no edge between V_1 and V_3. On the other hand $A_n = V_1 \cup V_2$ and $S_n - A_n = V_3 \cup V_4$ are the two parts of the bipartition of Γ_n. Hence vertices of V_1 could be only connected to vertices of V_4 and also vertices of V_3 could be only connected to vertices of V_2. Let E_i be the set of edges that are incident with some vertex $v \in V_i$

where $1 \leq i \leq 4$. Therefore $E_1 \subseteq E_4$ and $E_3 \subseteq E_2$. Since $|V_1| = |V_4| = m$ and $|V_2| = |V_3| = \frac{n!}{2} - m$ we have

$$|E_1| = \sum_{v \in V_1} deg(v) = m\binom{n}{2} = \sum_{v \in V_4} deg(v) = |E_4|$$

and

$$|E_2| = \sum_{v \in V_2} deg(v) = (\frac{n!}{2} - m)\binom{n}{2} = \sum_{v \in V_3} deg(v) = |E_3|.$$

So $E_1 = E_4$ and $E_2 = E_3$. Hence there are no edges between V_2 and V_4, and vertices of V_2 are only adjacent with vertices of V_3. Now observe that there is no edge between $V_1 \cup V_4$ and $V_2 \cup V_3$. But this contradicts part (ii) of Lemma 2.4. Hence we must have $S = A_n$ or $S_n - A_n$.

Second proof. In this proof we use part (iii) of Lemma 2.4. Since Γ_n is Hamiltonian, there is a cycle of length $n!$ in Γ_n. Let $f_1, f_2, \ldots, f_{n!}$ be consecutive vertices of this Hamiltonian cycle. So $d_{\Gamma_n}(f_i, f_{i+1}) = 1$ for any $1 \leq i \leq n!$ (setting $f_{n!+1} = f_1$). Now assume that f_{α_1}, f_{α_2}, ...,f_{α_m} are the elements of S and $\alpha_1 < \alpha_2 < \cdots < \alpha_m$ where $m = \frac{n!}{2}$.

Since any two permutations of S disagree on at least 3 points of $[n]$ we have

$$d_{\Gamma_n}(f_{\alpha_i}, f_{\alpha_{i+1}}) \geq 2 \ \ (\star)$$

for $1 \leq i \leq m$ where we set $f_{\alpha_{m+1}} = f_{\alpha_1}$.

On the other hand by considering the Hamiltonian cycle we have

$$d_{\Gamma_n}(f_{\alpha_1}, f_{\alpha_2}) + d_{\Gamma_n}(f_{\alpha_2}, f_{\alpha_3}) + \cdots + d_{\Gamma_n}(f_{\alpha_m}, f_{\alpha_1}) \leq n! \ \ (\star\star).$$

From $(\star)$ and $(\star\star)$ we conclude that

$$d_{\Gamma_n}(f_{\alpha_1}, f_{\alpha_2}) = d_{\Gamma_n}(f_{\alpha_2}, f_{\alpha_3}) = \cdots = d_{\Gamma_n}(f_{\alpha_m}, f_{\alpha_1}) = 2 \ \ (\star\star\star).$$

If $f_{\alpha_1} \in A_n$ then since $d_{\Gamma_n}(f_{\alpha_1}, f_{\alpha_2}) = 2$ we conclude that $f_{\alpha 2} \in A_n$ and similarly we have $f_{\alpha_i} \in A_n$(for $3 \leq i \leq m = \frac{n!}{2}$) and so $S = A_n$ (since $|S| = |A_n| = \frac{n!}{2}$). If f_{α_1} is an odd permutation then since $d_{\Gamma_n}(f_{\alpha_1}, f_{\alpha_2}) = 2$ we conclude that f_{α_2} is an odd permutation and hence $f_{\alpha_2} \in S_n - A_n$. Similarly we have $f_{\alpha_i} \in S_n - A_n$(for $3 \leq i \leq m = \frac{n!}{2}$). Hence $S = S_n - A_n$. $\square$

Let f and g be two permutations of $[n]$, $t \in \mathbb{N}$ and $t \leq n$. Then f and g are said to be *t-intersecting* if $|Fix(g^{-1}f)| \geq t$. Also a family S of permutations of $[n]$ is *t-intersecting* if $|Fix(g^{-1}f)| \geq t$ (Or equivalently $|\{ x \in [n] : f(x) = g(x)\}| \geq t$) for any two permutations f and g of S. Cameron and Ku proposed the following problem at the end of [4]:

Problem. *Given $t \geq 1$, is there a number $n_0(t)$ such that, if $n \geq n_0(t)$, then a t-intersecting subset of S_n has cardinality at most $(n-t)!$, and a set meeting the bound is a coset of the stabilizer of t points?*

The following theorems give some relations between intersecting subsets and sharply t-transitive subsets of S_n. Particularly, Theorems 1.1 and 3.2(vi) show that if there exists a sharply t-transitive subset of S_n, then any t-intersecting subset of S_n has at most $(n-t)!$ elements. Part (v) of the following theorem is proved by M. Deza and P. Frankl in the first lemma of [5].

Theorem 3.2. *Let $L \subseteq \{0,1,2,\ldots,t-1\}$, S be an L-intersecting subset and S_1 be a t-intersecting subset of S_n where $1 \leq t \leq n$. Then, for each $g \in S_n$,*

(i) $\{gf : f \in S\}$ and $\{fg : f \in S\}$ are L-intersecting,

(ii) $\{gf : f \in S_1\}$ and $\{fg : f \in S_1\}$ are t-intersecting,

(iii) $\{g_1 f : f \in S\} \cap \{g_2 f : f \in S\} = \varnothing$ for distinct permutations g_1 and g_2 of S_1,

(iv) $\{g_1 f : f \in S_1\} \cap \{g_2 f : f \in S_1\} = \varnothing$ for distinct permutations g_1 and g_2 of S,

(v) $|S_1||S| \leq n!$, and

(vi) If $L = \{0,1,2,\ldots,t-1\}$, and $|S| = \frac{n!}{(n-t)!}$ then $|S_1| \leq (n-t)!$.

Proof. The proofs of (i) and (ii) are immediate from the definitions. The proof of (iv) is similar to that of (iii), and (vi) follows from (v). Thus we only need to prove (iii) and (v).

(iii) Suppose $g_1 \neq g_2$ in S_1 and $\{g_1 f : f \in S\} \cap \{g_2 f : f \in S\} \neq \varnothing$. Then there exist f_1 and f_2 in S such that $g_1 f_1 = g_2 f_2$. From (i) we conclude that $g_1 f_1$ and $g_1 f_2$ disagree on at least $n+1-t$ points of $[n]$. Hence $g_2 f_2$ and $g_1 f_2$ also disagree on at least $n+1-t$ points of $[n]$, so $g_2 f_2$ and $g_1 f_2$ agree on at most $n-(n+1-t) = t-1$ points of $[n]$. But this contradicts part (ii). Hence we must have $\{g_1 f : f \in S\} \cap \{g_2 f : f \in S\} = \varnothing$.

(v) From (iv) it follows that

$$|\bigcup_{g \in S_1} \{gf : f \in S\}| = \sum_{g \in S_1} |\{gf : f \in S\}| = \sum_{g \in S_1} |S| = |S_1||S|.$$

On the other hand $\bigcup_{g \in S_1} \{gf : f \in S\} \subseteq S_n$. Hence $|S_1||S| \leq n!$. □

Finally we show that, wherever a sharply k-transitive subset of S_n exists, we can partition it into many t-intersecting families where $0 \leq t \leq k$ and each of them determines a sharply $(k-t)$-transitive subset of permutations.

Theorem 3.3. *Let $k, t \in \boldsymbol{Z}$, $0 \leq t \leq k \leq n$, S be a sharply k-transitive family of permutations of $[n]$ and A be an arbitrary t-subset of $[n]$. Then we can partition S into $N = \frac{n!}{(n-t)!}$ families $C_1, C_2, \ldots, C_N$ each of size $\frac{(n-t)!}{(n-k)!}$ such that for each $f \in C_i (1 \leq i \leq N)$, $f^{-1}C_i$ represents a sharply $(k-t)$-transitive subset of permutations of $[n]-A$.*

Proof. We define a relation R on S as follows:

$$g, f \in S : gRf \Longleftrightarrow \forall i \in A : f(i) = g(i).$$

Then R is an equivalence relation and partitions S into at most $N = \frac{n!}{(n-t)!}$ equivalence classes. Suppose C is one of these equivalence classes and $f \in C$.

Then for each $g \in C$,

$$i \in A \Longrightarrow g(i) = f(i) \Longrightarrow f^{-1}g(i) = i$$

and

$$i \in [n] - A \Longrightarrow f^{-1}g(i) \in [n] - A.$$

Thus each $g \in C$ determines a permutation of $X = [n] - A$, namely the restriction $f^{-1}g\,|_X$.

Now suppose that $A = \{x_1, x_2, x_3, \ldots, x_t\}$ and $u = (u_1, u_2, u_3, \ldots, u_{k-t})$ and $v = (v_1, v_2, v_3, \ldots, v_{k-t})$ are two $(k-t)$-tuples of distinct elements of $[n] - A$.

Because S is sharply k-transitive then by Theorem 3.2(i), $f^{-1}S = \{f^{-1}g | g \in S\}$ is also sharply k-transitive and so for two k-tuples

$$u' = (x_1, x_2, \ldots, x_t, u_1, u_2, u_3, \ldots, u_{k-t})$$

and

$$v' = (x_1, x_2, \ldots, x_t, v_1, v_2, v_3, \ldots, v_{k-t}),$$

there exists exactly one element of $f^{-1}S$ like h such that $u'^h = v'$.Then $u^h = v$ and $x_i^h = x_i (1 \leq i \leq t)$. So $h \in f^{-1}C$ and $C' = \{h\,|_X \; | h \in f^{-1}C\}$ is $(k-t)$-transitive set of permutations of X. In addition we can easily conclude sharpness of C' from the sharpness of $f^{-1}S$.

Thus from Theorem 1.1 we have

$$|C| = |f^{-1}C| = |C'| \leq \frac{(n-t)!}{((n-t)-(k-t))!} = \frac{(n-t)!}{(n-k)!}.$$

There are at most $N = \frac{n!}{(n-t)!}$ equivalence classes C, and so

$$\sum_C |C| \leq N.\frac{(n-t)!}{(n-k)!} = \frac{n!}{(n-k)!} \ (\star).$$

However $\sum_C |C| = |S| = \frac{n!}{(n-k)!}$ $(\star\star)$.

From $(\star)$ and $(\star\star)$ we conclude that there are exactly N non-empty classes, each of size $\frac{(n-t)!}{(n-k)!}$. □

In Theorem 3.3, if $t = k-1$, then $|C_i| = \frac{(n-k+1)!}{(n-k)!} = n-k+1$, and any equivalence class represents a Latin square of order $n-k+1$. Also if $t = 1$, then $N = \frac{n!}{(n-1)!} = n$ and $|C'| = \frac{(n-1)!}{(n-k)!}$ $(1 \leq i \leq N)$ and in this case, we can construct n sharply $(k-1)$-transitive subsets of permutations of $[n-1]$.

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References

1. A. Bonisoli, P. Quattrocchi, Each invertible sharply d-transitive finite permutation set with $d \geq 4$ is a group. *J. Algebraic Combin.*, **12**(3), (2000), 241–250.
2. P. J. Cameron, *Combinatorics: topics, techniques, algorithms*, Cambridge University Press, 1994.
3. P. J. Cameron, M. Deza, P. Frankl, Intersection Theorems in Permutation Groups, *Combinatorica*, **8**(3), (1988), 249–260.
4. P. J. Cameron, C. Y. Ku, Intersecting Families of Permutations, *European J. Combinatorics*, **24**(7), (2003), 881–890.
5. M. Deza, P. Frankl, On the maximum number of permutations with given maximal or minimal distance, *J. Combinatorial Theory (A)*, **22**(3), (1977), 352–360.