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On the Hyponormal Property of Operators

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Abstract. Let T be a bounded linear operator on a Hilbert space $\mathscr{H}$. We say that T has the hyponormal property if there exists a function f, continuous on an appropriate set so that $f(|T|) \geq f(|T^*|)$. We investigate the properties of such operators considering certain classes of functions on which our definition is constructed. For such a function f we introduce the f-Aluthge transform, $\tilde{T}_f$. Given two continuous functions f and g with the property $f(t)g(t) = t$, we also introduce the (f, g)-Aluthge transform, $\tilde{T}_{(f,g)}$. The features of these transforms are discussed as well.

Keywords: Hyponormal operators, Hyponormal property, Aluthge transform, Normal operator.

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1. Introduction

In this paper, $\mathbb{B}(\mathscr{H})$ denotes the algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$. An operator T is said to be self–adjoint if $T = T^*$. T is positive if it is self–adjoint and the points in the spectrum are all positive. The spectrum of the operator T is denoted by $\sigma(T)$. Let T be a bounded linear operator and $T = U|T|$ be the polar decomposition of T where U is a partial isometry and $|T| = (T^*T)^{\frac{1}{2}}$. This decomposition is unique as long as the kernel of U is the same as that of $|T|$. The operator T is invertible if and only if U is a unitary operator and $|T|$ is invertible. We denote by $\mathscr{R}(T)$

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and $\mathscr{N}(T)$ the range and the kernel of T, respectively; see [4]. If $T = U|T|$ is the polar decomposition, then $U^*U|T| = |T|$ and $U|T|U^* = |T^*|$. If U is unitary, then $Uf(|T|)U^* = f(|T^*|)$ for every function f that is continuous on $\sigma(T)$. If U is not necessarily a unitary operator then $Uf(|T|)U^* = f(|T^*|)$ is valid for those continuous functions which are approximated by polynomials without a constant term. If f is a continuous invertible function such that f and its inverse, f^{-1}, are both approximated by polynomials without a constant term, in this case $\mathscr{N}(f(|T|)) = \mathscr{N}(|T|) = \mathscr{N}(U)$ and $\mathscr{R}(f(|T|)) = \mathscr{R}(|T|)$ [4]. Let $\mathscr{D}(E)$ be the set of all increasing continuous positive functions f defined on E, for which there exist two sequences of polynomials without a constant term one of which converging to f and the other one converging to f^{-1}. For example if $f(x) = \frac{x}{1-x}$ we could easily see that $f \in \mathscr{D}(E)$ for some $E \subset \mathbb{R}$.

For $0 < \lambda < 1$, the $\lambda-$Aluthge transform of T is defined by $\tilde{T}_\lambda = |T|^\lambda U|T|^{1-\lambda}$. This notation was first introduced by Aluthge in the case when $\lambda = \frac{1}{2}$ in [1] during the investigating on the properties of p-hyponormal operators. We denote $\tilde{T}_{\frac{1}{2}}$ by $\tilde{T}$ and call it the Aluthge transform of T. The Aluthge transform of operators has received much attention today and it has been become a powerful tool in operator theory [2, 3, 5, 7, 8, 9, 10, 11, 15]. It follows easily from the definition that $\|\tilde{T}_\lambda\| \leq \|T\|$. An operator T is said to be p-hyponormal for some positive number p, if $(T^*T)^p \geq (TT^*)^p$. In the case when $p = 1$, A is called hyponormal. If T is invertible and $\log(|T|) \geq \log(|T^*|)$ then it is called log-hyponormal. Many authors study the properties of these types of operators as the classes of non-normal operators. For instance we cite here [1, 12, 14, 9, 13] from which this paper has been motivated. The classes of p-hyponormal and log-hyponormal operators are contained in the greater class of operators named the class of normaloid operators. An operator T is said to be normaloid whenever $r_{sp}(T) = \|T\|$ where $r_{sp}(T)$ is the spectral radius of T defined by $r_{sp}(T) = \sup\{|\lambda|; \lambda \in \sigma(T)\}$; see [1] and [14]. In [6] the authors discuss the polar decomposition of the Aluthge transform of operators. In that work an interesting result was stated for a special class of operators i.e for binormal operators. A bounded linear operator T is called binormal if $|T||T^*| = |T^*||T|$. We present some results related to the issue.

An interesting problem in operator theory is finding conditions on certain operators under which such operators become normal. For example in [9] the authors pay attention to this problem for p-hyponormal and log-hyponormal. In this paper, we apply the same method to another type of operators, introduced below to obtain similar results. We, in fact, introduce a new type of operator named the class of operators with the hyponormal property. It is noticed that we have p-hyponormal and log-hyponormal operators as spacial cases. Then, we try to investigate some properties of this class of operators. In this part, we essentially use the methods of [14]. We also generalize the notion of the λ-Aluthge transform of operators, which is close to this type of operators.

2. Results

We start this section with the following definition in which we generalize the notion of hyponormality of operators,

Definition 2.1. Let T be in $\mathbb{B}(\mathscr{H})$, T is said to have the hyponormal property if there exists an increasing function f, continuous on $\sigma(|T|)\cup\sigma(|T^*|)$, such that
$$f(|T|)\geq f(|T^*|).$$
We refer to such an operator T by f-hyponormal operator.

Note that log-hyponormal and p-hyponormal operators are the special cases of f-hyponormal operators for $f(t)=\log t$ and $f(t)=t^{2p}$ respectively. Associated with the f-hyponormal operators, we define the f-Aluthge transform of operators as follows;

Definition 2.2. Let $T=U|T|$ be the polar decomposition and f be continuous on $\sigma(|\ T\ |)$. The f-Aluthge transform of T, denoted by $\tilde{T}_f$, is defined by
$$\tilde{T}_f=(f(|T|))^{\frac{1}{2}}U(f(|T|))^{\frac{1}{2}}.$$

It is easy to see that for the f-hyponormal operator T, the operator $\tilde{T}_f$ is hyponormal. Henceforth, we assume that $T=U|T|$ is the polar decomposition of an f-hyponormal operator for some $f\in\mathscr{D}(\sigma(|T|)\cup\sigma(|T^*|))$ unless otherwise specified.

The following theorem is a generalization of the main result of [6] in which we explain the polar decomposition of the f-Aluthge transforms.

Theorem 2.3. *Let $T=U|T|$ be the polar decomposition of the operator T and let $f\in\mathscr{D}(\sigma(|T|)$ and $(f(|T|))^{\frac{1}{2}}(f(|T^*|))^{\frac{1}{2}}=V|(f(|T|))^{\frac{1}{2}}(f(|T^*|))^{\frac{1}{2}}|$ be the polar decomposition too. Then $\tilde{T}_f=VU|\tilde{T}_f|$ is the polar decomposition.*

Proof.
$$\begin{aligned}\tilde{T}_f &= (f(|T|))^{\frac{1}{2}}U(f(|T|))^{\frac{1}{2}}\\ &= (f(|T|))^{\frac{1}{2}}(f(|T^*|))^{\frac{1}{2}}U\\ &= V|(f(|T|))^{\frac{1}{2}}(f(|T^*|))^{\frac{1}{2}}|U\\ &= VU|\tilde{T}_f|U^*U\\ &= VU|\tilde{T}_f|.\end{aligned}$$
It is easy to check that
$$VU\xi=0\Leftrightarrow\tilde{T}_f\xi=0$$
which implies that $\mathscr{N}(VU)=\mathscr{N}(|\tilde{T}_f|)$.

Now it remains to show that VU is a partial isometry. We note that $\mathscr{N}(VU)^{\perp} = \mathscr{N}(|\tilde{T}_f|)^{\perp} = \overline{\mathscr{R}(|\tilde{T}_f|)}$. Let $\xi \in \mathscr{N}(VU)^{\perp}$. There exists a sequence $\{\eta_n\}$ in $\mathscr{H}$, so that $|\tilde{T}_f|\eta_n \to \xi$ as n goes to ∞. Thus

$$\|VU\xi\| = \|VU \lim |\tilde{T}_f|\eta_n\| = \lim \|VU|\tilde{T}_f|\eta_n\| = \|\lim \tilde{T}_f\eta_n\|$$

$$= \lim \|\tilde{T}_f\eta_n\| = \||\tilde{T}_f|\eta_n\| = \|\lim |\tilde{T}_f|\eta_n\| = \|\xi\|$$

which completes the proof. □

Corollary 2.4. *Let $T = U|T|$ be the polar decomposition of the invertible operator T. $\tilde{T}_f = U|\tilde{T}_f|$ if and only if T is binormal.*

Proof. The uniqueness of the polar decomposition $\tilde{T}_f = VU|\tilde{T}_f|$ in the previous theorem implies that $\tilde{T}_f = U|\tilde{T}_f|$ if and only if $V = P$, the projection onto the initial space of $(f(|T|))^{\frac{1}{2}}(f(|T^*|))^{\frac{1}{2}}$. This is equivalent to

$$(f(|T^*|))^{\frac{1}{2}}(f(|T|))^{\frac{1}{2}} = (f(|T|))^{\frac{1}{2}}(f(|T^*|))^{\frac{1}{2}}$$

which ensures that T is binormal if and only if $\tilde{T}_f = U|\tilde{T}_f|$. □

Here, we want to speak about another generalization of the λ-Aluthge transform.

Definition 2.5. Let $T = U|T|$ be the polar decomposition and f and g be two continuous functions on $\sigma(|T|)$. The (f,g)-Aluthge transform of T, denoted by $\tilde{T}_{(f,g)}$, is defined by

$$\tilde{T}_{(f,g)} = f(|T|)Ug(|T|).$$

Proposition 2.6. *Let $T = U|T|$ be the polar decomposition and let $f,\ g \in \mathscr{D}(\sigma(|T|))$ so that $f(t)g(t) = t$ for all $t \in \sigma(|T|)$. Then $\sigma(T) = \sigma(\tilde{T}_{(f,g)})$*

Proof. We first note that

$$\sigma(T) - \{0\} = \sigma(U|T|) - \{0\} = \sigma(Ug(|T|)f(|T|)) - \{0\} = \sigma(f(|T|)Ug(|T|)) - \{0\}.$$

Therefore it remains to show that T is invertible if and only if so is $\tilde{T}_{(f,g)}$. If T is invertible, then U is unitary and $|T|$ is invertible i.e. $\mathscr{N}(|T|) = 0$ and $\mathscr{R}(|T|) = \mathscr{H}$. So by our assumption $\mathscr{N}(f(|T|)) = 0$, $\mathscr{R}(f(|T|)) = \mathscr{H}$, $\mathscr{N}(g(|T|)) = 0$ and $\mathscr{R}(g(|T|)) = \mathscr{H}$. Thus $f(|T|)$ and $g(|T|)$ are invertible which imply that $\tilde{T}_{f,g}$ is invertible.

Now let $\tilde{T}_{(f,g)}$ is invertible. This implies that $\mathscr{R}(f(|T|)) = \mathscr{H}$ and $\mathscr{N}(g(|T|)) = 0$. So $\mathscr{R}(|T|) = \mathscr{H}$ and $\mathscr{N}(|T|) = 0$. Therefore $|T|$ is invertible. Hence $f(|T|)$ and $g(|T|)$ are invertible which by the invertibility of $\tilde{T}_{(f,g)}$ ensure that U is. Thus T is invertible. □

We prove the next two results by using some ideas of [9].

Theorem 2.7. *If $U^{n_0} = U^*$ for some positive integer n_0, then T is normal.*

Proof. Since T is f-hyponormal, we have $f(|T|) \geq f(|T^*|) = Uf(|T|)U^*$. multiplying both sides of this inequality by U and U^*, we reach $f(|T|) \geq Uf(|T|)U^* \geq U^2 f(|T)U^{*2}$. Continuing this process, we reach a string of inequalities as follows

$$f(|T|) \geq Uf(|T|)U^* \geq U^2 f(|T)U^{*2} \geq \cdots \geq U^{n_0+1} f(|T|)U^{(n_0+1)*} \geq \cdots . \quad (2.1)$$

Due to our assumption $U^*U = U^{n_0+1} = U^{(n_0+1)*}$ is the projection onto $\overline{\mathscr{R}(f(|T|))}$. So $f(|T|) = U^{n_0+1} f(|T|)U^{(n_0+1)*}$ which implies that $f(|T|) = f(|T^*|)$. Since f is increasing it has inverse f^{-1}, which implies that $f^{-1}f(|T|) = f^{-1}f(|T^*|)$. Thus the spectral mapping theorem ensures that $|T| = |T^*|$ i.e. T is normal. $\square$

Theorem 2.8. *If, either $U^{n_0} \to 0$ or $U^{(n_0)*} \to 0$ where the limits are taken in the strong operator topology, then T is normal.*

Proof. Let $\xi \in \mathscr{H}$. Since $f(|T|) > 0$, by (2.1) we have that

$$\|(f(|T|))^{\frac{1}{2}}\xi\| \geq \|(f(|T^*|))^{\frac{1}{2}}\xi\| = \|(f(|T|))^{\frac{1}{2}}U^*\xi\| \geq \cdots \geq \|(f(|T|))^{\frac{1}{2}}U^{*n}\xi\| \geq \cdots .$$

On the other hand

$$\Big| \|(f(|T|))^{\frac{1}{2}}U^{*n}\xi\| - \|(f(|T|))^{\frac{1}{2}}\xi\| \Big| \leq \|(f(|T|))^{\frac{1}{2}}\| \|U^{*n}\xi - \xi\| \to 0$$

as $n \to 0$. Thus we have $\|(f(|T|))^{\frac{1}{2}}\xi\| = \|(f(|T^*|))^{\frac{1}{2}}\xi\|$. Hence $f(|T|) = f(|T^*|)$ which implies that $|T| = |T^*|$. Therefore T is normal. $\square$

Theorem 2.9. *Let $\mathscr{N}(U) = \mathscr{N}(U^*)$. If $\tilde{T}$ is normal, then so is T.*

Proof. $\tilde{T}$ is normal so

$$|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}}$$

which implies that

$$|T|^{\frac{1}{2}}(U^*|T|U - U|T|U^*)|T|^{\frac{1}{2}} = 0.$$

Hence

$$|T|^{\frac{1}{2}}(U^*|T|U - U|T|U^*) = 0$$

on $\overline{\mathscr{R}(|T|)}$. Let $\xi \in \mathscr{N}(|T|)$. So $\xi \in \mathscr{N}(U) = \mathscr{N}(U^*)$ which yields that

$$|T|^{\frac{1}{2}}(U^*|T|U - U|T|U^*)\xi = 0.$$

Therefore $|T|^{\frac{1}{2}}(U^*|T|U - U|T|U^*) = 0$ on whole space $\mathscr{H}$. Taking adjoint we get $(U^*|T|U - U|T|U^*)|T|^{\frac{1}{2}} = 0$. So $U^*|T|U - U|T|U^* = 0$ on $\overline{\mathscr{R}(|T|)}$. Let $\xi \in \mathscr{N}(|T|)$. Similar to the argument stated above we have $U^*|T|U = U|T|U^*$. Using functional calculus we come to

$$U^*f(|T|)U = Uf(|T|)U^*.$$

On the other hand

$$f(|T|) \geq f(|T^*|) = Uf(|T|)U^* = U^*f(|T|)U$$

because of the assumption that T is f-hyponormal. Thus

$$f(|T^*|) = Uf(|T|)U^* \geq f(|T|) \geq f(|T^*|)$$

whence $f(|T|) = f(|T^*|)$, which implies that $|T| = |T^*|$. Therefore T is normal. $\square$

Theorem 2.10. *Let U be unitary, $\sigma(U)$ be contained in some open semicircle and let $\mathscr{N}(f(|T|))$ be a reducing subspace for U. Then $\tilde{T}_f$ is normal if and only if so is T.*

Proof. Let $\tilde{T}_f$ be normal. Thus

$$(f(|T|))^{\frac{1}{2}}Uf(|T|)U^*(f(|T|))^{\frac{1}{2}} = (f(|T|))^{\frac{1}{2}}U^*f(|T|)U(f(|T|))^{\frac{1}{2}}. \tag{2.2}$$

So

$$(f(|T|))^{\frac{1}{2}}\left(Uf(|T|)U^* - U^*f(|T|)U\right)(f(|T|))^{\frac{1}{2}} = 0.$$

Hence $(f(|T|))^{\frac{1}{2}}\left(Uf(|T|)U^* - U^*f(|T|)U\right) = 0$ on $\overline{\mathscr{R}(f(|T|)}$. Now let, $\xi \in \mathscr{N}(f(|T|)$. Thus $U\xi \in \mathscr{N}(f(|T|)$ and $U^*\xi \in \mathscr{N}(f(|T|)$ by the assumption. Hence

$$(f(|T|))^{\frac{1}{2}}(Uf(|T|)U^* - U^*f(|T|)U)\xi = 0.$$

We have just shown that

$$\langle (f(|T|))^{\frac{1}{2}}(Uf(|T|)U^* - U^*f(|T|)U)\xi, \xi\rangle = 0$$

for all $\xi \in \mathscr{H}$ which means that

$$(f(|T|))^{\frac{1}{2}}(Uf(|T|)U^* - U^*f(|T|)U) = 0.$$

Taking adjoint, we get

$$(Uf(|T|)U^* - U^*f(|T|)U)(f(|T|))^{\frac{1}{2}} = 0.$$

So $Uf(|T|)U^* - U^*f(|T|)U = 0$ on $\overline{\mathscr{R}(f(|T|)}$. Let $\xi \in \mathscr{N}(f(|T|)$. Therefore $U\xi \in \mathscr{N}(f(|T|)$ and $U^*\xi \in \mathscr{N}(f(|T|)$ by our assumption. Thus $U^*f(|T|)U\xi = Uf(|T|)U^*\xi = 0$ which implies that $Uf(|T|)U^* = U^*f(|T|)U$. Invoking functional calculus we see that $U|T|U^* = U^*|T|U$. Hence $|T|U^2 = U^2|T|$. Since $\sigma(U)$ is contained in some open semicircle, we observe that $|T|U = U|T|$. This completes the proof because U is unitary. $\square$

The conclusion of the following theorem has been already proved for p-hyponormal operators in [1, 14].

Theorem 2.11. *If U is unitary, then the eigenspaces of U reduce T.*

Proof. Let

$$Q := f(|T|) - f(|T^*|) = f(|T|) - Uf(|T|)U^*.$$

Thus $Q \geq 0$ by our assumption. Let $\lambda \in \sigma_p(U)$ and $M_\lambda = \{\xi \in \mathscr{H}; U\xi = \lambda\xi\}$. U is unitary thus $U^*\xi = \bar{\lambda}\xi$ for any $\xi \in M_\lambda$. Hence

$$\begin{aligned}\langle Q\xi, \xi\rangle &= \langle f(|T|)\xi, \xi\rangle - \langle Uf(|T|)U^*\xi, \xi\rangle \\ &= \langle f(|T|)\xi, \xi\rangle - \langle f(|T|)\bar{\lambda}\xi, \bar{\lambda}\xi\rangle = 0.\end{aligned}$$

Since Q is positive we have that $Q\xi = 0$. So $f(|T|)\xi = Uf(|T|)U^*\xi$ or equivalently $Uf(|T|)\xi = \lambda f(|T|)\xi$. This implies that $f(|T|)\xi \in M_\lambda$. So $(f(|T|))^n\xi \in M_\lambda$ for all positive integers n and hence $p(f(|T|))\xi \in M_\lambda$ for all polynomials p. But, there exists a sequence of polynomials p_n, without a constant term, converging to f^{-1} uniformly. Therefore we have that $|T|\xi \in M_\lambda$ □

In the next lemma, T is not necessarily assumed to be an f-hyponormal operator.

Lemma 2.12. *Let $T = X + \mathrm{i}Y$ be the Cartesian decomposition of operator T where X is self-adjoint and $Y \geq 0$ and let T_0 be another operator defined by $T_0 = X + \mathrm{i}f(Y)$. If T_0 is hyponormal, then the eigenspaces of Y reduce X.*

Proof. Let $y \in \sigma_p(Y)$ and $M_y = \{\xi \in \mathscr{H}; Y\xi = y\xi\}$ be the eigenspace corresponding to y. Then for any $\xi \in M_y$, we have that $f(Y)\xi = f(y)\xi$. Hence

$$\begin{aligned}\langle \mathrm{i}[X, f(Y)]\xi, \xi\rangle &= \mathrm{i}(\langle Xf(Y)\xi, \xi\rangle - \langle f(Y)X\xi, \xi\rangle) \\ &= \mathrm{i}(\langle Xf(y)\xi, \xi\rangle - \langle X\xi, f(y)\xi\rangle) = 0.\end{aligned}$$

Since $\mathrm{i}[X, f(Y)]$ is positive we see that $[X, f(Y)]\xi = 0$ which implies that $f(Y)X\xi = Xf(Y)\xi = f(y)X\xi$. This implies that $YX\xi = yX\xi$. So M_y reduces X. □

Theorem 2.13. *Let U be unitary. if $\sigma(U) \neq \{z; |z| = 1\}$, then the eigenspaces of $|T|$ reduce T.*

Proof. Without loss of generality, we may assume that $1 \notin \sigma(U)$ and consider the inverse Cayley transform of U by $B = \mathrm{i}(U + I)(U - I)^{-1}$. Let $Q := f(|T|) - f(|T^*|) = f(|T|) - Uf(|T|)U^*$. So

$$\mathrm{i}[B, f(|T|)] = 2(U - I)^{-1}Q(U^* - I)^{-1},$$

(see [14]) which is positive by our assumption. This shows that the eigenspaces of $|T|$ reduce B and consequently they reduce U. So they reduce T as well. □

In the following, we want to speak about symbols introduced by Xia in [14] which is useful for the problem that if f-hyponormal operators are normaloid. This makes sense by knowing the fact that p-hyponormal and log-hyponormal operators are normaloid; see [1, 12, 14].

Suppose B is a contraction. Denote

$$B^{[n]} = \begin{cases} B^n & , n \geq 0 \\ B^{n*} & , n < 0. \end{cases}$$

If $S_B^{\pm}(T) := st-\lim_{m\to\mp\infty} B^{[-m]}TB^{[m]}$ exist, then the operators $S_B^{\pm}$ are called the polar symbols of T related to B. Given operator B, denote

$$S_B^{\pm} = \{T \in \mathbb{B}(\mathscr{H}); S_B^{\pm}(T) \ exists\}.$$

The following lemmata are held,

Lemma 2.14. [14] *Let $T = U|T|$ be the polar decomposition. If U is a unitary operator and $|T| \in S_U^{\pm}$ then $S_U^{\pm}(|T|)$ are positive,*

$$|S_U^{\pm}(T)| = S_U^{\pm}(|T|),$$

and $US_U^{\pm}(|T|) = S_U^{\pm}(T)$ are normal.

Lemma 2.15. [14] *Let T be a normal operator and f be a continuous function on $\sigma(T)$. If B is unitary and $T \in S_B^{\pm} \cap (S_B^{\pm})^*$, then $f(T) \in S_B^{\pm}$ and*

$$f(S_B^{\pm}(T)) = S_B^{\pm}(f(T)).$$

Lemma 2.16. *Let $T = U|T|$ be the polar decomposition of the f-hyponormal operator T and let U be unitary. Then the operator symbols*

$$S_U^{\pm} := \lim_{m\to\mp\infty} U^{m*}TU^m$$

exist.

Proof. f-hyponormality of T implies that $f(|T|) \geq Uf(|T|)U^*$. Multiplying both sides by U and U^*, we reach

$$U^*f(|T|)U \geq f(|T|) \geq Uf(|T|)U^*.$$

Let n be a positive integer. To continue this process, we come to a string of inequalities as follows

$$U^{n*}f(|T|)U^n \geq \cdots \geq U^*f(|T|)U \geq f(|T|) \geq Uf(|T|)U^* \geq \cdots \geq U^nf(|T|)U^{n*}.$$

Thus the sequence $\{U^{n*}f(|T|)U^n\}$ is bounded and increasing and $\{U^nf(|T|)U^{*n}\}$ is bounded and decreasing which imply that

$$S_U^{\pm}(f(|T|)) := \lim_{m\to\mp\infty} U^{m*}f(|T|)U^m$$

exist. Therefore $S_U^{\pm}(T)$ exist and

$$S_U^{\pm}(T) = \lim_{m\to\mp\infty} U^{m*}TU^m = Ug[S_U^{\pm}(f(|T|))].$$

$\square$

In the following, we show that f-hyponormal operators are normaloid for a certain class of functions f. Let $\mathscr{C}\mathscr{P}(E)$ consists of those continuous functions f, for which there exists a sequence of polynomials, with positive coefficients, without a constant term converging to f, uniformly.

Lemma 2.17. *Let A be a positive operator and let $f \in \mathscr{C}\mathscr{P}(\sigma(A))$. Then $\|f(A)\| \leq f(\|A\|)$.*

Proof. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x$ be a polynomial where a_is are all positive. We have that

$$\begin{aligned} \|p(A)\| &= \|a_n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A\| \\ &\leq a_n\|A\|^n + a_{n-1}\|A\|^{n-1} + \cdots + a_1\|A\| \\ &= p(\|A\|). \end{aligned}$$

Now the result is obviously concluded from these equations. □

Theorem 2.18. *Let g be the inverse of f and $g \in \mathscr{C}\mathscr{P}(\sigma(A))$. If U is unitary, then $r_{sp}(T) = \|T\|$.*

Proof. By the previous Lemma we see that $f(|T|) \leq S_U^+(f(|T|)) \leq \|f(|T|)\|$. therefore $\|S_U^+(f(|T|))\| = \|f(|T|)\|$. Since $S_U^+(f(|T|))$ is positive

$$\|S_U^+(f(|T|))\| = \|f(|T|)\| \in \sigma(S_U^+(f(|T|))).$$

Thus

$$g(\|f(|T|)\|) \in \sigma\left(g[(S_U^+(f(|T|)))]\right) = \sigma(S_U^+(|T|)) \tag{2.3}$$

But $\|T\| \leq g(\|f(|T|)\|)$ and

$$r_{sp}(S_U^+(|T|)) = \|S_U^+(|T|)\| \leq \|T\| \leq g(\|f(|T|)\|)$$

which by (2.3) yields that $\|T\| = g(\|f(|T|)\|)$. So $\|T\| \in \sigma(S_U^+(|T|))$ and the rest of the proof is similar to the proof of [1, Theorem 9]. □

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