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Sums of Strongly z-Ideals and Prime Ideals in $\mathcal{R}L$

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Abstract. It is well known that the sum of two z-ideals in $C(X)$ is either $C(X)$ or a z-ideal. The main aim of this paper is to study the sum of strongly z-ideals in $\mathcal{R}L$, the ring of real-valued continuous functions on a frame L. For every ideal I in $\mathcal{R}L$, we introduce the biggest strongly z-ideal included in I and the smallest strongly z-ideal containing I, denoted by I^{sz} and I_{sz}, respectively. We study some properties of I^{sz} and I_{sz}. Also, it is observed that the sum of any family of minimal prime ideals in the ring $\mathcal{R}L$ is either $\mathcal{R}L$ or a prime strongly z-ideal in $\mathcal{R}L$. In particular, we show that the sum of two prime ideals in $\mathcal{R}L$ which are not chains is a prime strongly z-ideal.

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1. Introduction

An ideal I of a ring A (the term "ring" means a commutative ring with identity) is a *z-ideal* if whenever two elements of A are in the same of maximal ideals and I contains one of the elements, then it also contains the other. This algebraic definition of z-ideal was coined in the context of rings of continuous functions by Kohls in [16] and is also recorded as Problem 4A.5 in the text *Rings of continuous functions* by Gillman and Jerison [10]. Also, Estaji in [6] introduced and studied z-weak ideals and prime weak ideals in the rings of continuous functions on a topological space. A study of z-ideals in rings generally has been carried out by Mason in the article [17]. In pointfree topology, z-ideals were introduced by Dube in [4] where he showed that the algebraic definition agree with the "topological" definition in terms of the cozero map. It was shown in [10, 20] that if B is an absolutely convex subring of the ring of all continuous functions on a topological space, then a sum of two z-ideals of B is a z-ideal. If B is a ring (or a module) and K is an ideal (or a submodule) of B, let $B(K) = \{(a, b) \in B \times B : a - b \in K\}$. In [11], this construction is used to find a lattice-ordered subring of the ring $C(\mathbb{R})$ of all continuous real-valued functions on the real line $\mathbb{R}$ with two z-ideals whose sum is not even semiprime. Therefore sum of two z-ideals in $\mathcal{R}L$ may not be a z-ideal, and thus in this paper, we discuss on sum of strongly z-ideals in the ring $\mathcal{R}L$. The concept of zero-sets and strongly z-ideals in $\mathcal{R}L$ is introduced in [7]. An ideal I in $\mathcal{R}L$ is called strongly z-ideal if $Z(\alpha) \in Z[I]$ implies $\alpha \in I$, where $Z(\alpha)$ is the zero-set of α in $\mathcal{R}L$.

This paper is organized as follows. In Section 2, we review some basic notions and properties of a frame and the pointfree version of the ring of continuous real-valued functions. Also, we recall some properties of z-ideals and strongly z-ideals in $\mathcal{R}L$.

In Section 3, we study the sum of strongly z-ideals in $\mathcal{R}L$ and we show that, under some conditions, the sum of strongly z-ideals is a strongly z-ideal (Theorem 3.2).

In Section 4, for every ideal I in $\mathcal{R}L$, we introduce the biggest strongly z-ideal included in I, denoted by I^{sz} and the smallest strongly z-ideal containing I, denoted by I_{sz}, and we study I^{sz} and I_{sz}. Similar to $C(X)$, we show that the sum of a family of minimal prime ideals in the ring $\mathcal{R}L$ is either $\mathcal{R}L$ or a prime ideal in $\mathcal{R}L$ (Corollary 4.4). Finally, we show that the sum of two prime ideals in $\mathcal{R}L$ which are not chains, is a prime strongly z-ideal (Proposition 4.19).

2. Preliminaries

In this section, we collect some notations from the literature on frames and the ring of continuous real-valued functions on a frame. Our references for frames are [14, 18] and for the ring $\mathcal{R}L$ are [1, 2].

A *frame* is a complete lattice L in which the distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s : s \in S\}$$

holds for all $x \in L$ and $S \subseteq L$. We denote the top element and the bottom element of L by $\top$ and $\bot$, respectively.

A *frame homomorphism* (or a *frame map*) is a map between frames which preserves finite meets, containing the top element, and arbitrary joins, containing the bottom element. An element $p \in L$ is said to be *prime* if $p < \top$ and $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. A lattice ordered ring A is called an *f-ring*, if $(f \wedge g)h = fh \wedge gh$ for every $f, g, h \in A$ and every $0 \leq h \in A$.

Recall the contravariant *functor* Σ from **Frm** to the category **Top** of topological spaces which assigns to each frame L its *spectrum* ΣL of prime elements with $\Sigma_a = \{p \in \Sigma L | a \not\leq p\}$ $(a \in L)$ as its open sets.

An element a of a frame L is said to be *completely below* b, written $a \prec\!\prec b$, if there exists a sequence $\{c_q\}$, $q \in \mathbb{Q} \cap [0,1]$, where $c_0 = a,\ c_1 = b$, and $c_p \prec c_q$ if $p < q$ where $u \prec v$ means $u^* \vee v = \top$ where $u^* = \bigvee\{x \in L : x \wedge u = \bot\}$. A frame L is called *completely regular* if each $a \in L$ is a join of elements completely below it.

Regarding the *frame* $\mathcal{L}(\mathbb{R})$ *of reals* and the f-ring $\mathcal{R}L$ of *continuous real functions* on L, we use the notation of [2]. See also [1]

The *cozero map* is the map $\mathrm{coz} : \mathcal{R}L \to L$, defined by

$$\mathrm{coz}(\alpha) = \bigvee\{\alpha(p,0) \vee \alpha(0,q) : p, q \in \mathbb{Q}\}.$$

A *cozero element* of L is an element of the form $\mathrm{coz}(\alpha)$ for some $\alpha \in \mathcal{R}L$ (see [2]). The cozero part of L is denoted by $\mathrm{Coz}\, L$. It is known that L is completely regular if and only if $\mathrm{Coz}\, L$ generates L. A frame L is called coz-dense if whenever $\Sigma_{\mathrm{coz}(\alpha)} = \emptyset$, then $\alpha = \mathbf{0}$ (see [15]).

Here we recall some notations from [5]. Let $a \in L$ and $\alpha \in \mathcal{R}L$. The sets $\{r \in \mathbb{Q} : \alpha(-,r) \leq a\}$ and $\{s \in \mathbb{Q} : \alpha(s,-) \leq a\}$ are denoted by $L(a,\alpha)$ and $U(a,\alpha)$, respectively. For $a \neq \top$ it is obvious that for each $r \in L(a,\alpha)$ and $s \in U(a,\alpha)$, $r \leq s$. In fact, we have that if $p \in \Sigma L$ and $\alpha \in \mathcal{R}L$, then $(L(p,\alpha), U(p,\alpha))$ is a Dedekind cut for a real number which is denoted by $\widetilde{p}(\alpha)$ (see [5]). Throughout this paper, for every $\alpha \in \mathcal{R}L$ we define $\alpha[p] = \widetilde{p}(\alpha)$ where p is a prime element of L.

Recall from [7] that for $\alpha \in \mathcal{R}L$, $Z(\alpha) = \{p \in \Sigma L : \alpha[p] = 0\}$ is called the *zero-set* of α. For every $A \subseteq \mathcal{R}L$, we write $Z[A] = \{Z(\alpha) : \alpha \in A\}$. Also we let $Z[\mathcal{R}L] = Z[L]$ for simplicity. An ideal I in $\mathcal{R}L$ is called a strongly z-ideal if $Z(\alpha) \in Z[I]$ implies that $\alpha \in I$, that is $I = Z^{\leftarrow}[Z[I]]$, where $Z^{\leftarrow}[Z[I]] = \{\alpha \in \mathcal{R}L : Z(\alpha) \in Z[I]\}$ (see [7, 8]). Note that the intersection of an arbitrary family of strongly z-ideals is a strongly z-ideal. Also, in the ring $\mathcal{R}L$, every strongly z-ideal is a z-ideal (see [7, Proposition 5.6]).

For every $f \in C(\Sigma L)$, let us recall that there exists a unique frame map $\widehat{f} : \mathcal{L}(\mathbb{R}) \to L$ such that

$$\widehat{f}(p, q) = \bigvee \{a \in L : f(\Sigma_a) \subseteq \rrbracket p, q \llbracket \},$$

for every $p, q \in \mathbb{Q}$, where $\rrbracket p, q\llbracket = \{x \in \mathbb{R} : p < x < q\}$. In addition, we have $Z(\widehat{f}) = Z(f)$ (see [15]). For every $\alpha \in \mathcal{R}L$, we define $\overline{\alpha} : \Sigma L \to \mathbb{R}$ given by $\overline{\alpha}(p) = \alpha[p]$, for $p \in \Sigma L$. It is clear that $Z(\alpha) = Z(\overline{\alpha})$. Also, we have:

Proposition 2.1. [9] *Let L be a frame. Let $\varphi : C(\Sigma L) \to \mathcal{R}L$ with $\varphi(f) = \widehat{f}$ and $\psi : \mathcal{R}L \to C(\Sigma L)$ with $\psi(\alpha) = \overline{\alpha}$. Then ψ is an f-ring homomorphism and a monomorphism. If L is a coz-dense frame, then ψ is an isomorphism, and $\psi^{-1} = \varphi$.*

For every $\alpha \in \mathcal{R}L$, we put $M_\alpha := \{\beta \in \mathcal{R}L : Z(\alpha) \subseteq Z(\beta)\}$. In addition we have

Proposition 2.2. *For every $\alpha \in \mathcal{R}L$, M_α is a strongly z-ideal of $\mathcal{R}L$.*

3. On Sum of Strongly z-Ideals in $\mathcal{R}L$

As is well-known, the sum of two z-ideals in $C(X)$ is either $C(X)$ or a z-ideal, see [10, Lemma 14.8]. Fortunately, the proof of this result in [20] can be modified for $\mathcal{R}L$ and is presented below.

Lemma 3.1. *Let $\alpha, \beta, \gamma \in \mathcal{R}L$ and $Z(\alpha) \supseteq Z(\beta) \cap Z(\gamma)$. Define*

$$h(p) = \begin{cases} 0 & p \in Z(\beta) \cap Z(\gamma), \\ \frac{\overline{\alpha}(p)\overline{\beta}^2(p)}{\overline{\gamma}^2(p)+\overline{\beta}^2(p)} & p \notin Z(\beta) \cap Z(\gamma) \end{cases}$$

and

$$k(p) = \begin{cases} 0 & p \in Z(\beta) \cap Z(\gamma), \\ \frac{\overline{\alpha}(p)\overline{\gamma}^2(p)}{\overline{\gamma}^2(p)+\overline{\beta}^2(p)} & p \notin Z(\beta) \cap Z(\gamma). \end{cases}$$

Then we have the following facts.

1. $|h| \vee |k| \le |\overline{\alpha}|$
2. $\overline{\alpha} = h + k$.
3. $\overline{\alpha}\overline{\beta}^2 = h(\overline{\beta}^2 + \overline{\gamma}^2)$ *and* $\overline{\alpha}\,\overline{\gamma}^2 = k(\overline{\beta}^2 + \overline{\gamma}^2)$.
4. $h, k \in C(\Sigma L)$.

Proof. Since $\overline{\alpha}, \overline{\beta}, \overline{\gamma} : \Sigma L \to \mathbb{R}$ are continuous functions and $\overline{\gamma}^2(p) + \overline{\beta}^2(p) \neq 0$ for every $p \notin Z(\beta) \cap Z(\gamma)$, we infer that h and k are continuous. Also,

$$(h + k)(p) = (\frac{\overline{\alpha}\overline{\beta}^2}{\overline{\gamma}^2 + \overline{\beta}^2} + \frac{\overline{\alpha}\,\overline{\gamma}^2}{\overline{\gamma}^2 + \overline{\beta}^2})(p) = \overline{\alpha}(p).$$

for every $p \in \Sigma L$. Therefore $h + k = \overline{\alpha}$. It is evident that $|h| \le |\overline{\alpha}|$ and $|k| \le |\overline{\alpha}|$, hence $|h| \vee |k| \le |\overline{\alpha}|$. Clearly $\overline{\alpha}\,\overline{\gamma}^2 = k(\overline{\beta}^2 + \overline{\gamma}^2)$ and $\overline{\alpha}\overline{\beta}^2 = h(\overline{\beta}^2 + \overline{\gamma}^2)$. □

In what follows, all frames are assumed to be coz-dense.

Theorem 3.2. *Let I and J be two strongly z-ideals of $\mathcal{R}L$. Then $I+J=\mathcal{R}L$ or $I+J$ is a strongly z-ideal.*

Proof. Let $I+J \neq \mathcal{R}L$ and $\alpha \in \mathcal{R}L$ be an element with $Z(\alpha)=Z(\beta)$, where $\beta \in I+J$. We show that $\alpha \in I+J$. But $\beta=\beta_1+\beta_2$, where $\beta_1 \in I$ and $\beta_2 \in J$. Clearly,

$$Z(\alpha)=Z(\beta) \supseteq Z(\beta_1) \cap Z(\beta_2).$$

Let h and k be as in the previous lemma, then $h+k=\overline{\alpha}$. But $Z(\beta_1)=Z(\overline{\beta_1}) \subseteq Z(h)$ and $Z(\beta_2)=Z(\overline{\beta_2}) \subseteq Z(k)$. Now, let $\overline{I}=\{\overline{\delta}|\delta \in I\} \subseteq C(\Sigma L)$ and $\overline{J}=\{\overline{\sigma}|\sigma \in J\} \subseteq C(\Sigma L)$. Since I and J are strongly z-ideals of $\mathcal{R}L$ then $\overline{I}$ and $\overline{J}$ are strongly z-ideals of $C(\Sigma L)$. Also, $\overline{I}+\overline{J}$ is a z-ideal of $C(\Sigma L)$. Therefore $h \in \overline{I}$ and $k \in \overline{J}$. So $\overline{\alpha}=h+k \in \overline{I}+\overline{J}$. Thus, by Proposition 2.1,

$$\alpha=\widehat{\overline{\alpha}} \in \widehat{\overline{I}+\overline{J}}=\widehat{\overline{I}}+\widehat{\overline{J}}=I+J.$$

Hence $\alpha \in I+J$ and we are through. □

Corollary 3.3. *Let $F=\{I_\lambda\}_{\lambda\in\Lambda}$ be a family of strongly z-ideals in $\mathcal{R}L$. Then either $\Sigma_{\lambda\in\Lambda}I_\lambda=\mathcal{R}L$ or $\Sigma_{\lambda\in\Lambda}I_\lambda$ is a strongly z-ideal.*

Corollary 3.4. *If $\alpha,\beta \in \mathcal{R}L$, then $M_\alpha+M_\beta=M_{\alpha^2+\beta^2}$.*

Proof. Let $\gamma \in M_{\alpha^2+\beta^2}$, then $Z(\alpha^2+\beta^2) \subseteq Z(\gamma)$. Since, by [7, Proposition 3.3], $\alpha^2 \in M_\alpha$ and $\beta^2 \in M_\beta$, we conclude that $\alpha^2+\beta^2 \in M_\alpha+M_\beta$. Also, by Proposition 2.2 and Theorem 3.2, $M_\alpha+M_\beta$ is a strongly z-ideal, then $\gamma \in M_\alpha+M_\beta$. Hence $M_{\alpha^2+\beta^2} \subseteq M_\alpha+M_\beta$. Conversely, let $\delta \in M_\alpha, \eta \in M_\beta$ and $\gamma=\delta+\eta \in M_\alpha+M_\beta$. Then

$$Z(\alpha^2+\beta^2)=Z(\alpha)\cap Z(\beta) \subseteq Z(\delta)\cap Z(\eta) \subseteq Z(\gamma),$$

hence $\gamma \in M_{\alpha^2+\beta^2}$, that is $M_\alpha+M_\beta \subseteq M_{\alpha^2+\beta^2}$ □

Remark 3.5. Let $\alpha,\beta \in \mathcal{R}L$. Then $M_\alpha M_\beta=M_\alpha\cap M_\beta=M_{\alpha\beta}$. For, by Proposition 2.2, [7, Proposition 5.6] and [12, Lemma 7.2.2], $M_\alpha M_\beta=M_\alpha\cap M_\beta$. Also, by [7, Proposition 3.3], we have

$$\gamma \in M_\alpha\cap M_\beta \Leftrightarrow Z(\alpha)\cup Z(\beta) \subseteq Z(\gamma) \Leftrightarrow Z(\alpha\beta) \subseteq Z(\gamma) \Leftrightarrow \gamma \in M_{\alpha\beta}.$$

4. Strongly z-Ideals I_{sz} and I^{sz}

Let I be an ideal of $\mathcal{R}L$. It is clear that $Z^{\leftarrow}[Z[I]]$ is a strongly z-ideal containing I. It is observed that this ideal is the intersection of all the strongly z-ideals containing I. So it is the smallest strongly z-ideal containing I. We denote it by I_{sz}. Also, by Theorem 3.2, the sum of strongly z-ideals included in I is a strongly z-ideal and it is the biggest strongly z-ideal included in I. We denote it by I^{sz}. Therefore $I^{sz} \subseteq I \subseteq I_{sz}$ show that every ideal I in $\mathcal{R}L$ stand between two strongly z-ideals. In this section, we study some properties of strongly z-ideals I_{sz} and I^{sz} as the biggest strongly z-ideal and the smallest strongly z-ideal included in and containing I, respectively.

Lemma 4.1. *Let I and J be ideals of $\mathcal{R}L$ such that $I \subseteq J$, then*

(1) $I^{sz} \subseteq J^{sz}$.
(2) $I_{sz} \subseteq J_{sz}$.

Proof. It is evident. □

Proposition 4.2. *If I is a strongly z-ideal of $\mathcal{R}L$ and P is a minimal prime ideal over I, then P is a strongly z-ideal of $\mathcal{R}L$.*

Proof. Suppose that P is not a strongly z-ideal. Then there exist $\alpha, \beta \in \mathcal{R}L$ such that $Z(\alpha) = Z(\beta)$, $\alpha \in P$ and $\beta \notin P$. Put $S = (\mathcal{R}L \backslash P) \cup \{\gamma\alpha^n : \gamma \notin P, n \in \mathbb{N}\}$. The S is a multiplicatively closed subset and $S \cap I = \emptyset$. Therefore there exists a prime ideal, say P', such that $I \subseteq P'$ and $P' \cap S = \emptyset$ (see [13, Theorem 3.44]). Now, if $\delta \in P'$, then $\delta \notin S$ and so $\delta \in P$, that is, $P' \subseteq P$. Also, $\alpha \in P$ but $\alpha \notin P'$. Hence $P' \subset P$, which is a contradiction. □

Corollary 4.3. *Every minimal prime ideal of $\mathcal{R}L$ is a strongly z-ideal.*

Proof. Let P be a minimal prime ideal of $\mathcal{R}L$. Clearly, the ideal $(\mathbf{0})$ is a strongly z-ideal and it is included in every ideal. Thus, by Proposition 4.2, P is a strongly z-ideal. □

Corollary 4.4. *Let $F = \{P_\lambda\}_{\lambda \in \Lambda}$ be a family of minimal prime ideals in $\mathcal{R}L$. Then $\Sigma_{\lambda \in \Lambda} P_\lambda = \mathcal{R}L$ or $P = \Sigma_{\lambda \in \Lambda} P_\lambda$ is a prime ideal in $\mathcal{R}L$.*

Proof. It is a consequence of Corollary 4.3, Theorem 3.2, and [7, Theorem 5.11]. □

Proposition 4.5. *Let P be a prime ideal in $\mathcal{R}L$. Then P^{sz} and P_{sz} are prime ideals.*

Proof. Let P be a prime idea. Then P_{sz} is a strongly z-ideal containing P. Hence, by [7, Theorem 5.11], P_{sz} is prime. On the other hand, P contains a minimal prime ideal, say Q. But, by Corollary 4.3, Q is a strongly z-ideal. Since P^{sz} is the biggest strongly z-ideal included in P, we infer that $Q \subseteq P^{sz}$. Thus, by [7, Theorem 5.11], P^{sz} is prime. Hence $P^{sz} \subseteq P \subseteq P_{sz}$ says that every prime ideal of $\mathcal{R}L$ stands between two prime strongly z-ideals. □

Lemma 4.6. *Let $\alpha, \beta \in \mathcal{R}L$, then the following statements hold:*

(1) $M_\alpha \subseteq M_\beta$ *if and only if* $Z(\beta) \subseteq Z(\alpha)$.
(2) $M_\alpha = M_\beta$ *if and only if* $Z(\beta) = Z(\alpha)$.

Proof. It is evident. □

Proposition 4.7. *Let I be an ideal in $\mathcal{R}L$. Then*

(1) $I^{sz} = \{\alpha \in \mathcal{R}L : M_\alpha \subseteq I\}$.
(2) $I_{sz} = \{\beta \in \mathcal{R}L : \beta \in M_\alpha \textit{ for some } \alpha \in I\}$.

Proof. (1) First, we show that $J = \{\alpha \in \mathcal{R}L : M_\alpha \subseteq I\}$ is an ideal. To do this, suppose that $\alpha, \beta \in J$. So $M_\alpha \subseteq I$ and $M_\beta \subseteq I$. Then, by Corollary 3.4,

$$M_{\alpha^2+\beta^2} = M_\alpha + M_\beta \subseteq I.$$

Again, by Lemma 4.6, $Z(\alpha^2+\beta^2) \subseteq Z(\alpha+\beta)$ implies that $M_{\alpha+\beta} \subseteq M_{\alpha^2+\beta^2}$. Thus $M_{\alpha+\beta} \subseteq I$ and hence $\alpha+\beta \in J$. Now, suppose that $\alpha \in J$ and $\beta \in \mathcal{R}L$. So $M_\alpha \subseteq I$. Also we have $\alpha \in M_\alpha \subseteq I$. Since I is an ideal we infer that $\alpha\beta \in I$. Now, by Remark 3.5,

$$M_{\alpha\beta} = M_\alpha \cap M_\beta \subseteq M_\alpha \subseteq I.$$

Therefore $\alpha\beta \in J$ and thus J is an ideal. Now, we show that J is a strongly z-ideal. Suppose that $Z(\beta) \subseteq Z(\gamma)$ where $\beta \in J$ and $\gamma \in \mathcal{R}L$. So, by Lemma 4.6, $M_\gamma \subseteq M_\beta$. Since $\beta \in J$, it implies that $M_\beta \subseteq I$ and hence $M_\gamma \subseteq I$. Therefore $\gamma \in J$. Thus J is a strongly z-ideal.

Finally, we show that J is the biggest strongly z-ideal included in I. It is clear that $J \subseteq I$, because if $\alpha \in J$ then $M_\alpha \subseteq I$. But $\alpha \in M_\alpha$ implies that $\alpha \in I$. Now suppose that K is a strongly z-ideal such that $K \subseteq I$. Let $\beta \in K$. Since K is a strongly z-ideal, $M_\beta \subseteq K$. But $K \subseteq I$, therefore $M_\beta \subseteq I$ and so $\beta \in J$. Hence $K \subseteq J$. Thus $J = I^{sz}$.

(2) First, we show that $J = \{\beta \in \mathcal{R}L : \beta \in M_\alpha \text{ for some } \alpha \in I\}$ is an ideal. For doing this, suppose that $\beta, \gamma \in J$. Then there exist $\alpha_1, \alpha_2 \in I$ such that $\beta \in M_{\alpha_1}$ and $\gamma \in M_{\alpha_2}$. Now, by Corollary 3.4,

$$\beta + \gamma \in M_{\alpha_1} + M_{\alpha_2} = M_{\alpha_1^2+\alpha_2^2}.$$

Therefore $\beta+\gamma \in J$. Now, let $\beta \in J$ and $\gamma \in \mathcal{R}L$. Since $\beta \in J$, there is an element α in I such that $\beta \in M_\alpha$. Then $Z(\alpha) \subseteq Z(\beta)$ and $Z(\gamma) \subseteq Z(\gamma)$ and so $Z(\alpha) \subseteq Z(\alpha\gamma) \subseteq Z(\beta\gamma)$. Also, since $\alpha \in M_\alpha$ and M_α is a strongly z-ideal we conclude that $\beta\gamma \in M_\alpha$. Therefore $\beta\gamma \in J$ and thus J is an ideal. Now, we show that J is a strongly z-ideal. To do this, suppose that $Z(\beta) \subseteq Z(\gamma)$ where $\beta \in J$ and $\gamma \in \mathcal{R}L$. Then, $\beta \in J$ implies that there exists an element α in I such that $\beta \in M_\alpha$. Hence $Z(\alpha) \subseteq Z(\beta)$, and so $Z(\alpha) \subseteq Z(\gamma)$. Also, since $\alpha \in M_\alpha$ and M_α is a strongly z-ideal, we infer that $\gamma \in M_\alpha$. Hence $\gamma \in J$. Therefore J is a strongly z-ideal.

Finally, we show that J is the smallest strongly z-ideal containing I. It is clear that $I \subseteq J$. Now, suppose that K is a strongly z-ideal such that $I \subseteq K$. Let $\beta \in J$. Then there exists an element α in I such that $\beta \in M_\alpha$, and hence $Z(\alpha) \subseteq Z(\beta)$. Since $\alpha \in K$ and K is a strongly z-ideal, it follows that $\beta \in K$. Therefore $J \subseteq K$. Thus $J = I_{sz}$ and the proof is complete. □

Proposition 4.8. *Let I be an ideal in $\mathcal{R}L$ and $\alpha \in \mathcal{R}L$. Then*

(1) $I^{sz} = \Sigma_{M_\alpha \subseteq I} M_\alpha$.

(2) $I_{sz} = \Sigma_{\alpha \in I} M_\alpha$.

Proof. (1) Since, by Proposition 2.2, every M_α is a strongly z-ideal, we infer from Theorem 3.2 that $\Sigma_{M_\alpha \subseteq I} M_\alpha$ is a strongly z-ideal. Also, it is clear that $\Sigma_{M_\alpha \subseteq I} M_\alpha \subseteq I$. Now, we show that $\Sigma_{M_\alpha \subseteq I} M_\alpha$ is the biggest strongly z-ideal included in I. Let K be a strongly z-ideal such that $K \subseteq I$. Let $\beta \in K$. Then $M_\beta \subseteq K \subseteq I$ and so $\beta \in \Sigma_{M_\alpha \subseteq I} M_\alpha$. Thus $K \subseteq \Sigma_{M_\alpha \subseteq I} M_\alpha$. Therefore $I^{sz} = \Sigma_{M_\alpha \subseteq I} M_\alpha$.

(2) Since, by Proposition 2.2, every M_α is a strongly z-ideal, we can conclude from Theorem 3.2 that $\Sigma_{\alpha \in I} M_\alpha$ is a strongly z-ideal. Clearly, $I \subseteq \Sigma_{\alpha \in I} M_\alpha$. Now, we show that $\Sigma_{\alpha \in I} M_\alpha$ is the smallest strongly z-ideal containing I. Suppose that $\beta \in \Sigma_{\alpha \in I} M_\alpha$. Then there exist $\alpha_1, \ldots, \alpha_n \in I$ such that $\beta = M_{\alpha_1} + \cdots + M_{\alpha_n}$. Now, by Corollary 3.4, $\beta \in M_{\alpha_1^2 + \cdots + \alpha_n^2}$. Let $\alpha_1^2 + \cdots + \alpha_n^2 = \gamma$, so $\gamma \in I$. Hence, by Proposition 4.7, $\beta \in I_{sz}$. Therefore $I_{sz} = \Sigma_{\alpha \in I} M_\alpha$. $\square$

Proposition 4.9. *Let I be an ideal in $\mathcal{R}L$ and $\alpha \in \mathcal{R}L$. Then*

(1) $I^{sz} = \bigcup_{M_\alpha \subseteq I} M_\alpha$.
(2) $I_{sz} = \bigcup_{\alpha \in I} M_\alpha$.

Proof. (1) By Proposition 4.8, it is enough to show that $\bigcup_{M_\alpha \subseteq I} M_\alpha = \Sigma_{M_\alpha \subseteq I} M_\alpha$. Since for every $M_\alpha \subseteq I$, $M_\alpha \subseteq \Sigma_{M_\alpha \subseteq I} M_\alpha$, we have $\bigcup_{M_\alpha \subseteq I} M_\alpha \subseteq \Sigma_{M_\alpha \subseteq I} M_\alpha$. Now, let $\beta \in \Sigma_{M_\alpha \subseteq I} M_\alpha$. So $\beta \in M_{\alpha_1} + \cdots + M_{\alpha_n}$, where $M_{\alpha_i} \subseteq I$ for $i = 1, 2, ..., n$. Now, by Corollary 3.4, we have $M_{\alpha_1} + \cdots + M_{\alpha_n} = M_{\alpha_1^2 + \cdots + \alpha_n^2}$. Let $\gamma = \alpha_1^2 + \cdots + \alpha_n^2$. Therefore $\beta \in M_\gamma$ and $M_\gamma \subseteq I$. Thus $\beta \in \bigcup_{M_\alpha \subseteq I} M_\alpha$.

(2) By Proposition 4.8, it is enough to show that $\bigcup_{\alpha \in I} M_\alpha = \Sigma_{\alpha \in I} M_\alpha$. For every $\alpha \in I$ since $M_\alpha \subseteq \Sigma_{\alpha \in I} M_\alpha$, then $\bigcup_{\alpha \in I} M_\alpha \subseteq \Sigma_{\alpha \in I} M_\alpha$. Now, suppose that $\beta \in \Sigma_{\alpha \in I} M_\alpha$. So $\beta \in M_{\alpha_1} + \cdots + M_{\alpha_n}$ where $\alpha_i \in I$ for $i = 1, 2, ..., n$. Now, by Corollary 3.4, we have $\beta \in M_{\alpha_1^2 + \cdots + \alpha_n^2}$. Let $\gamma = \alpha_1^2 + \cdots + \alpha_n^2 \in I$. Therefore $\beta \in M_\gamma$ and thus $\beta \in \bigcup_{\alpha \in I} M_\alpha$. Hence $\Sigma_{\alpha \in I} M_\alpha \subseteq \bigcup_{\alpha \in I} M_\alpha$. $\square$

Corollary 4.10. *Let I be an ideal in $\mathcal{R}L$. Then the following statements are equivalent:*

(1) *I is a strongly z-ideal.*
(2) $I = \Sigma_{\alpha \in I} M_\alpha = \{\beta \in \mathcal{R}L : M_\beta \subseteq I\}$.

Proof. (1) $\Rightarrow$ (2) Suppose that I is a strongly z-ideal. Then $I^{sz} = I = I_{sz}$ and, by Propositions 4.7 and 4.8, we have $I = \Sigma_{\alpha \in I} M_\alpha = \{\beta \in \mathcal{R}L : M_\beta \subseteq I\}$.

(2) $\Rightarrow$ (1) Let $I = \Sigma_{\alpha \in I} M_\alpha$. Since every M_α is a strongly z-ideal and, by Theorem 3.2, we infer that I is a strongly z-ideal. $\square$

Proposition 4.11. *Let I be an ideal of $\mathcal{R}L$ and $\alpha \in \mathcal{R}L$. If $M_\alpha \subseteq \sqrt{I}$ then $M_\alpha \subseteq I$.*

Proof. Let $\beta \in M_\alpha \subseteq \sqrt{I}$. Without loss of generality, we may assume that $|\beta| \leq \mathbf{1}$. Define $\gamma = \Sigma_{n=1}^{\infty} 2^{-n} \beta^{\frac{1}{n}}$. Clearly $\gamma \in \mathcal{R}L$ and, since $Z(\beta) = Z(\gamma)$

and M_α is a strongly z-ideal, then $\gamma \in M_\alpha$. Hence $\gamma \in \sqrt{I}$ and so there exists an element m in $\mathbb{N}$ such that $\gamma^m \in I$. Furthermore, since for every $n \in \mathbb{N}$, $2^{-n}\beta^{\frac{1}{n}} \leq \gamma$, we have $2^{-2m}\beta^{\frac{1}{2m}} \leq \gamma$ which implies that

$$(2^{-2m}\beta^{\frac{1}{2m}})^m \leq \gamma^m$$

and hence $2^{-2m^2}\beta^{\frac{1}{2}} \leq \gamma^m$. Now, by [12, Lemma 7.2.1], there exists an element δ in $\mathcal{R}L$ such that $\beta = \delta\gamma^m$. This shows $\beta \in I$ and hence $M_\alpha \subseteq I$. □

Corollary 4.12. *Let I and J be two ideals in $\mathcal{R}L$ and J be a strongly z-ideal. If $J \subseteq \sqrt{I}$, then $J \subseteq I$.*

By Corollary 4.10, $J = \Sigma_{\alpha \in J} M_\alpha \subseteq \sqrt{I}$. Hence $M_\alpha \subseteq \sqrt{I}$, for every $\alpha \in J$ and, by Proposition 4.11, $M_\alpha \subseteq I$ for every $\alpha \in J$, that is, $J \subseteq I$.

Corollary 4.13. *Let I be an ideal in $\mathcal{R}L$. Then the following statements hold:*

(1) $(\sqrt{I})^{sz} = I^{sz}$.
(2) $(\sqrt{I})_{sz} = I_{sz}$.

Proof. By Proposition 4.8 and Proposition 4.11, we have

$$(\sqrt{I})^{sz} = \Sigma_{M_\alpha \subseteq \sqrt{I}} M_\alpha = \Sigma_{M_\alpha \subseteq I} M_\alpha = I^{sz}.$$

Similarly, by Proposition 4.8 and Proposition 4.12, we have

$$(\sqrt{I})_{sz} = \Sigma_{\alpha \in \sqrt{I}} M_\alpha = \Sigma_{\alpha \in I} M_\alpha = I_{sz}$$

and the proof is complete. □

Corollary 4.14. *Let I be an ideal in $\mathcal{R}L$. Then $\sqrt{I}$ is a strongly z-ideal if and only if I is a strongly z-ideal.*

Proof. Let I be a strongly z-ideal. Then, by Remark 3.5 and [7, Proposition 3.3], $\sqrt{I} = I$ and hence $\sqrt{I}$ is a strongly z-ideal.

Conversely, let $\sqrt{I}$ be a strongly z-ideal. Then by Corollary 4.13,

$$I^{sz} \subseteq I \subseteq \sqrt{I} = (\sqrt{I})^{sz} = I^{sz},$$

so $I = I^{sz}$ and hence I is a strongly z-ideal. □

Corollary 4.15. *If I is a proper ideal in $\mathcal{R}L$, then I is a strongly z-ideal if and only if every minimal prime ideal over I is a strongly z-ideal.*

Proof. If every prime ideal minimal over I is a strongly z-ideal, then, by Lemma 4.2, $\sqrt{I}$ is a strongly z-ideal and hence, by Corollary 4.14, I is a strongly z-ideal. Conversely, let P be a minimal prime ideal over a strongly z-ideal I. Then $I = I^{sz} \subseteq P^{sz} \subseteq P$ and minimality of P implies that $P = P^{sz}$. Thus P is a strongly z-ideal. □

Remark 4.16. The next example shows that the converse of Lemma 4.1 is not true in general.

EXAMPLE 4.17. Let I be an ideal of $\mathcal{R}L$ which is not semiprime. Put $J = \sqrt{I}$, then $I \neq J$. But, by Corollary 4.13, $J^{sz} = I^{sz}$ and $J_{sz} = I_{sz}$.

Proposition 4.18. *Let I, J be two ideals and $\{I_\lambda\}_{\lambda\in\Lambda}$ be a family of ideals of $\mathcal{R}L$. Then*

(1) $(IJ)_{sz} = I_{sz}J_{sz}$.
(2) $(IJ)^{sz} = I^{sz}J^{sz}$.
(3) $(\bigcap_{\lambda\in\Lambda} I_\lambda)_{sz} = \bigcap_{\lambda\in\Lambda} I_{\lambda_{sz}}$.
(4) $(\bigcap_{\lambda\in\Lambda} I_\lambda)^{sz} = \bigcap_{\lambda\in\Lambda} I_\lambda^{sz}$.
(5) $(I+J)_{sz} = I_{sz} + J_{sz}$.
(6) $(I+J)_{sz} = (I_{sz} + J_{sz})_{sz}$.
(7) $I^{sz} + J^{sz} \subseteq (I+J)^{sz}$.

Proof. For ideals I and J we have $I^{sz} \subseteq I \subseteq I_{sz}$ and $J^{sz} \subseteq J \subseteq J_{sz}$.

(1) We have $IJ \subseteq (IJ)_{sz}$ and $IJ \subseteq I_{sz}J_{sz}$. Since $(IJ)_{sz}$ is the smallest strongly z-ideal containing IJ, we conclude that $(IJ)_{sz} \subseteq I_{sz}J_{sz}$. Now, suppose that $\alpha \in I_{sz}J_{sz}$. Then $\alpha = \alpha_1\alpha_2$ where $\alpha_1 \in I_{sz}$ and $\alpha_2 \in J_{sz}$. So, by Proposition 4.7, there exist $\beta_1 \in I$ and $\beta_2 \in J$ such that $\alpha_1 \in M_{\beta_1}$ and $\alpha_2 \in M_{\beta_2}$. Therefore $Z(\beta_1) \subseteq Z(\alpha_1)$ and $Z(\beta_2) \subseteq Z(\alpha_2)$ and hence

$$Z(\beta_1\beta_2) \subseteq Z(\alpha_1\alpha_2) = Z(\alpha).$$

Thus, $\alpha \in M_{\beta_1\beta_2}$ and $\beta_1\beta_2 \in IJ$. So, by Proposition 4.7, $\alpha \in (IJ)_{sz}$. Therefore $I_{sz}J_{sz} \subseteq (IJ)_{sz}$.

(2) We have $(IJ)^{sz} \subseteq IJ$ and $I^{sz}J^{sz} \subseteq IJ$. Since $(IJ)^{sz}$ is the biggest strongly z-ideal included in IJ, we infer that $I^{sz}J^{sz} \subseteq (IJ)^{sz}$. Also, we have $(IJ)^{sz} \subseteq I^{sz}$ and $(IJ)^{sz} \subseteq J^{sz}$. Hence $(IJ)^{sz} \subseteq I^{sz} \cap J^{sz}$. Therefore, by [7, Proposition 5.6] and [12, Lemma 7.2.2],

$$(IJ)^{sz} \subseteq I^{sz} \cap J^{sz} = I^{sz}J^{sz}.$$

Thus $(IJ)^{sz} = I^{sz}J^{sz}$.

(3) We have $\bigcap I_\lambda \subseteq I_\lambda$, for every $\lambda \in \Lambda$. Then $(\bigcap I_\lambda)_{sz} \subseteq I_{\lambda_{sz}}$ for every $\lambda \in \Lambda$. So $(\bigcap I_\lambda)_{sz} \subseteq \bigcap I_{\lambda_{sz}}$. Now let $\beta \in \bigcap I_{\lambda_{sz}}$. Then $\beta \in I_{\lambda_{sz}}$ for every $\lambda \in \Lambda$. So, by Proposition 4.7, for every $\lambda \in \Lambda$ there exists an element α in I_λ such that $\beta \in M_\alpha$. Therefore $\alpha \in \bigcap I_\lambda$ and $\beta \in M_\alpha$. Now, by Proposition 4.7, $\beta \in (\bigcap I_\lambda)_{sz}$. Hence $\bigcap I_{\lambda_{sz}} \subseteq (\bigcap I_\lambda)_{sz}$.

(4) We have $\bigcap I_\lambda \subseteq I_\lambda$, for every $\lambda \in \Lambda$. Then $(\bigcap I_\lambda)^{sz} \subseteq I_\lambda^{sz}$ for every $\lambda \in \Lambda$. So $(\bigcap I_\lambda)^{sz} \subseteq \bigcap I_\lambda^{sz}$. Now let $\beta \in \bigcap I_\lambda^{sz}$, then $\beta \in I_\lambda^{sz}$ for every $\lambda \in \Lambda$. So, by Proposition 4.7, $M_\beta \subseteq I_\lambda$ for every $\lambda \in \Lambda$. Therefore $M_\beta \subseteq \bigcap I_\lambda$. Again, by Proposition 4.7, $\beta \in (\bigcap I_\lambda)^{sz}$.

(5) We have $(I+J) \subseteq (I+J)_{sz}$ and $I+J \subseteq I_{sz}+J_{sz}$. Since $(I+J)_{sz}$ is the smallest strongly z-ideal containing $I+J$, we infer that $(I+J)_{sz} \subseteq I_{sz}+J_{sz}$. Now, suppose that $\alpha \in I_{sz}+J_{sz}$. Then $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 \in I_{sz}$ and $\alpha_2 \in J_{sz}$. Thus, by Proposition 4.7, there exist $\beta_1 \in I$ and $\beta_2 \in J$ such that

$\alpha_1 \in M_{\beta_1}$ and $\alpha_2 \in M_{\beta_2}$. Therefore $Z(\beta_1) \subseteq Z(\alpha_1)$ and $Z(\beta_2) \subseteq Z(\alpha_2)$. So

$$Z(\beta_1^2 + \beta_2^2) = Z(\beta_1) \cap Z(\beta_2) \subseteq Z(\alpha_1) \cap Z(\alpha_2) \subseteq Z(\alpha).$$

Hence $\alpha \in M_{\beta_1^2+\beta_2^2}$. Since $\beta_1^2 + \beta_2^2 \in I + J$, by Proposition 4.7, $\alpha \in (I + J)_{sz}$. Therefore $I_{sz} + J_{sz} \subseteq (I + J)_{sz}$.

(6) From (5) we have $(I + J)_{sz} \subseteq (I_{sz} + J_{sz})_{sz}$. Now, let $\beta \in (I_{sz} + J_{sz})_{sz}$. Then, by Proposition 4.7, there exists an element α in $I_{sz} + J_{sz}$ such that $\beta \in M_\alpha$. So there exist $\alpha_1 \in I_{sz}$ and $\alpha_2 \in J_{sz}$ such that $\alpha = \alpha_1 + \alpha_2$. Again, by Proposition 4.7, there exist $\beta_1 \in I$ and $\beta_2 \in J$ such that $\alpha_1 \in M_{\beta_1}$ and $\alpha_2 \in M_{\beta_2}$. Therefore, by Corollary 3.4,

$$\alpha = \alpha_1 + \alpha_2 \in M_{\beta_1} + M_{\beta_2} = M_{\beta_1^2+\beta_2^2}.$$

Since

$$Z(\beta_1^2 + \beta_2^2) \subseteq Z(\alpha_1 + \alpha_2) = Z(\alpha) \subseteq Z(\beta),$$

we conclude that $\beta \in M_{\beta_1^2+\beta_2^2}$. Also, $\beta_1^2 + \beta_2^2 \in I + J$ and, by Proposition 4.7 we infer that $\beta \in (I + J)_{sz}$. Thus $(I_{sz} + J_{sz})_{sz} \subseteq (I + J)_{sz}$.

(7) We have $I^{sz} + J^{sz} \subseteq I + J$. Then, since $(I + J)^{sz}$ is the biggest strongly z-ideal included in $I + J$, we infer that $I^{sz} + J^{sz} \subseteq (I + J)^{sz}$. □

Proposition 4.19. *Let P and Q be two prime ideals in $\mathcal{R}L$ which are not chains. If P_m and Q_m are minimal prime ideals such that $P_m \subseteq P$ and $Q_m \subseteq Q$, then $P + Q = P_m + Q_m$. In particular, $P + Q$ is a prime strongly z-ideal.*

Proof. Clearly $P_m + Q_m \subseteq P + Q$. Now, since P_m and Q_m are minimal prime ideals, we conclude from Corollary 4.3 that P_m and Q_m are strongly z-ideals. By Theorem 3.2, P_m+Q_m is a strongly z-ideal and, since P_m+Q_m contains the prime ideal P_m, we infer from [7, Theorem 5.11] that $P_m + Q_m$ is prime. Since the prime ideals $P_m + Q_m$ and P contain the prime ideal P_m, we conclude from [3, Proposition 3.7] that $P_m + Q_m$ and P form a chain; that is, $P \subseteq P_m + Q_m$ or $P_m + Q_m \subseteq P$. If $P_m + Q_m \subseteq P$, then the prime ideals P and Q contain the prime ideal Q_m and, by [3, Proposition 3.7], P and Q form a chain, which is a contradiction. Hence $P \subseteq P_m + Q_m$. Similarly, $Q \subseteq P_m + Q_m$. Therefore $P + Q \subseteq P_m + Q_m$. Thus $P + Q = P_m + Q_m$. Hence $P + Q$ is a strongly z-ideal and, by [7, Theorem 5.11], we conclude that $P + Q$ is a prime strongly z-ideal. □

Corollary 4.20. *Let P and Q be two prime ideals in $\mathcal{R}L$. Then $(P + Q)^{sz} = P^{sz} + Q^{sz}$.*

Proof. If P and Q are chains, we are through. So, suppose that P and Q are not chains. Let P_m and Q_m are minimal prime ideals such that $P_m \subseteq P$ and $Q_m \subseteq Q$. Therefore, by Proposition 4.19, $P + Q = P_m + Q_m$ is a strongly z-ideals. By Corollary 4.3, P_m and Q_m are strongly z-ideals, which follows that $P_m \subseteq P^{sz}$ and $Q_m \subseteq Q^{sz}$. So

$$P_m + Q_m \subseteq P^{sz} + Q^{sz} \subseteq P + Q = P_m + Q_m.$$

Therefore, by Theorem 3.2, we have

$$(P+Q)^{sz} = (P_m + Q_m)^{sz} = P_m + Q_m = P^{sz} + Q^{sz}$$

and the proof is complete. $\square$

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References

1. R.N. Ball, and J. Walters-Wayland, C and C^* quotients on pointfree topology, *Dissertations Mathematicae* (*Rozprawy Math.*), **412**, Warszawa, (2002).
2. B. Banaschewski, The real numbers in pointfree topology, *Textos de Mathematica (Series B)*, **12**, University of Coimbra, Departmento de Mathematica, Coimbra, 1997.
3. T. Dube, Some algebraic characterizations of F-frames, *Algebra Univers.*, **62**, (2009), 273–288.
4. T. Dube, Concerning P-frames, essential P-frames and strongly zero-dimensional frames, *Algebra Univers.*, **61**, (2009), 115–138.
5. M.M. Ebrahimi, and A. Karimi Feizabadi, Pointfree prime representation of real Riesz maps, *Algebra Univers.*, **54**, (2005), 291–299.
6. A.A. Estaji, z-weak ideals and prime weak ideals, *Iranian Journal of Mathematical Sciences and Informatics*, **7(2)**, (2012), 53–62.
7. A.A. Estaji, A. Karimi Feizabadi, and M. Abedi, Zero sets in pointfree topology and strongly z-ideals, *Bull. Iranian Math. Soc.*, **41**, (2015), 1071–1084.
8. A.A. Estaji, A. Karimi Feizabadi, and M. Abedi, Strongly fixed ideals in $C(L)$ and compact frames, *Arch. Math. (Brno)*, **51**, (2015), 1–12.
9. A.A. Estaji, A. Karimi Feizabadi, and M. Abedi, Intersection of essential ideals in the ring of real-valued continuous functions on a frame, *J. Algebr. Syst.*, **5(2)**, (2017), 149–161.
10. L. Gillman, and M. Jerison, *Rings of continuous functions*, Springer-Verlag, 1976.
11. M. Henriksen and F.A. Smith, Sums of z-ideals and semiprime ideals, *General Topology and Its Relations to Modern Analysis and Algebra*, **5**, (1982), 272–278.
12. O. Ighedo, *Concerning ideals of pointfree function rings*, Ph.D. Thesis, University of South Africa, 2013.
13. R.Y. Sharp, *Steps in commutative algebra*, Cambridge Univ. Press, Cambridge, 2000.
14. P.T. Johnstone, *Stone spaces*, Cambridge Univ. Press, Cambridge, 1982.
15. A. Karimi Feizabadi, A. A. Estaji and M. Abedi, On minimal ideals in the ring of real-valued continuous functions on a frame, *Arch. Math. (Brno)*, **54(1)**, (2018), 1–13.
16. C. Kohls, Ideals in rings of continuous functions, *Fund. Math.*, **45**, (1957), 28–50.
17. G. Mason, z-ideals and prime ideals, *J. Algebra*, **26**, (1973), 280–297.
18. J. Picado and A. Pultr, *Frames and Locales: Topology without Points*, Frontiers in Mathematics, Birkhäuser/Springer, Basel AG, Basel, 2012.
19. A. Rezaei Aliabad and M. Parsinia, z_R-Ideals and z°_R-Ideals in subrings of $\mathbb{R}^X$, *Iranian Journal of Mathematical Sciences and Informatics*, **14**(1), (2019), 55–67.
20. D. Rudd, On two sum theorems for ideals of $C(X)$, *Mich. Math. J.*, **17**, (1970), 139–141.