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Bounds on $m_r(2, 29)$

Rumen Daskalov*, Elena Metodieva

Department of Mathematics, Technical University of Gabrovo, Bulgaria.

E-mail: daskalov@tugab.bg
E-mail: metodieva@tugab.bg

Abstract. An (n, r)-arc is a set of n points of a projective plane such that some r, but no $r + 1$ of them, are collinear. The maximum size of an (n, r)-arc in PG(2, q) is denoted by $m_r(2, q)$. In this paper thirteen new (n, r)-arc in PG$(2, 29)$ and a table with the best known lower and upper bounds on $m_r(2, 29)$ are presented. The results are obtained by non-exhaustive local computer search.

Keywords: Finite projective plane, (n, r)-Arc in a projective plane, (l, t)-Blocking set in a projective plane, Maximum size of an (n, r)-arc.

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1. Introduction

Let GF(q) denote the Galois field of q elements and V$(3, \mathrm{q})$ be the vector space of row vectors of length three with entries in GF(q). Let PG$(2, \mathrm{q})$ be the corresponding projective plane. The *points* (x_1, x_2, x_3) of PG$(2, \mathrm{q})$ are the 1-dimensional subspaces of V$(3, \mathrm{q})$. Subspaces of dimension two are called *lines*. The number of points and the number of lines in PG$(2, \mathrm{q})$ is $q^2 + q + 1$. There are $q + 1$ points on every line and $q + 1$ lines through every point.

For both an intrinsic understanding of the plane PG$(2, \mathrm{q})$ and for applications, for example, in coding theory, it is essential to characterize certain

*Corresponding Author

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subsets of the plane. Some of the essential subsets of the plane are arcs and blocking sets.

Definition 1.1. An (n,r)-arc is a set of n points of a projective plane such that some r, but no $r+1$ of them, are collinear. An (n,r)-arc is complete if there is no $(n+1,r)$-arc containing it.

Note that an $(n,2)$-arc is called an n-arc as well. The maximum size of an (n,r)-arc in PG$(2,\mathrm{q})$ is denoted by $m_r(2,q)$.

Definition 1.2. An (l,t)-blocking set S in PG(2, q) is a set of l points such that every line of PG(2, q) intersects S in at least t points, and there is a line intersecting S in exactly t points.

Note that an (n,r)-arc is the complement of a $(q^2+q+1-n, q+1-r)$-blocking set in a projective plane, and conversely.

Definition 1.3. Let M be a set of points in any plane. An i-secant is a line meeting M in exactly i points. Define τ_i as the number of i-secants to a set M.

In terms of τ_i the definitions of an (n,r)-arc and an (l,t)-blocking set become:

An (n,r)-arc is a set of n points of a projective plane for which $\tau_i \geq 0$ for $i<r$, $\tau_r>0$, and $\tau_i=0$ when $i>r$.

An (l,t)-blocking set is a set of l points of a projective plane for which $\tau_i=0$ for $i<t$, $\tau_t>0$, and $\tau_i\geq 0$ when $i>t$.

For an introduction to projective geometries over finite fields and further information on the geometrical properties of arcs and blocking sets, we refer to [15].

In 1947 Bose [6] proved that

$$m_2(2,q)=q+1 \text{ for } q \text{ odd}$$
$$m_2(2,q)=q+2 \text{ for } q \text{ even.}$$

From the results of Barlotti [5] and Ball [1] it follows that for q odd and

$$r=(q+1)/2,\ r=(q+3)/2$$
$$m_r(2,q)=(r-1)q+1.$$

The following two theorems are proved by Ball and Daskalov respectively.

Theorem 1.4. [1] *Let K be an (n,r)-arc in* PG(2, q) *where q is prime.*

1. *If $r \le (q+1)/2$ then $m_r(2,q) \le (r-1)q+1$.*

2. *If $r \ge (q+3)/2$ then $m_r(2,q) \le (r-1)q+r-(q+1)/2$.*

Theorem 1.5. [9] *Let K be an (n,r)-arc in* PG(2, q) *with $r > (q+3)/2$ and $q \le 29$ is prime. Then*

$$m_r(2,q) \le (r-1)q+r-(q+3)/2.$$

A survey on (n,r)-arcs, containing the best results known at the time, was presented in [16]. Afterwards, many improvements were made by [10], [11] and [7]. Those were summarized by Ball and Hirschfeld [3], who presented a new table with bounds on $m_r(2,q)$ for $q \le 19$. As indicated by that table [3], the exact values of $m_r(2,q)$ are determined only for $q \le 9$.

r\q	3	4	5	7	8	9
2	4	6	6	8	10	10
3		9	11	15	15	17
4			16	22	28	28
5				29	33	37
6				36	42	48
7					49	55
8						65

Table 1: Values of $m_r(2,q)$

Several new improvements have been obtained in recent years. A $(79,6)$-arc in PG(2,17) and a $(126,8)$-arc in PG(2,19) were given in [17]. In addition, (n,r)-arcs with the following parameters were constructed in [12] : $(95,7)$, $(183,12)$ and $(205,13)$ in PG(2,17), and $(243,14)$ and $(264,15)$ in PG(2,19). In 2011 T. A. Gulliver [14] produced an optimal $(78,8)$-arc in PG(2,11). A summary of the new improvements can be found in the online table for $m_r(2,q)$, $q \le 19$, maintained by S. Ball [2]. New results and tables with lower and upper bounds on $m_r(2,q)$ for $q=23$, and $q=25,27$ are presented in [12] and [13], respectively.

In [8] a full classification (up to equivalence) of all complete k-arcs in the Desarguesian projective planes of order 27 and 29 is obtained.

To obtain good (n,r)-arcs we apply local search techniques. The neighborhood structure is simple one. Given an arc, then its neighborhood consists of all arcs that can be obtained from the given arc by adding new points or deleting some points. The choice of a starting solution is based on some heuristic observations. The cost function is chosen to favor as local optima arcs with a small number of r-secants. The computation times are in order of several minutes up to a few hours on a PC. Similar techniques are employed for construction of (l, t)-blocking sets.

In order to present the results in a more concise form, the points in PG(2,29) are in lexicographic order and each point is associated with its number. For example some of the points in PG(2,29) and their numbers are given in the next Table 2.

Number	Point	Number	Point	Number	Point
1	(0,0,1)	155	(0,4,8)	...	...
2	(0,1,0)	...	...	743	(1,24,24)
3	(0,1,1)	403	(1,12,24)	...	...
...	...	404	(1,12,25)	852	(1,28,9)

Table 2: Some points in PG(2,29)

In the following Table 3 some of the lines and their numbers are presented.

Number	a, b, c	List of points of the line $ax + by + cz = 0$
1	0,0,1	2 31 60 89 118 147 176 205 234 263 292 321 350 379 408 437 466 495 524 553 582 611 640 669 698 727 756 785 814 843
210	1,6,5	24 54 76 98 120 171 193 215 237 288 310 332 354 405 427 449 471 522 544 566 588 639 661 683 705 727 778 800 822 844
644	1,21,4	4 38 69 100 131 162 193 224 255 286 317 348 350 381 412 443 474 505 536 567 598 629 660 691 722 753 784 786 817 848
712	1,23,14	19 33 79 96 142 159 176 222 239 285 302 348 365 382 428 445 491 508 525 571 588 634 651 697 714 731 777 794 840 857

Table 3: Some lines in PG(2,29)

According to [17] the next Table 4 holds:

r	2	3	4	5	6	7	8	9	10
$m_r(2, 29)$	30	43	70	94	126	146	181	201	233
r	11	12	13	14	15	16	17	18	19
$m_r(2, 29)$	255	291	324	342	407	436	451	471	499
r	20	21	22	23	24	25	26	27	28
$m_r(2, 29)$	521	563	580	619	658	683	715	755	784

Table 4: Lower bounds on $m_r(2, 29)$ in [17]

In this paper we improve the lower bounds for $r = 11, 13, 14, 17 - 26$. For $r = 14, 20, 22, 23, 26$ the improvements are significant.

2. New Arcs in PG(2, 29)

Theorem 2.1. *There exist a* $(258, 11)$*-arc, a* $(325, 13)$*-arc, a* $(361, 14)$*-arc, a* $(452, 17)$*-arc, a* $(474, 18)$*-arc, a* $(500, 19)$*-arc, a* $(531, 20)$*-arc, a* $(564, 21)$*-arc, a* $(593, 22)$*-arc, a* $(625, 23)$*-arc, a* $(662, 24)$*-arc, a* $(694, 25)$*-arc and a* $(725, 26)$*-arc in* PG(2,29).

Proof. 1. The set of points having numbers 64 65 79 83 84 86 88 89 90 92 93 102 105 110 117 118 124 131 134 135 136 138 141 143 144 153 155 156 157 162 164 166 168 170 174 179 180 182 184 186 187 194 195 201 202 207 208 210 212 215 224 226 227 229 231 235 236 239 241 250 257 258 262 271 272 273 276 279 289 290 295 301 303 304 306 307 309 310 318 320 323 325 327 332 335 339 344 346 347 348 349 350 351 352 356 357 358 362 367 371 375 377 379 382 384 385 390 397 400 401 402 403 405 409 410 416 419 421 424 425 426 429 435 437 438 442 443 447 448 455 457 460 464 465 466 468 470 474 481 483 487 491 499 500 506 507 508 511 513 520 523 524 525 526 527 531 537 540 546 550 551 553 554 558 565 570 572 577 579 582 587 588 592 594 595 599 601 604 605 606 612 616 617 618 619 624 627 628 632 633 639 640 644 648 651 654 660 661 662 664 665 671 672 674 680 681 682 684 685 686 687 693 700 703 705 707 714 718 720 722 723 724 733 734 737 740 742 747 749 750 752 757 760 762 764 765 771 777 779 780 786 792 795 801 803 804 806 813 816 818 825 833 839 846 847 853 862 868 forms a (258,11)-arc $\mathcal{K}_1$ in PG(2,29) with secant distribution

$$\tau_0 = 29,\ \tau_1 = 5,\ \tau_2 = 2,\ \tau_3 = 3,\ \tau_4 = 8,\ \tau_5 = 26,$$

$$\tau_6 = 24,\ \tau_7 = 76,\ \tau_8 = 95,\ \tau_9 = 143,\ \tau_{10} = 223,\ \tau_{11} = 237$$

2. Deleting from $\mathcal{K}_1$ the points with numbers 79 102 110 141 179 195 201 295 303 349 448 506 511 513 527 540 546 582 587 616 627 648 671 703 734 825 and adding the points having numbers 67 70 94 97 108 113 114 115 122 129 130 149 167 190 197 199 213 216 232 240 247 252 256 261 263 266 282 283 284 293 300 312 321 372 373 387 420 427 436 439 456 461 471 478 479 480 493 504 512 519 533 538 563 567 568 573 581 596 598 620 623 634 647 649 658 683 695 696 702 711 725 731 736 746 761 770 781 784 793 796 798 800 807 821 823 824 826 831 832 834 836 841 857 we obtain a (325,13)-arc $\mathcal{K}_2$ in PG(2,29) with secant distribution

$$\tau_0 = 31,\ \tau_4 = 2,\ \tau_5 = 2,\ \tau_6 = 4,\ \tau_7 = 9,\ \tau_8 = 21,$$

$$\tau_9 = 33,\ \tau_{10} = 81,\ \tau_{11} = 172,\ \tau_{12} = 230,\ \tau_{13} = 286$$

3. From $\mathcal{K}_2$ by deleting points with numbers 538 620 649 824 and adding points with numbers 66 79 102 110 121 141 195 245 265 269 295 303 313 324 336 344 378 396 397 448 490 506 511 513 527 556 582 587 627 631 648 671 703

734 743 799 814 825 850 870 we obtain a (361,14)-arc in PG(2,29). The secant distribution of this arc is

$$\tau_0 = 31,\ \tau_8 = 2,\ \tau_9 = 14,\ \tau_{10} = 26,\ \tau_{11} = 54,\ \tau_{12} = 142,\ \tau_{13} = 298,\ \tau_{14} = 304$$

4. The set of points having numbers 1 2 19 20 21 22 23 24 25 27 28 29 30 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 62 68 69 71 73 74 75 76 78 80 81 87 91 95 96 98 100 101 103 104 106 107 109 111 112 116 119 120 123 125 126 128 132 133 137 139 140 142 145 146 147 148 150 151 154 158 160 163 165 169 172 173 175 176 177 178 181 183 185 188 193 196 198 200 203 204 205 206 209 211 217 218 219 220 221 222 228 230 233 234 237 238 242 243 244 246 251 253 254 255 259 260 264 267 268 270 274 275 277 278 280 281 285 287 288 291 292 294 296 297 298 302 305 308 311 315 316 317 319 326 328 329 330 331 333 334 337 338 340 341 342 343 345 353 355 359 360 361 363 364 365 366 368 369 370 374 376 380 381 383 388 389 392 393 394 395 398 399 404 406 407 408 411 413 414 415 417 422 423 428 430 431 432 434 440 441 444 445 449 450 451 452 453 454 458 459 462 463 467 469 472 473 475 476 477 484 485 486 488 489 492 494 495 497 498 501 502 503 509 510 516 517 518 521 522 528 529 530 532 534 535 536 541 542 543 545 547 548 549 555 557 559 560 561 564 566 569 571 574 575 576 578 580 583 584 585 586 590 591 593 600 602 603 607 608 609 610 611 613 615 616 621 622 625 626 629 630 635 636 638 641 642 643 646 650 652 653 656 657 659 663 666 667 668 669 670 673 676 677 678 679 688 689 690 691 694 697 698 699 704 708 709 710 712 713 715 716 717 721 726 727 728 729 730 732 735 739 744 748 751 753 754 755 756 759 763 766 767 768 769 772 773 774 775 778 782 787 788 789 790 791 794 797 802 805 808 809 810 811 812 815 817 819 820 822 827 828 829 830 835 837 838 840 842 843 848 849 851 852 854 856 859 861 863 864 866 867 forms a (419,13)-blocking set in PG(2,29) with secant distribution

$$\tau_{13} = 239,\ \tau_{14} = 406\ \tau_{15} = 197,\ \tau_{28} = 17,\ \tau_{29} = 12$$

The complement of this blocking set is a $(452, 17)$-arc in PG(2, 29).

5. The set of points having numbers 7 10 19 21 22 23 24 25 27 28 29 30 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 51 52 54 55 56 57 58 59 60 62 68 69 71 73 74 75 78 80 81 87 91 96 98 100 101 104 106 107 109 111 112 116 119 120 123 125 126 128 133 137 139 140 142 145 146 147 148 150 151 154 158 160 163 165 172 173 175 176 178 181 183 185 188 193 196 198 200 203 204 205 206 209 211 217 218 219 220 221 222 228 230 233 234 237 238 242 243 244 246 253 254 255 259 260 264 267 268 270 274 275 277 278 280 281 285 287 291 292 294 297 298 302 305 308 311 315 316 317 319 326 328 329 330 333 334 337 338 340 341 342 343 345 353 355 359 360 361 363 364 365 366 368 369 374 376 380 381 383 388 389 392 393 394 395 398 399 404 406 407 408 411 413 414 415 417 422 423 428 430 431 432 434 440 441 444 445 449 450 451 453 454 458 459 462

463 467 469 472 473 475 476 477 484 485 486 488 492 494 495 497 498 501 502 503 509 510 516 517 518 521 522 523 529 530 532 535 536 541 542 543 545 547 548 549 555 557 559 560 561 564 566 569 571 574 575 576 578 580 583 584 585 586 590 591 593 600 602 603 607 609 610 611 613 615 621 622 625 626 629 630 635 636 638 641 642 643 646 650 652 653 656 657 659 663 666 667 668 669 670 673 676 677 678 679 688 689 690 691 694 697 698 699 704 708 709 710 712 713 715 716 717 721 726 727 728 729 730 732 739 744 748 751 753 754 755 756 759 763 766 767 768 769 772 773 774 775 778 782 787 788 789 790 791 794 797 802 805 808 810 811 812 815 817 819 820 822 827 828 829 830 835 837 838 840 842 843 848 849 851 852 856 859 861 863 864 866 867 forms a (397,12)-blocking set in PG(2,29) with secant distribution

$$\tau_{12} = 214, \tau_{13} = 323,\ \tau_{14} = 218,\ \tau_{15} = 85,\ \tau_{16} = 3,$$

$$\tau_{25} = 1,\ \tau_{26} = 4,\ \tau_{27} = 11,\ \tau_{28} = 6, ,\ \tau_{29} = 6$$

The complement of this blocking set is a $(474, 18)$-arc in PG(2, 29).

6. The set of points having numbers 1 2 4 8 10 15 16 17 18 20 23 25 29 31 33 37 39 44 45 46 47 52 54 58 65 66 67 71 74 75 78 82 83 88 90 91 94 96 98 101 106 109 111 113 117 123 125 126 127 128 130 135 137 138 139 140 142 143 151 152 153 155 157 158 159 164 165 166 168 170 171 172 175 179 180 181 182 185 189 196 199 200 201 202 206 209 210 212 216 223 227 229 230 233 235 236 237 240 244 246 247 250 251 253 257 260 261 262 263 266 267 271 272 273 275 280 282 283 284 288 289 292 293 294 295 303 310 316 318 319 320 324 325 328 329 333 334 337 338 342 343 346 347 350 351 354 356 359 362 363 364 365 366 367 373 378 381 384 385 387 388 392 393 394 395 397 399 400 402 403 406 408 410 411 415 418 421 424 427 430 434 435 438 443 447 448 449 451 452 454 455 456 460 463 464 465 467 472 476 477 478 480 481 483 484 485 489 494 495 498 502 505 508 511 514 517 521 522 526 529 530 532 533 535 536 538 539 540 542 544 545 547 548 551 553 554 557 559 562 565 566 567 568 569 570 576 578 585 586 589 590 594 595 598 599 603 604 607 608 611 612 613 614 616 629 635 637 638 639 640 641 643 644 648 649 650 652 654 655 657 659 660 661 665 666 670 671 672 675 679 681 682 685 686 688 691 692 695 696 697 699 702 703 705 709 716 720 722 723 726 730 731 732 733 736 747 750 751 752 753 759 760 761 762 763 764 766 767 768 771 773 774 775 777 779 780 781 790 792 793 794 795 797 802 804 805 806 807 809 815 816 819 821 823 826 831 834 836 838 841 842 843 849 850 854 857 858 861 865 866 867 forms a (371,11)-blocking set in PG(2,29) with secant distribution

$$\tau_{11} = 247,\ \tau_{12} = 261,\ \tau_{13} = 178,\ \tau_{14} = 93,\ \tau_{15} = 39,\ \tau_{16} = 22,$$

$$\tau_{17} = 11,\ \tau_{18} = 3,\ \tau_{19} = 1,\ \tau_{28} = 4,\ \tau_{29} = 4,\ \tau_{30} = 8$$

The complement of this blocking set is a $(500, 19)$-arc in PG(2, 29).

7. The set of points having numbers 1 4 10 11 15 16 17 18 23 25 31 33 39 44 45 46 47 52 58 66 67 71 74 75 78 82 83 84 90 91 94 96 98 101 106 109 111 113 116 117 123 126 127 128 130 135 137 138 139 140 142 151 153 155 157 158 159 164 165 168 170 171 172 179 180 181 185 196 199 200 201 202 209 210 212 216 223 227 229 230 233 235 236 237 240 244 246 247 250 251 253 255 257 260 261 263 266 267 271 272 273 274 275 277 278 280 282 283 284 288 292 293 294 295 297 303 310 318 320 324 325 328 329 333 334 337 338 342 343 347 350 351 354 356 359 362 363 364 365 366 367 372 375 378 381 384 385 387 388 390 392 393 394 395 399 402 403 406 408 410 411 415 418 421 424 434 435 438 441 443 447 448 449 451 452 455 456 460 465 467 472 476 480 481 483 484 485 489 494 495 497 498 502 508 511 514 517 521 522 526 529 530 533 537 538 539 540 542 544 545 547 548 551 553 554 557 560 565 566 567 568 569 570 573 576 578 581 585 589 590 594 595 597 598 599 603 604 607 608 611 612 614 622 629 635 637 638 639 640 644 648 649 650 652 654 655 657 658 659 660 661 665 666 671 672 675 679 681 682 685 686 688 692 695 696 697 699 702 703 705 709 716 720 722 723 727 730 731 732 733 736 747 748 751 752 753 760 761 762 764 767 768 773 774 775 777 779 781 790 792 793 794 795 797 802 804 805 806 815 816 819 821 823 826 830 831 834 836 838 841 842 848 849 850 854 857 858 861 865 866 forms a (340,10)-blocking set in PG(2,29) with secant distribution

$$\tau_{10} = 244,\ \tau_{11} = 270,\ \tau_{12} = 175,\ \tau_{13} = 108,\ \tau_{14} = 37,$$

$$\tau_{15} = 17,\ \tau_{16} = 4,\ \tau_{18} = 2,\ \tau_{28} = 1,\ \tau_{29} = 5,\ \tau_{30} = 8.$$

The complement of this blocking set is a $(531, 20)$-arc in PG(2, 29).

8. The set of points having numbers 1 2 3 4 10 11 22 23 29 30 39 42 44 45 46 47 49 58 59 61 62 66 67 69 71 74 78 80 82 83 87 88 89 90 91 96 98 109 111 116 117 125 127 128 131 134 137 138 140 151 152 155 157 158 159 164 165 171 172 178 179 180 181 187 200 201 202 203 205 206 207 209 210 212 219 220 227 229 230 232 233 235 240 246 247 250 251 257 262 266 267 271 272 283 284 288 289 292 293 297 300 304 309 316 319 320 328 329 335 336 342 343 346 347 351 356 359 363 364 365 366 370 373 378 379 380 381 382 383 384 385 386 387 388 389 390 391 392 393 394 395 396 397 398 399 400 401 402 403 404 405 406 407 410 411 418 420 421 424 425 427 434 435 438 445 447 448 451 452 454 455 456 458 465 467 474 476 477 478 483 484 485 487 494 497 498 505 507 508 511 512 514 521 522 527 529 530 532 534 543 545 547 548 550 554 559 566 567 568 569 576 581 586 589 590 596 597 603 604 607 611 612 613 616 623 628 629 635 638 639 640 643 644 648 649 658 660 661 665 666 671 675 681 682 685 686 692 696 702 703 705 712 713 720 722 723 726 729 730 731 732 733 750 751 752 753 754 756 757 760 764 767 768 773 774 777 781 786 793 794 795 798 801 804 805 814 815 816 821 826 831 836 841 842 845 846 849 850 852 854 857 858 861 863 865 866 870 forms a (307,9)-blocking set in PG(2,29) with secant distribution

$$\tau_9 = 272,\ \tau_{10} = 269,\ \tau_{11} = 192,\ \tau_{12} = 65, \tau_{13} = 35,$$

$$\tau_{14} = 18,\ \tau_{15} = 4,\ \tau_{16} = 4,\ \tau_{19} = 1,\ \tau_{30} = 11$$

The complement of this blocking set is a $(564, 21)$-arc in PG(2, 29).

9. The set of points having numbers 1 2 11 13 14 19 22 25 29 31 40 42 43 48 49 51 60 62 69 72 77 80 87 89 91 94 96 101 102 106 111 113 116 124 125 127 130 131 134 135 138 140 142 151 153 157 158 164 165 166 171 172 178 179 181 182 188 193 200 203 212 217 219 220 222 223 227 233 237 240 244 246 247 249 250 251 256 257 260 262 266 267 272 273 275 278 283 284 288 289 295 297 300 304 309 313 316 324 325 329 333 335 336 338 342 343 346 351 356 362 363 364 366 367 373 375 377 378 380 382 386 389 391 396 398 402 404 405 407 415 418 419 420 425 426 427 437 438 443 445 449 454 455 458 460 465 466 467 472 474 477 478 483 487 489 490 494 496 497 505 506 507 512 513 517 525 527 528 530 534 536 541 543 546 550 552 554 557 559 565 566 568 569 576 581 586 589 590 594 596 597 599 603 604 607 608 616 619 623 628 632 635 637 640 643 644 648 649 652 657 659 660 665 666 670 672 675 681 682 683 685 686 688 692 695 699 705 710 712 713 715 720 729 732 739 744 750 751 754 760 761 766 767 768 770 773 774 775 779 781 790 792 794 797 798 801 802 804 805 807 814 816 819 821 825 826 830 831 836 838 841 845 852 855 857 860 863 870 forms a (278,8)-blocking set in PG(2,29) with secant distribution

$$\tau_8 = 268,\ \tau_9 = 264,\ \tau_{10} = 182,\ \tau_{11} = 82,\ \tau_{12} = 48,$$

$$\tau_{13} = 13,\ \tau_{14} = 3,\ \tau_{15} = 1,\ \tau_{28} = 1,\ \tau_{29} = 2,\ \tau_{30} = 7.$$

The complement of this blocking set is a $(593, 22)$-arc in PG(2, 29).

10. The set of points having numbers 1 2 3 13 14 19 27 29 31 39 40 43 46 48 51 62 63 72 75 77 80 90 91 94 101 102 104 106 109 113 117 118 125 126 129 130 131 135 140 142 153 157 159 163 164 166 171 172 178 182 188 189 193 203 206 217 219 220 222 233 237 244 246 248 251 252 256 260 262 268 269 272 273 275 280 283 284 295 300 304 309 313 315 324 325 329 332 333 335 338 343 344 346 362 363 364 366 367 375 376 378 379 380 381 382 383 384 385 386 387 388 389 390 391 392 393 394 395 396 397 398 399 400 401 402 403 404 405 406 407 412 415 419 420 423 425 426 437 443 449 454 455 458 460 466 472 475 477 478 483 489 506 507 512 513 517 521 525 528 536 538 539 541 546 552 555 557 565 566 568 569 570 571 578 586 594 597 599 601 602 603 607 608 618 619 623 628 632 637 648 649 652 657 659 660 664 667 670 672 681 683 686 688 695 698 699 710 712 713 715 729 734 739 744 750 754 761 766 767 768 770 773 775 779 790 792 797 801 802 807 819 824 826 830 831 833 838 841 842 852 855 860 862 867 870 forms a (245,7)-blocking set in PG(2,29) with secant distribution

$$\tau_7 = 286,\ \tau_8 = 282,\ \tau_9 = 168,\ \tau_{10} = 79,\ \tau_{11} = 34,$$

$$\tau_{12} = 11,\ \tau_{13} = 2,\ \tau_{18} = 1,\ \tau_{30} = 8.$$

The complement of this blocking set is a $(626, 23)$-arc in PG(2, 29).

11. The set of points having numbers 1 2 3 13 14 19 29 31 40 43 48 51 62 72 75 77 80 90 91 94 101 102 106 113 117 118 125 130 131 134 135 140 142 153 157 159 164 166 171 178 182 188 193 203 217 219 220 222 233 237 244 246 251 252 256 260 262 272 273 275 280 283 284 295 300 304 309 313 324 325 329 333 335 338 346 362 363 364 366 367 375 378 379 380 381 382 383 384 385 386 387 388 389 390 391 392 393 394 395 396 397 398 399 400 401 402 403 404 405 406 407 415 419 420 425 426 437 443 449 454 455 460 466 472 477 478 483 489 506 507 512 513 517 525 528 536 541 546 552 557 565 566 568 569 570 578 586 594 597 599 602 603 607 608 619 623 628 632 637 648 649 652 657 659 660 670 672 676 681 686 688 695 699 710 712 713 715 729 739 744 750 754 761 766 767 768 770 773 775 779 790 792 797 800 801 802 807 819 822 826 830 831 838 841 852 855 857 860 870 forms a (209,6)-blocking set in PG(2,29) with secant distribution

$$\tau_6 = 332,\ \tau_7 = 291,\ \tau_8 = 160,\ \tau_9 = 64,\ \tau_{10} = 13,\ \tau_{11} = 3,\ \tau_{12} = 1,\ \tau_{30} = 7.$$

The complement of this blocking set is a $(662, 24)$-arc in PG(2, 29).

12. The set of points having numbers 1 4 15 16 17 18 29 33 37 39 44 47 52 58 67 71 78 82 89 94 96 98 101 102 105 109 111 113 125 126 139 140 152 153 155 168 170 171 179 181 185 196 200 202 206 209 210 216 223 229 233 235 236 244 253 261 262 266 267 277 278 288 289 293 297 303 310 316 320 324 328 333 338 343 347 354 359 362 367 370 375 381 388 390 397 399 406 415 421 424 430 438 443 451 452 460 465 472 476 478 480 481 483 485 489 495 498 505 514 521 524 532 537 538 539 540 545 553 554 559 576 581 586 595 598 607 613 614 637 638 640 650 652 657 659 671 672 675 692 695 696 702 705 709 716 720 723 731 733 736 747 750 752 760 761 764 766 775 777 780 781 790 795 797 802 804 809 815 823 826 831 834 842 849 857 858 866 forms a (177,5)-blocking set in PG(2,29) with secant distribution

$$\tau_5 = 294,\ \tau_6 = 374,\ \tau_7 = 173,\ \tau_8 = 16,\ \tau_9 = 5,\ \tau_{10} = 1,\ \tau_{11} = 2,\ \tau_{30} = 6.$$

The complement of this blocking set is a $(694, 25)$-arc in PG(2, 29).

13. The set of points having numbers 1 2 14 19 31 48 49 72 77 88 94 106 113 125 131 135 140 157 164 166 178 193 203 219 220 222 237 251 260 272 280 283 300 309 313 325 338 346 363 366 367 379 380 381 382 383 384 385 386 387 388 389 390 391 392 393 394 395 396 397 398 399 400 401 402 403 404 405 406 407 419 425 426 438 443 454 460 472 483 489 506 512 513 524 525 526 527 528 529 530 531 532 533 534 535 536 537 538 539 540 541 542 543 544 545 546 547 548 549 550 551 552 566 569 570 586 599 607 619 628 632 649 657 660 672 686 695 712 713 715 729 744 754 766 773 775 792 802 807 819 831 838 841 843 855

860 forms a (146,4)-blocking set in PG(2,29) with secant distribution

$$\tau_4 = 235,\ \tau_5 = 509,\ \tau_6 = 111,\ \tau_7 = 9,\ \tau_8 = 2,\ \tau_{30} = 5.$$

The complement of this blocking set is a $(725, 26)$-arc in PG(2, 29).

□

r			r			r	
2	30		11	258-291		20	531-555
3	43-59		12	291-320		21	564-585
4	70-88		13	325-349		22	593-615
5	94-117		14	361-378		23	625-645
6	126-146		15	407		24	662-675
7	146-175		16	436		25	694-705
8	181-204		17	452-465		26	725-735
9	201-233		18	474-495		27	755-765
10	233-262		19	500-525		28	784-795

Table 5: Lower and upper bounds on $m_r(2,29)$

The upper bounds in Table 5 come from Theorems 1.1 and 1.2.

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