

## Approximation by $(p, q)$ -Lupaş Stancu Operators

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**ABSTRACT.** In this paper,  $(p, q)$ -Lupas Bernstein Stancu operators are constructed. Statistical as well as other approximation properties of  $(p, q)$ -Lupaş Stancu operators are studied. Rate of statistical convergence by means of modulus of continuity and Lipschitz type maximal functions has been investigated.

**Keywords:**  $(p, q)$ -Integers, Lupaş  $(p, q)$ -Bernstein Stancu operators, Statistical approximation, Korovkin's type approximation.

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### 1. INTRODUCTION AND PRELIMINARIES

In 1912, S.N. Bernstein [6] introduced his famous operators  $B_n : C[0, 1] \rightarrow C[0, 1]$  defined for any  $n \in \mathbb{N}$  and for any function  $f \in C[0, 1]$

$$B_n(f; x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0, 1]. \quad (1.1)$$

and named it Bernstein polynomials to provide a constructive proof of the Weierstrass theorem [20].

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Further, based on  $q$ -integers, Lupaş [21] introduced the first  $q$ -Bernstein operators [6] and investigated its approximating and shape-preserving properties. Another  $q$ -analogue of the Bernstein polynomials is due to Phillips [38]. Since then several generalizations of well-known positive linear operators based on  $q$ -integers have been introduced and their approximation properties studied.

Recently, the applications of  $(p, q)$ -calculus (post quantum calculus) emerged as a new area in the field of approximation theory [20]. The development of post quantum calculus has led to the discovery of various generalizations of Bernstein polynomials involving  $(p, q)$ -integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design [7] and solutions of differential equations.

Mursaleen *et al* [27] introduced the concept of post quantum calculus in approximation theory and constructed the  $(p, q)$ -analogue of Bernstein operators defined as follows for  $0 < q < p \leq 1$ :

$$B_{n,p,q}(f; x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^n \left[ \begin{matrix} n \\ k \end{matrix} \right]_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right), \quad x \in [0, 1]. \quad (1.2)$$

Note when  $p = 1$ ,  $(p, q)$ -Bernstein Operators given by (1.2) turns out to be Phillips  $q$ -Bernstein Operators [38].

Also, we have

$$\begin{aligned} (1-x)_{p,q}^n &= \prod_{s=0}^{n-1} (p^s - q^s x) = (1-x)(p-qx)(p^2-q^2x)\dots(p^{n-1}-q^{n-1}x) \\ &= \sum_{k=0}^n (-1)^k p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[ \begin{matrix} n \\ k \end{matrix} \right]_{p,q} x^k. \end{aligned}$$

Further, they applied the concept of  $(p, q)$ -calculus in approximation theory and studied approximation properties based on  $(p, q)$ -integers for Bernstein-Stancu operators,  $(p, q)$ -analogue of Bernstein-Kantorovich,  $(p, q)$ -analogue of Bernstein-Shurer operators,  $(p, q)$ -analogue of Bleimann-Butzer-Hahn operators and  $(p, q)$ -analogue of Lorentz polynomials on a compact disk in [28, 31, 32, 33, 35].

On the other hand, Khalid and Lobiyal defined  $(p, q)$ -analogue of Lupaş Bernstein operators [17] as follows :

For any  $p > 0$  and  $q > 0$ , the linear operators  $L_{p,q}^n : C[0, 1] \rightarrow C[0, 1]$  as

$$L_{p,q}^n(f; x) = \sum_{k=0}^n \frac{f\left(\frac{p^{n-k} [k]_{p,q}}{[n]_{p,q}}\right) \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} x^k (1-x)^{n-k}}{\prod_{j=1}^n \{p^{j-1}(1-x) + q^{j-1}x\}}, \tag{1.3}$$

are  $(p, q)$ -analogue of Lupaş Bernstein operators.

Again when  $p = 1$ , Lupaş  $(p, q)$ -Bernstein operators turns out to be Lupaş  $q$ -Bernstein operators as given in [22, 37].

When  $p = q = 1$ , Lupaş  $(p, q)$ -Bernstein operators turns out to be classical Bernstein operators [6].

They studied two different techniques as de-Casteljau’s algorithm and Korovkin’s type approximation properties [17]: de-Casteljau’s algorithm and related results of degree elevation reduction for Bèzier curves and surfaces holds for all  $p > 0$  and  $q > 0$ . However to study Korovkin’s type approximation properties for Lupaş  $(p, q)$ -analogue of the Bernstein operators,  $0 < q < p \leq 1$  is needed.

Based on Korovkin’s type approximation, they proved that the sequence of  $(p, q)$ -analogue of Lupaş Bernstein operators  $L_{p_n, q_n}^n(f, x)$  converges uniformly to  $f(x) \in C[0, 1]$  if and only if  $0 < q_n < p_n \leq 1$  such that  $\lim_{n \rightarrow \infty} q_n = 1$ ,  $\lim_{n \rightarrow \infty} p_n = 1$  and  $\lim_{n \rightarrow \infty} q_n^n = 1$ . On the other hand, for any  $p > 0$  fixed and  $p \neq 1$ , the sequence  $L_{p,q}^n(f, x)$  converges uniformly to  $f(x) \in C[0, 1]$  if and only if  $f(x) = ax + b$  for some  $a, b \in \mathbb{R}$ .

Furthermore, in comparison to  $q$ -Bèzier curves and surfaces based on Lupaş  $q$ -Bernstein rational functions, their generalization gives more flexibility in controlling the shapes of curves and surfaces.

Some advantages of using the extra parameter  $p$  have been discussed in the field of approximations on compact disk [35] and in computer aided geometric design [17].

For more details related to approximation theory [20], one can refer [1, 2, 3, 5, 8, 9, 12, 13, 14, 15, 18, 19, 22, 23, 24, 34, 36, 39, 40, 42, 43, 44, 45, 46, 47, 48].

Let us recall certain notations of  $(p, q)$ -calculus.

For any  $p > 0$  and  $q > 0$ , the  $(p, q)$  integers  $[n]_{p,q}$  are defined by

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \dots + pq^{n-2} + q^{n-1} = \begin{cases} \frac{p^n - q^n}{p - q}, & \text{when } p \neq q \neq 1 \\ n p^{n-1}, & \text{when } p = q \neq 1 \\ [n]_q, & \text{when } p = 1 \\ n, & \text{when } p = q = 1 \end{cases}$$

where  $[n]_q$  denotes the  $q$ -integers and  $n = 0, 1, 2, \dots$ .

The formula for  $(p, q)$ -binomial expansion is as follows:

$$(ax + by)_{p,q}^n := \sum_{k=0}^n p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k} b^k x^{n-k} y^k,$$

$$(x + y)_{p,q}^n = (x + y)(px + qy)(p^2x + q^2y) \cdots (p^{n-1}x + q^{n-1}y),$$

$$(1 - x)_{p,q}^n = (1 - x)(p - qx)(p^2 - q^2x) \cdots (p^{n-1} - q^{n-1}x),$$

where  $(p, q)$ -binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}! [n-k]_{p,q}!}.$$

Details on  $(p, q)$ -calculus can be found in [10, 11, 27].

Also, we have  $(p, q)$ -analogue of Euler's identity as:

$$(1 - x)_{p,q}^n = \prod_{s=0}^{n-1} (p^s - q^s x) = (1 - x)(p - qx)(p^2 - q^2x) \cdots (p^{n-1} - q^{n-1}x)$$

$$= \sum_{k=0}^n (-1)^k p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k.$$

Again by some simple calculations and using the property of  $(p, q)$ -integers, we get  $(p, q)$ -analogue of Pascal's relation as follows:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} + p^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} \quad (1.4)$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = p^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q}. \quad (1.5)$$

We recall some results from [17] for Lupas  $(p, q)$ -Bernstein operators, which reproduces linear and constant functions.

**Some auxillary results:**

$$(1) L_{p,q}^n(1, \frac{u}{u+1}) = 1$$

$$(2) L_{p,q}^n(t, \frac{u}{u+1}) = \frac{u}{u+1}$$

$$(3) L_{p,q}^n(t^2, \frac{u}{u+1}) = \frac{u}{u+1} \frac{p^{n-1}}{[n]_{p,q}} + \frac{qu}{u+1} (\frac{qu}{p+qu}) \frac{[n-1]_{p,q}}{[n]_{p,q}}$$

or equivalently for  $x = \frac{u}{u+1}$

$$L_{p,q}^n(1, x) = 1, \tag{1.6}$$

$$L_{p,q}^n(t, x) = x, \tag{1.7}$$

$$L_{p,q}^n(t^2, x) = \frac{p^{n-1}x}{[n]_{p,q}} + \frac{q^2 x^2}{p(1-x) + qx} \frac{[n-1]_{p,q}}{[n]_{p,q}}. \tag{1.8}$$

**2. CONSTRUCTION OF  $(p, q)$ -LUPAŞ STANCU OPERATORS**

In this section, we introduce  $(p, q)$ -Lupaş Stancu operators as follows:

For any  $p > 0$  and  $q > 0$ , the linear operators  $L_{p,q}^n : C[0, 1] \rightarrow C[0, 1]$

$$L_{n,p,q}^{\alpha,\beta}(f; x) = \sum_{k=0}^n f\left(\frac{p^{n-k} [k]_{p,q} + \alpha}{[n]_{p,q} + \beta}\right) b_{p,q}^{k,n}(t) \tag{2.1}$$

and  $b_{p,q}^{k,n}(t)$  is given by

$$b_{p,q}^{k,n}(t) = \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=1}^n \{p^{j-1}(1-t) + q^{j-1}t\}}, \tag{2.2}$$

where  $0 < \alpha < \beta$ .

We give some equalities for operators (2.1) in the following lemma.

**Lemma 4.1.** The following equalities are true:

- (i)  $L_{n,p,q}^{\alpha,\beta}(1; x) = 1,$
- (ii)  $L_{n,p,q}^{\alpha,\beta}(t; x) = \frac{[n]_{p,q}x + \alpha}{[n]_{p,q} + \beta},$
- (iii)  $L_{n,p,q}^{\alpha,\beta}(t^2; x) = \frac{1}{([n]_{p,q} + \beta)^2} \frac{q^2 [n]_{p,q} [n-1]_{p,q}}{p(1-x) + qx} x^2 + \frac{[n]_{p,q}(2\alpha + p^{n-1})}{([n]_{p,q} + \beta)^2} x + \frac{\alpha^2}{([n]_{p,q} + \beta)^2}.$

*Proof.* Proof of part (i) is obvious.

$$\begin{aligned} L_{n,p,q}^{\alpha,\beta}(t; x) &= \sum_{k=0}^n \left( \frac{p^{n-k} [k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right) b_{p,q}^{k,n}(t) \\ &= \frac{[n]_{p,q}}{[n]_{p,q} + [\beta]} L_{p,q}^n(t; x) + \frac{[\alpha]}{[n]_{p,q} + [\beta]} L_{p,q}^n(1; x). \end{aligned}$$

So from inequalities (1.6) and (1.7), we get the result.

Proof (iii)

$$\begin{aligned} L_{n,p,q}^{\alpha,\beta}(t^2; x) &= \sum_{k=0}^n \left( \frac{p^{n-k} [k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right) b_{p,q}^{k,n}(t) \\ &= \frac{1}{([n]_{p,q} + \beta)^2} \left[ p^{2n-2k} [k]_{p,q}^2 b_{p,q}^{k,n}(t) \right. \\ &\quad \left. + 2\alpha p^{n-k} [k]_{p,q} b_{p,q}^{k,n}(t) + \alpha^2 b_{p,q}^{k,n}(t) \right] \\ &= \frac{1}{([n]_{p,q} + \beta)^2} [A + B + C]. \end{aligned}$$

$$A = p^{2n} \sum_{k=0}^n \frac{[k]_{p,q}^2}{p^{2k}} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=1}^n \{p^{j-1}(1-t) + q^{j-1}t\}}$$

$$A = [n] p^{2n} \sum_{k=1}^n \frac{[k]_{p,q}}{p^{2k}} \frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} u^k}{\prod_{j=1}^n \{p^{j-1} + q^{j-1}u\}}.$$

On shifting the limits and on replacing  $k$  by  $k+1$ , we get

$$\begin{aligned} A &= [n] p^{2n} \sum_{k=1}^n \frac{[k+1]_{p,q}}{p^{2k+2}} \frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k+1)}{2}} u^k}{\prod_{j=1}^{n-1} \{p^j + q^j u\}}, \\ &= [n] p^n \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{[k+1]_{p,q}}{p^{k+2}} \frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k+1)}{2}} \left(\frac{qu}{p}\right)^k}{\prod_{j=0}^{n-2} \{p^j + q^j \left(\frac{qu}{p}\right)\}}. \end{aligned}$$

Using  $[k + 1]_{p,q} = p^k + q[k]_{p,q}$ , we get our desired result:

$$\begin{aligned}
 A &= [n]p^n \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{[p^k + q[k]]}{p^{k+2}} \frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k+1)}{2}} \left(\frac{qu}{p}\right)^k}{\prod_{j=0}^{n-2} \{p^j + q^j(\frac{qu}{p})\}}, \\
 &= [n]_{p,q} p^{n-1} \frac{u}{u+1} + \frac{q^2 u^2 [n]_{p,q} [n-1]_{p,q}}{(u+1)(p+qu)},
 \end{aligned}$$

equivalently

$$A = [n]_{p,q} p^{n-1} x + \frac{q^2 [n]_{p,q} [n-1]_{p,q}}{(p(1-x) + qx)} x^2.$$

Similarly

$$\begin{aligned}
 B &= 2\alpha p^n \sum_{k=0}^n \frac{[k]}{p^k} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=1}^n \{p^{j-1}(1-t) + q^{j-1}t\}}, \\
 &= 2\alpha [n]p^n \sum_{k=1}^n \frac{1}{p^k} \frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} u^k}{\prod_{j=1}^n \{p^{j-1} + q^{j-1}u\}}.
 \end{aligned}$$

After shifting the limits and on replacing  $k$  by  $k + 1$ , we get

$$B = 2\alpha [n]p^n \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{1}{p} \frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k-1)}{2}} \left(\frac{qu}{p}\right)^k}{\prod_{j=0}^{n-2} \{p^j + q^j(\frac{qu}{p})\}},$$

which implies

$$B = 2\alpha [n]_{p,q} x.$$

Similarly

$$\begin{aligned}
 C &= \alpha^2 \sum_{k=0}^n \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=1}^n \{p^{j-1}(1-t) + q^{j-1}t\}}, \\
 &= \alpha^2.
 \end{aligned}$$

□

**Theorem 2.1.** Let  $0 < q_n < p_n \leq 1$  such that  $\lim_{n \rightarrow \infty} p_n = 1$ ,  $\lim_{n \rightarrow \infty} q_n = 1$ ,  $\lim_{n \rightarrow \infty} q_n^n = 1$  and for  $f \in C[0, 1]$ , we have  $\lim_n |L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| = 0$ .

*Proof.* Let us recall the following Korovkin's theorem see [20]. Let  $(T_n)$  be a sequence of positive linear operators from  $C[a, b]$  into  $C[a, b]$ . Then  $\lim \|T_n(f, x) - f(x)\|_{C[a,b]} = 0$ , for all  $f \in C[a, b]$  if and only if  $\lim_n \|T_n(f_i, x) - f_i(x)\|_{C[a,b]} = 0$ , for  $i = 0, 1, 2$ , where  $f_0(t) = 1$ ,  $f_1(t) = t$  and  $f_2(t) = t^2$ .

□

### 3. THE RATE OF CONVERGENCE

In this section, we compute the rates of convergence of the operators  $L_{n,p,q}^{\alpha,\beta}(f; x)$  to the functions  $f$  by means of modulus of continuity, elements of Lipschitz class and peetre's K-functional.

Let  $f \in C[0, 1]$ . The modulus of continuity of  $f$  denoted by  $\omega(f, \delta)$  is defined as:

$$\omega(f, \delta) = \sup_{y,x \in [0,1], |y-x| < \delta} |f(y) - f(x)|.$$

where  $w(f; \delta)$  satisfies the following conditions: for all  $f \in C[0, 1]$ ,

$$\lim_{\delta \rightarrow 0} w(f; \delta) = 0. \quad (3.1)$$

and

$$|f(y) - f(x)| \leq w(f; \delta) \left( \frac{|y-x|}{\delta} + 1 \right). \quad (3.2)$$

**Theorem 3.1.** Let  $0 < q < p \leq 1$ , and  $f \in C[0, 1]$ , and  $\delta > 0$ , we have

$$\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq 2\omega(f; \delta_n)$$

where

$$\delta_n = \left[ \left( \frac{q^2 [n]_{p,q} [n-1]_{p,q}}{([n]_{p,q} + \beta)^2 (p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \left( \frac{p^{n-1} [n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right) + \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \right]^{\frac{1}{2}}.$$



*Proof.* From lemma (4.1) we have

$$|L_{n,p,q}^{\alpha,\beta}(t-x)^2; x| = \left( \frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) x^2 + \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right) x + \frac{\alpha^2}{([n]_{p,q} + \beta)^2}. \tag{3.3}$$

For  $x \in [0, 1]$ , we take

$$|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq w(f; \delta) \left\{ 1 + \frac{1}{\delta} (L_{n,p,q}^{\alpha,\beta}(t-x)^2 : x)^{\frac{1}{2}} \right\},$$

then we get

$$\begin{aligned} \|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} &\leq w(f; \delta) \left\{ 1 + \frac{1}{\delta} (L_{n,p,q}^{\alpha,\beta}(t-x)^2 : x)^{\frac{1}{2}} \right\} \\ &\leq w(f; \delta) \left\{ 1 + \frac{1}{\delta} \left( \frac{1}{([n]_{p,q} + \beta)^2} \frac{q^2[n]_{p,q}[n-1]_{p,q}}{p(1-x) + qx} \right. \right. \\ &\quad \left. \left. - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right. \right. \\ &\quad \left. \left. + \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \right)^{\frac{1}{2}} \right\}. \end{aligned}$$

If we choose

$$\begin{aligned} \delta_n &= \left[ \left( \frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) \right. \\ &\quad \left. + \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right) + \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \right]^{\frac{1}{2}}. \end{aligned}$$

Then we have

$$\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq 2\omega(f; \delta_n).$$

So we have the desired result. □

Now we compute the approximation order of operator  $L_{n,p,q}^{\alpha,\beta}$  in term of the elements of the usual Lipschitz class.

Let  $f \in C[0, 1]$  and  $0 < \rho \leq 1$ . We recall that  $f$  belongs to  $Lip_M(\rho)$  if the inequality

$$|f(x) - f(y)| \leq M|x - y|^\rho; \text{ for all } x, y \in [0, 1] \quad (3.4)$$

holds.

**Theorem 3.2.** For all  $f \in Lip_M(\rho)$

$$\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq M\delta_n^\rho$$

where

$$\delta_n = \left[ \left( \frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right) + \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \right]^{\frac{1}{2}}$$

and  $M$  is a positive constant.

*Proof.* Let  $f \in Lip_M(\rho)$  and  $0 < \rho \leq 1$ . by (3.4) and linearity and monotonicity of  $L_{n,p,q}^{\alpha,\beta}$  then we have

$$\begin{aligned} |L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| &\leq L_{n,p,q}^{\alpha,\beta}(|f(t) - f(x)|; x) \\ &\leq L_{n,p,q}^{\alpha,\beta}(|t - x|^\rho; x). \end{aligned}$$

Applying the Holder inequality with  $m = \frac{2}{\rho}$  and  $n = \frac{2}{2-\rho}$ , we get

$$|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq (L_{n,p,q}^{\alpha,\beta}((t-x)^2; x))^{\frac{\rho}{2}}. \quad (3.5)$$

if we choose  $\delta = \delta_n$  as above, then proof is completed.

Finally, we will study the rate of convergence of the positive linear operators  $L_{n,p,q}^{\alpha,\beta}$  by means of the Peetre's K-functionals.

$C^2[0, 1]$  : The space of those functions  $f$  for which  $f, f', f'' \in C[0, 1]$ . we recall the following norm in the space  $C^2[0, 1]$  :

$$\|f\|_{C^2[0,1]} = \|f\|_{C[0,1]} + \|f'\|_{C[0,1]} + \|f''\|_{C[0,1]}.$$

We consider the following Peetre's K-functional

$$K(f, \delta) := \inf_{g \in C^2[0,1]} \left\{ \|f - g\|_{C[0,1]} + \delta \|g\|_{C^2[0,1]} \right\}.$$

□

**Theorem 3.3.** *Let  $f \in C[0, 1]$ . Then we have*

$$\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq 2K(f; \delta_n)$$

Where  $K(f; \delta_n)$  is Peetre's functional and

$$\begin{aligned} \delta_n = & \frac{1}{4} \left( \frac{q^2 [n]_{p,q} [n-1]_{p,q}}{([n]_{p,q} + \beta)^2 (p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) \\ & + \frac{1}{4} \left( \frac{p^{n-1} [n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right) + \frac{1}{4} \frac{\alpha^2}{([n]_{p,q} + \beta)^2} + \frac{1}{2} \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]}. \end{aligned}$$

*Proof.* Let  $g \in C^2[0, 1]$ . If we use the Taylor's expansion of the function  $g$  at  $s = x$ , we have

$$g(s) = g(x) + (s-x)g'(x) + \frac{(s-x)^2}{2}g''(x).$$

Hence we get

$$\begin{aligned} \|L_{n,p,q}^{\alpha,\beta}(g; x) - g(x)\|_{C[0,1]} \leq & \|L_{n,p,q}^{\alpha,\beta}((s-x); x)\|_{C[0,1]} \|g(x)\|_{C^2[0,1]} \\ & + \frac{1}{2} \|L_{n,p,q}^{\alpha,\beta}((s-x)^2; x)\|_{C[0,1]} \|g(x)\|_{C^2[0,1]}. \end{aligned} \tag{3.6}$$

From the lemma (2.1) we have

$$\|L_{n,p,q}^{\alpha,\beta}((s-x); x)\|_{C[0,1]} \leq \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]}. \tag{3.7}$$

So if we use (3.3) and (3.7) in (3.6), then we get

$$\begin{aligned} \|L_{n,p,q}^{\alpha,\beta}(g; x) - g(x)\|_{C[0,1]} \leq & \left[ \frac{1}{2} \left( \frac{q^2 [n]_{p,q} [n-1]_{p,q}}{([n]_{p,q} + \beta)^2 (p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) \right. \\ & + \left( \frac{1}{2} \frac{p^{n-1} [n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right) + \frac{1}{2} \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \tag{3.8} \\ & \left. + \frac{[\alpha] + [\beta]}{[n] + [\beta]} \right] \|g(x)\|_{C[0,1]}. \end{aligned} \tag{3.9}$$

On the other hand, we can write

$$\begin{aligned} |L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| &\leq |L_{n,p,q}^{\alpha,\beta}(f - g; x)| + |L_{n,p,q}^{\alpha,\beta}(g; x) - g(x)| \\ &\quad + |f(x) - g(x)|. \end{aligned}$$

If we take the maximum on  $[0, 1]$ , we have

$$\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq 2\|f - g\|_{C[0,1]} + \|L_{n,p,q}^{\alpha,\beta}(g; x) - g(x)\|_{C[0,1]}. \quad (3.10)$$

If we consider (3.8) in (3.10), we obtain

$$\begin{aligned} \|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} &\leq 2\|f - g\|_{C[0,1]} + \left[ \frac{1}{4} \left( \frac{q[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} \right. \right. \\ &\quad \left. \left. - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \left( \frac{1}{4} \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right) \right. \\ &\quad \left. + \frac{1}{4} \frac{\alpha^2}{([n]_{p,q} + \beta)^2} + \frac{1}{2} \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]} \right] \|g(x)\|_{C^2[0,1]}. \end{aligned}$$

If we choose

$$\begin{aligned} \delta_n &= \frac{1}{4} \left( \frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) \\ &\quad + \left( \frac{1}{4} \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right) + \frac{1}{4} \frac{\alpha^2}{([n]_{p,q} + \beta)^2} + \frac{1}{2} \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]}, \end{aligned}$$

then we get

$$\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq 2 \left\{ \|f - g\|_{C[0,1]} + \delta_n \|g(x)\|_{C^2[0,1]} \right\}. \quad (3.11)$$

Finally, one can observe that if we take the infimum of both side of above inequality for the function  $g \in C^2[0, 1]$ , we can find

$$\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq 2K(f, \delta_n).$$

□

#### 4. THE RATES OF STATISTICAL CONVERGENCE

At this point, let us recall the concept of statistical convergence. The statistical convergence which was introduced by Fast [41] in 1951, is an important research area in approximation theory. In [41], Gadjiev and Orhan used the concept of statistical convergence in approximation theory. They proved a Bohman-Korovkin type theorem for statistical convergence.

Recently, statistical approximation properties of many operators are investigated in [4, 25, 26, 29, 30].

A sequence  $x = (x_k)$  is said to be statistically convergent to a number  $L$  if for every  $\epsilon > 0$ ,

$$\delta\{K \in \mathbf{N} : |x_k - L| \geq \epsilon\} = 0,$$

where  $\delta(K)$  is the natural density of the set  $K \subseteq \mathbf{N}$ .

The density of subset  $K \subseteq N$  is defined by

$$\delta(K) := \lim_n \frac{1}{n} \{\text{the number } k \leq n : k \in K\}$$

whenever the limit exists.

For instance,  $\delta(\mathbf{N}) = 1$ ,  $\delta\{2K : k \in \mathbf{N}\} = \frac{1}{2}$  and  $\delta\{k^2 : K \in \mathbf{N}\} = 0$ .

To emphasize the importance of the statistical convergence, we have an example: The sequence

$$X_k = \begin{cases} L_1; & \text{if } k = m^2, \\ L_2; & \text{if } k \neq m^2. \end{cases} \quad \text{where } m \in \mathbf{N} \quad (4.1)$$

is statistically convergent to  $L_2$  but not convergent in ordinary sense when  $L_1 \neq L_2$ . We note that any convergent sequence is statistically convergent but not conversely.

Now we consider sequences  $q = q_n$  and  $p = p_n$  such that:

$$st - \lim_n q_n = 1, \quad st - \lim_n p_n = 1, \quad \text{and} \quad st - \lim_n q_n^n = 1. \quad (4.2)$$

Gadjiev and Orhan [41] gave the following theorem for linear positive operators which is about statistically Korovkin type theorem. Now, we recall this theorem.

**Theorem 4.1.** If  $A_n$  be the sequence of linear positive operators from  $C[a, b]$  to  $C[a, b]$  satisfies the conditions

$$st - \lim_n \|A_n((t^\nu; x)) - (x)^\nu\|_{C[0, 1]} = 0 \text{ for } \nu = 0, 1, 2.$$

then for any function  $f \in C[a, b]$ ,

$$st - \lim_n \|A_n(f; \cdot) - f\|_{C[a, b]} = 0.$$

Now we will discuss the rates of statistical convergence of  $L_{n,p,q}^{\alpha,\beta}$  operators.

**Remark 4.2.** For  $q \in (0, 1)$  and  $p \in (q, 1]$ , it is obvious that

$$\lim_{n \rightarrow \infty} [n]_{p,q} = \begin{cases} 0, & \text{when } p, q \in (0, 1) \\ \frac{1}{1-q}, & \text{when } p = 1 \text{ and } q \in (0, 1). \end{cases}$$

In order to reach to convergence results of the operator  $L_{p,q}^n(f; x)$ , we take a sequence  $q_n \in (0, 1)$  and  $p_n \in (q_n, 1]$  such that  $\lim_{n \rightarrow \infty} p_n = 1$ ,  $\lim_{n \rightarrow \infty} q_n = 1$ . So we get  $\lim_{n \rightarrow \infty} [n]_{p_n, q_n} = \infty$ .

**Theorem 4.3.** Let  $L_{n,p,q}^{\alpha,\beta}$  be the sequence of operators and the sequences  $p = p_n$  and  $q = q_n$  satisfies Remark 4.2 then for any function  $f \in C[0, 1]$

$$st - \lim_n \|L_{n,p_n,q_n}^{\alpha,\beta}(f, \cdot) - f\| = 0. \quad (4.3)$$

*Proof.* Clearly for  $\nu = 0$ ,

$$L_{n,p_n,q_n}^{\alpha,\beta}(1, x) = 1,$$

which implies

$$st - \lim_n \|L_{n,p_n,q_n}^{\alpha,\beta}(1; x) - 1\| = 0.$$

For  $\nu = 1$

$$\begin{aligned} \|L_{n,p_n,q_n}^{\alpha,\beta}(t; x) - x\| &\leq \left| \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta} x + \frac{\alpha}{[n]_{p_n,q_n} + \beta} - x \right| \\ &= \left| \left( \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta} - 1 \right) x + \frac{\alpha}{[n]_{p_n,q_n} + \beta} \right| \\ &\leq \left| \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta} - 1 \right| + \left| \frac{\alpha}{[n]_{p_n,q_n} + \beta} \right|. \end{aligned}$$

For a given  $\epsilon > 0$ , let us define the following sets.

$$\begin{aligned}
 U &= \{n : \|L_{n,p_n,q_n}^{\alpha,\beta}(t; x) - x\| \geq \epsilon\} \\
 U' &= \{n : 1 - \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta} \geq \epsilon\} \\
 U'' &= \{n : \frac{\alpha}{[n]_{p_n,q_n} + \beta} \geq \epsilon\}.
 \end{aligned}$$

It is obvious that  $U \subseteq U'' \cup U'$ ,

So using

$$\delta\{k \leq n : 1 - \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta} \geq \epsilon\},$$

then we get

$$st - \lim_n \|L_{n,p_n,q_n}^{\alpha,\beta}(t; x) - x\| = 0. \tag{4.4}$$

Lastly for  $\nu = 2$ , we have

$$\begin{aligned}
 \|L_{n,p_n,q_n}^{\alpha,\beta}(t^2; x) - x^2\| &\leq \left| \frac{q^2[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{p(1-x) + qx} \frac{1}{([n]_{p_n,q_n} + \beta)^2} - 1 \right| \\
 &\quad + \left| \frac{[n]_{p_n,q_n}(2\alpha + p^{n-1})^2}{[n]_{p_n,q_n} + \beta} x \right| + \left| \frac{\alpha^2}{([n]_{p_n,q_n} + \beta)^2} \right|.
 \end{aligned}$$

If we choose

$$\begin{aligned}
 \alpha_n &= \frac{q^2[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{p(1-x) + qx} \frac{1}{([n]_{p_n,q_n} + \beta)^2} - 1 \\
 \beta_n &= \frac{[n]_{p_n,q_n}(2\alpha + p^{n-1})^2}{[n]_{p_n,q_n} + \beta} \\
 \gamma_n &= \frac{\alpha^2}{([n]_{p_n,q_n} + \beta)^2}.
 \end{aligned}$$

Then

$$st - \lim_n \alpha_n = st - \lim_n \beta_n = st - \lim_n \gamma_n = 0.$$

Now given  $\epsilon > 0$ , we define the following four sets:

$$\begin{aligned}
 U &= \|L_{n,p_n,q_n}^{\alpha,\beta}(t^2; x) - x^2\| \geq \epsilon, \\
 U_1 &= \{n : \alpha_n \geq \frac{\epsilon}{3}\}, \\
 U_2 &= \{n : \beta_n \geq \frac{\epsilon}{3}\},
 \end{aligned}$$

$$U_3 = \{n : \gamma_n \geq \frac{\epsilon}{3}\}.$$

It is obvious that  $U \subseteq U_1 \cup U_2 \cup U_3$ . Thus we obtain

$$\begin{aligned} & \delta\{K \leq n : \|L_{n,p,q}^{\alpha,\beta}(t^2 : x) - x^2\| \geq \epsilon\} \\ & \leq \delta\{K \leq n : \alpha_n \geq \frac{\epsilon}{3}\} + \delta\{K \leq n : \beta_n \geq \frac{\epsilon}{3}\} + \delta\{K \leq n : \gamma_n \geq \frac{\epsilon}{3}\}. \end{aligned}$$

So the right hand side of the inequalities is zero.

Then

$$st - \lim_n \|L_{n,p_n,q_n}^{\alpha,\beta}(t; x) - x\| = 0$$

holds and thus the proof is completed. □

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